

# 7

# Indeterminate Forms

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## INTRODUCTION

We know that if  $\lim_{x \rightarrow c} f(x)$ ,  $\lim_{x \rightarrow c} g(x)$  both exist and  $\lim_{x \rightarrow c} g(x) \neq 0$ , then  $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} g(x)}$ . The question arises, what happens if  $\lim_{x \rightarrow c} g(x) = 0$ . It is easy to see that if  $\lim_{x \rightarrow c} g(x) = 0$ , then the necessary condition for  $\lim_{x \rightarrow c} \frac{f(x)}{g(x)}$  to exist (finitely) is that  $\lim_{x \rightarrow c} f(x) = 0$ .

In fact, if  $\lim_{x \rightarrow c} \frac{f(x)}{g(x)}$  exists, say  $l$ , then

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} \frac{f(x)}{g(x)} \cdot g(x) = \lim_{x \rightarrow c} \frac{f(x)}{g(x)} \cdot \lim_{x \rightarrow c} g(x) = l \cdot 0 = 0.$$

If  $\lim_{x \rightarrow c} f(x) = 0 = \lim_{x \rightarrow c} g(x)$ , then  $\frac{f(x)}{g(x)}$  is said to assume **indeterminate form**  $\frac{0}{0}$  as  $x \rightarrow c$ .

We also have some other *indeterminate forms* such as  $\frac{\infty}{\infty}$ ,  $\infty - \infty$ ,  $0 \cdot \infty$ ,  $0^\circ$ ,  $\infty^\circ$  and  $1^\infty$  etc.

## 7.1 INDETERMINATE FORM $\frac{0}{0}$

### L' Hôpital's rule

If  $f(x)$ ,  $g(x)$  are differentiable and  $g'(x) \neq 0$  for all  $x$  in  $(c - \delta, c + \delta)$  except possibly at  $x = c$ ,

$\lim_{x \rightarrow c} f(x) = 0 = \lim_{x \rightarrow c} g(x)$  and  $\lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}$  exists (finitely or infinitely), then  $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}$ .

(We accept it without proof.)

**Remark.** L' Hôpital's rule remains valid when  $\lim_{x \rightarrow c}$  is replaced by one sided limits

$\lim_{x \rightarrow c^-}$  or  $\lim_{x \rightarrow c^+}$ .

## ILLUSTRATIVE EXAMPLES

**Example 1.** Evaluate the following limits :

$$(i) \lim_{x \rightarrow 4} \frac{x^4 - 256}{x^2 - 16} \quad (ii) \lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos x}{\frac{\pi}{2} - x} \quad (iii) \lim_{x \rightarrow \frac{\pi}{4}} \frac{1 - \tan x}{\cos 2x}. \quad (\text{I.S.C. 2009})$$

**Solution.** (i)  $\lim_{x \rightarrow 4} \frac{x^4 - 256}{x^2 - 16}$   $\left( \frac{0}{0} \text{ form, use L'Hopital's rule} \right)$

$$= \lim_{x \rightarrow 4} \frac{4x^3 - 0}{2x - 0} = \lim_{x \rightarrow 4} 2x^2 = 2 \cdot 4^2 = 32.$$

(ii)  $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos x}{\frac{\pi}{2} - x}$   $\left( \frac{0}{0} \text{ form, use L'Hopital's rule} \right)$

$$= \lim_{x \rightarrow \frac{\pi}{2}} \frac{-\sin x}{-1} = \lim_{x \rightarrow \frac{\pi}{2}} \sin x = \sin \frac{\pi}{2} = 1.$$

(iii)  $\lim_{x \rightarrow \frac{\pi}{4}} \frac{1 - \tan x}{\cos 2x}$   $\left( \frac{0}{0} \text{ form, use L'Hopital's rule} \right)$

$$= \lim_{x \rightarrow \frac{\pi}{4}} \frac{0 - \sec^2 x}{-\sin 2x \cdot 2} = \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sec^2 x}{2 \sin 2x}$$

$$= \frac{\sec^2 \frac{\pi}{4}}{2 \sin \frac{\pi}{2}} = \frac{(\sqrt{2})^2}{2 \cdot 1} = \frac{2}{2} = 1.$$

**Example 2.** Evaluate the following limits :

(i)  $\lim_{x \rightarrow 0} \frac{\sin x - x + \frac{1}{6}x^3}{x^3}$  (I.S.C. 2001)      (ii)  $\lim_{x \rightarrow 0} \frac{x e^x - \log(1+x)}{x^2}$ .

**Solution.** (i)  $\lim_{x \rightarrow 0} \frac{\sin x - x + \frac{1}{6}x^3}{x^3}$   $\left( \frac{0}{0} \text{ form, use L'Hopital's rule} \right)$

$$= \lim_{x \rightarrow 0} \frac{\cos x - 1 + \frac{1}{2}x^2}{3x^2} \quad \left( \frac{0}{0} \text{ form} \right)$$

$$= \lim_{x \rightarrow 0} \frac{-\sin x + x}{6x} \quad \left( \frac{0}{0} \text{ form} \right)$$

$$= \lim_{x \rightarrow 0} \frac{-\cos x + 1}{6} = \frac{-1 + 1}{6} = 0.$$

(ii)  $\lim_{x \rightarrow 0} \frac{x e^x - \log(1+x)}{x^2}$   $\left( \frac{0}{0} \text{ form, use L'Hopital's rule} \right)$

$$\begin{aligned} &= \lim_{x \rightarrow 0} \frac{x \cdot e^x + e^x \cdot 1 - \frac{1}{1+x}}{2x} \\ &= \lim_{x \rightarrow 0} \frac{(x+1)e^x - \frac{1}{1+x}}{2x} \quad \left( \frac{0}{0} \text{ form, use L'Hopital's rule} \right) \end{aligned}$$

$$= \lim_{x \rightarrow 0} \frac{(x+1) \cdot e^x + e^x \cdot 1 + \frac{1}{(1+x)^2}}{2} = \frac{(0+1) \cdot 1 + 1 + \frac{1}{1}}{2} = \frac{3}{2}.$$

**Example 3.** Evaluate the following limits :

(i)  $\lim_{x \rightarrow 1} \frac{1 + \log x - x}{1 - 2x + x^2}$       (ii)  $\lim_{x \rightarrow 0} \frac{e^x + e^{-x} + 2 \cos x - 4}{x^4}$ .

**Solution.** (i)  $\lim_{x \rightarrow 1} \frac{1 + \log x - x}{1 - 2x + x^2}$   $\left( \frac{0}{0} \text{ form, use L'Hopital's rule} \right)$

$$= \lim_{x \rightarrow 1} \frac{0 + \frac{1}{x} - 1}{0 - 2 + 2x} = \lim_{x \rightarrow 1} \frac{1-x}{-2x(1-x)} = \lim_{x \rightarrow 1} \frac{1}{-2x} = -\frac{1}{2}.$$

$$\begin{aligned}
 (ii) \quad & \text{Lt}_{x \rightarrow 0} \frac{e^x + e^{-x} + 2 \cos x - 4}{x^4} && \left( \frac{0}{0} \text{ form, use L'Hopital's rule} \right) \\
 &= \text{Lt}_{x \rightarrow 0} \frac{e^x - e^{-x} - 2 \sin x}{4x^3} && \left( \frac{0}{0} \text{ form} \right) \\
 &= \text{Lt}_{x \rightarrow 0} \frac{e^x + e^{-x} - 2 \cos x}{12x^2} && \left( \frac{0}{0} \text{ form} \right) \\
 &= \text{Lt}_{x \rightarrow 0} \frac{e^x - e^{-x} + 2 \sin x}{24x} && \left( \frac{0}{0} \text{ form} \right) \\
 &= \text{Lt}_{x \rightarrow 0} \frac{e^x + e^{-x} + 2 \cos x}{24} = \frac{1+1+2}{24} = \frac{1}{6}.
 \end{aligned}$$

**Example 4.** Evaluate the following limits :

$$(i) \quad \text{Lt}_{x \rightarrow \pi^+} \frac{\sin x}{\sqrt{x-\pi}}$$

$$(ii) \quad \text{Lt}_{x \rightarrow 0} \frac{\log(1+x^3)}{\sin^3 x}.$$

$$\begin{aligned}
 \text{Solution. } (i) \quad & \text{Lt}_{x \rightarrow \pi^+} \frac{\sin x}{\sqrt{x-\pi}} && \left( \frac{0}{0} \text{ form, use L'Hopital's rule} \right) \\
 &= \text{Lt}_{x \rightarrow \pi^+} \frac{\cos x}{\frac{1}{2}(x-\pi)^{-1/2} \cdot 1} = \text{Lt}_{x \rightarrow \pi^+} 2\sqrt{x-\pi} \cos x = 0.
 \end{aligned}$$

It may be noted that  $\sqrt{x-\pi}$  is not defined on the left of  $\pi$  so that the left limit does not exist.

$$\begin{aligned}
 (ii) \quad & \text{Lt}_{x \rightarrow 0} \frac{\log(1+x^3)}{\sin^3 x} = \text{Lt}_{x \rightarrow 0} \frac{\log(1+x^3) \cdot \left(\frac{x}{\sin x}\right)^3}{x^3} \\
 &= \text{Lt}_{x \rightarrow 0} \frac{\log(1+x^3)}{x^3} \cdot 1^3 && \left( \because \text{Lt}_{x \rightarrow 0} \frac{x}{\sin x} = 1 \right) \\
 &= \text{Lt}_{x \rightarrow 0} \frac{\log(1+x^3)}{x^3} && \left( \frac{0}{0} \text{ form, use L'Hopital's rule} \right) \\
 &= \text{Lt}_{x \rightarrow 0} \frac{\frac{1}{1+x^3} \cdot 3x^2}{3x^2} = \text{Lt}_{x \rightarrow 0} \frac{1}{1+x^3} = \frac{1}{1+0} = 1.
 \end{aligned}$$

**Example 5.** Evaluate the following limits :

$$(i) \quad \text{Lt}_{x \rightarrow 0} \frac{x - \tan^{-1} x}{x - \sin x} \quad (\text{I.S.C. 2011}) \quad (ii) \quad \text{Lt}_{x \rightarrow 0} \frac{1 + \sin x - \cos x + \log(1-x)}{x \tan^2 x}.$$

$$\begin{aligned}
 \text{Solution. } (i) \quad & \text{Lt}_{x \rightarrow 0} \frac{x - \tan^{-1} x}{x - \sin x} && \left( \frac{0}{0} \text{ form, use L'Hopital's rule} \right) \\
 &= \text{Lt}_{x \rightarrow 0} \frac{\frac{1}{1+x^2}}{1 - \cos x} && \left( \frac{0}{0} \text{ form} \right) \\
 &= \text{Lt}_{x \rightarrow 0} \frac{0 - (-1)(1+x^2)^{-2} \cdot 2x}{\sin x} = \text{Lt}_{x \rightarrow 0} \frac{2x}{(1+x^2)^2 \sin x} \\
 &= \text{Lt}_{x \rightarrow 0} \frac{2}{(1+x^2)^2} \cdot \left(\frac{x}{\sin x}\right) = \frac{2}{(1+0)^2} \cdot 1 = 2.
 \end{aligned}$$

$$\begin{aligned}
 (ii) \quad & \text{Lt}_{x \rightarrow 0} \frac{1 + \sin x - \cos x + \log(1-x)}{x \tan^2 x} = \text{Lt}_{x \rightarrow 0} \frac{1 + \sin x - \cos x + \log(1-x)}{x^3} \cdot \left(\frac{x}{\tan x}\right)^2 \\
 &= \text{Lt}_{x \rightarrow 0} \frac{1 + \sin x - \cos x + \log(1-x)}{x^3} \cdot 1^2 && \left( \because \text{Lt}_{x \rightarrow 0} \frac{x}{\tan x} = 1 \right) \\
 &= \text{Lt}_{x \rightarrow 0} \frac{1 + \sin x - \cos x + \log(1-x)}{x^3} && \left( \frac{0}{0} \text{ form} \right) \\
 &= \text{Lt}_{x \rightarrow 0} \frac{\cos x + \sin x + \frac{1}{1-x}(-1)}{3x^2} && \left( \frac{0}{0} \text{ form} \right) \\
 &= \text{Lt}_{x \rightarrow 0} \frac{-\sin x + \cos x - \frac{1}{(1-x)^2}}{6x} && \left( \frac{0}{0} \text{ form} \right) \\
 &= \text{Lt}_{x \rightarrow 0} \frac{-\cos x - \sin x - \frac{2}{(1-x)^3}}{6} = \frac{-1-0-2}{6} = -\frac{1}{2}.
 \end{aligned}$$

**Example 6.** What is the fallacy in the following use of L' Hôpital's rule?

$$\lim_{x \rightarrow 1} \frac{x^3 + 3x - 4}{2x^2 + x - 3} = \lim_{x \rightarrow 1} \frac{3x^2 + 3}{4x + 1} = \lim_{x \rightarrow 1} \frac{6x}{4} = \frac{3}{2}$$

**Solution.** The function  $\frac{3x^2 + 3}{4x + 1}$  is not of the form  $\frac{0}{0}$  as  $x \rightarrow 1$ , therefore, L'Hôpital's rule is not applicable to evaluate  $\lim_{x \rightarrow 1} \frac{3x^2 + 3}{4x + 1}$ .

In fact, we have

$$\lim_{x \rightarrow 1} \frac{3x^2 + 3}{4x + 1} = \frac{3 \cdot 1^2 + 3}{4 \cdot 1 + 1} = \frac{6}{5} \text{ and hence } \lim_{x \rightarrow 1} \frac{x^3 + 3x - 4}{2x^2 + x - 3} = \frac{6}{5}.$$

## EXERCISE 7.1

Evaluate the following (1 to 13) limits :

- |   |   |
|---|---|
| 1. (i) $\lim_{x \rightarrow 3} \frac{x^4 - 81}{x - 3}$                              | (ii) $\lim_{x \rightarrow 0} \frac{(1+x)^n - 1}{x}$ .                       |
| 2. (i) $\lim_{x \rightarrow 0} \frac{\sin ax}{\sin bx}$                             | (ii) $\lim_{x \rightarrow 2} \frac{e^x - e^2}{x - 2}$ .                     |
| 3. (i) $\lim_{x \rightarrow 0} \frac{x e^x}{1 - e^x}$                               | (ii) $\lim_{x \rightarrow 0} \frac{e^x - 1}{\tan 2x}$ .                     |
| 4. (i) $\lim_{x \rightarrow 0} \frac{e^x - (1+x)}{x^2}$                             | (ii) $\lim_{x \rightarrow 1} \frac{x^2 - x \log x + \log x - 1}{x - 1}$ .   |
| 5. (i) $\lim_{x \rightarrow 0} \frac{\cos x - 1}{\cos 2x - 1}$                      | (ii) $\lim_{x \rightarrow 0} \frac{8^x - 2^x}{4x}$ .                        |
| 6. (i) $\lim_{x \rightarrow 0} \frac{x - \tan x}{x - \sin x}$                       | (ii) $\lim_{x \rightarrow 0} \frac{2 \tan^{-1} x - x}{2x - \sin^{-1} x}$ .  |
| 7. (i) $\lim_{x \rightarrow 0} \frac{\log \sec 2x}{\log \sec x}$                    | (ii) $\lim_{x \rightarrow 0} \frac{\cos 2x - \cos x}{\sin^2 x}$ .           |
| 8. (i) $\lim_{x \rightarrow 0} \frac{x - \sin x}{x^3}$                              | (ii) $\lim_{x \rightarrow 0} \frac{e^x - e^{-x} - 2 \log(1+x)}{x \sin x}$ . |
| 9. (i) $\lim_{x \rightarrow 0} \frac{e^x + e^{-x} + 2 \cos x - 4}{x^3}$             | (ii) $\lim_{x \rightarrow 0} \frac{x^2 + 2 \cos x - 2}{x \sin^3 x}$ .       |
| 10. (i) $\lim_{x \rightarrow 0} \frac{\log(1-x)}{\tan \frac{\pi}{2} x}$             | (ii) $\lim_{x \rightarrow 0} \frac{(\tan^{-1} x)^2}{\log(1+x^2)}$ .         |
| 11. (i) $\lim_{x \rightarrow 0} \frac{e^x - e^{-x} - 2x}{x - \sin x}$ (I.S.C. 2013) | (ii) $\lim_{x \rightarrow 0} \frac{e^x - e^{\sin x}}{x - \sin x}$ .         |
| 12. (i) $\lim_{x \rightarrow 0} \frac{\log(1-x^2)}{\log \cos x}$                    | (ii) $\lim_{x \rightarrow 0} \frac{e^x \sin x - x - x^2}{x^3}$ .            |
| 13. (i) $\lim_{x \rightarrow 0^+} \frac{3^x - 2^x}{\sqrt{x}}$                       | (ii) $\lim_{x \rightarrow 0} \frac{(1+x)^n - nx - 1}{x^2}, n > 1$ .         |

14. What is the fallacy in the following use of L'Hôpital's rule ?

$$\lim_{x \rightarrow 2} \frac{x^3 - x^2 - x - 2}{x^3 - 3x^2 + 3x - 2} = \lim_{x \rightarrow 2} \frac{3x^2 - 2x - 1}{3x^2 - 6x + 3} = \lim_{x \rightarrow 2} \frac{6x - 2}{6x - 6} = \lim_{x \rightarrow 2} \frac{6}{6} = 1$$

## 7.2 INDETERMINATE FORM $\frac{\infty}{\infty}$

### L' Hôpital's rule

If  $f(x), g(x)$  are differentiable and  $g'(x) \neq 0$  for all  $x$  in  $(c - \delta, c + \delta)$  except possibly at  $x = c$ ,  $\lim_{x \rightarrow c} f(x) \rightarrow \infty$ ,  $\lim_{x \rightarrow c} g(x) \rightarrow \infty$  and  $\lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}$  exists (finitely or infinitely), then  $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}$ .

(We accept it without proof.)

Analogously, we have L'Hôpital's rule when  $x \rightarrow -\infty$ .

### 7.2.1 Indeterminate forms $\infty - \infty$ and $0 \cdot \infty$

These may be handled by first transforming to one of the forms  $\frac{0}{0}$  or  $\frac{\infty}{\infty}$ . See examples 3 and 4 (below).

## ILLUSTRATIVE EXAMPLES

**Example 1.** Evaluate the following limits:

$$(i) \quad \underset{x \rightarrow \frac{\pi}{2}}{\text{Lt}} \frac{\tan x}{\tan 3x}$$

$$(ii) \quad \underset{x \rightarrow \infty}{\text{Lt}} \frac{e^x + 3x^3}{4e^x + 2x^2}.$$

**Solution.** (i)  $\underset{x \rightarrow \frac{\pi}{2}}{\text{Lt}} \frac{\tan x}{\tan 3x}$  ( $\frac{\infty}{\infty}$  form, use L'Hopital's rule)

$$\begin{aligned} &= \underset{x \rightarrow \frac{\pi}{2}}{\text{Lt}} \frac{\sec^2 x}{\sec^2 3x \cdot 3} = \underset{x \rightarrow \frac{\pi}{2}}{\text{Lt}} \frac{\cos^2 3x}{3 \cos^2 x} = \underset{x \rightarrow \frac{\pi}{2}}{\text{Lt}} \frac{(4 \cos^3 x - 3 \cos x)^2}{3 \cos^2 x} \\ &= \underset{x \rightarrow \frac{\pi}{2}}{\text{Lt}} \frac{(4 \cos^2 x - 3)^2}{3} = \underset{x \rightarrow \frac{\pi}{2}}{\text{Lt}} \frac{(4.0 - 3)^2}{3} = \frac{9}{3} = 3. \end{aligned}$$

(ii)  $\underset{x \rightarrow \infty}{\text{Lt}} \frac{e^x + 3x^3}{4e^x + 2x^2}$  ( $\frac{\infty}{\infty}$  form)

$$\begin{aligned} &= \underset{x \rightarrow \infty}{\text{Lt}} \frac{e^x + 9x^2}{4e^x + 4x} \\ &= \underset{x \rightarrow \infty}{\text{Lt}} \frac{e^x + 18x}{4e^x + 4} \\ &= \underset{x \rightarrow \infty}{\text{Lt}} \frac{e^x + 18}{4e^x} \\ &= \underset{x \rightarrow \infty}{\text{Lt}} \frac{e^x}{4e^x} = \frac{1}{4}. \end{aligned}$$

**Example 2.** Evaluate the following limits:

$$(i) \quad \underset{x \rightarrow c^+}{\text{Lt}} \frac{\log(x-c)}{\log(e^x - e^c)}$$

$$(ii) \quad \underset{x \rightarrow 0^+}{\text{Lt}} \log_{\sin 2x} \sin x.$$

**Solution.** (i)  $\underset{x \rightarrow c^+}{\text{Lt}} \frac{\log(x-c)}{\log(e^x - e^c)}$  ( $\frac{\infty}{\infty}$  form)

$$\begin{aligned} &= \underset{x \rightarrow c^+}{\text{Lt}} \frac{\frac{1}{x-c}}{\frac{1}{e^x - e^c} \cdot e^x} = \underset{x \rightarrow c^+}{\text{Lt}} \frac{e^x - e^c}{(x-c)e^x} \\ &= \underset{x \rightarrow c^+}{\text{Lt}} \frac{e^x}{e^x \cdot 1 + (x-c)e^x} = \underset{x \rightarrow c^+}{\text{Lt}} \frac{1}{1+x-c} = \frac{1}{1} = 1. \end{aligned}$$

(ii)  $\underset{x \rightarrow 0^+}{\text{Lt}} \log_{\sin 2x} \sin x = \underset{x \rightarrow 0^+}{\text{Lt}} \frac{\log \sin x}{\log \sin 2x}$  ( $\frac{\infty}{\infty}$  form)

$$\begin{aligned} &= \underset{x \rightarrow 0^+}{\text{Lt}} \frac{\frac{1}{\sin x} \cdot \cos x}{\frac{1}{\sin 2x} \cdot \cos 2x \cdot 2} = \underset{x \rightarrow 0^+}{\text{Lt}} \frac{\cos x}{\sin x} \cdot \frac{2 \sin x \cos x}{2 \cos 2x} \\ &= \underset{x \rightarrow 0^+}{\text{Lt}} \frac{\cos^2 x}{\cos 2x} = \frac{1}{1} = 1. \end{aligned}$$

**Example 3.** Evaluate the following limits:

$$(i) \quad \underset{x \rightarrow 1}{\text{Lt}} \left( \frac{x}{x-1} - \frac{1}{\log x} \right)$$

$$(ii) \quad \underset{x \rightarrow 0}{\text{Lt}} \left( \operatorname{cosec} x - \frac{1}{x} \right)$$

$$(iii) \quad \underset{x \rightarrow 0}{\text{Lt}} \left( \frac{1}{x^2} - \frac{\cot x}{x} \right)$$

$$(iv) \quad \underset{x \rightarrow \frac{\pi}{2}}{\text{Lt}} \left( x \tan x - \frac{\pi}{2} \sec x \right).$$

(I.S.C. 2010)

7. Evaluate the following limits :

$$(i) \lim_{x \rightarrow 1} (2-x)^{\frac{\tan \pi x}{2}} \quad (ii) \lim_{x \rightarrow 1} x^{\frac{1}{x-1}}.$$

**Hint.** (ii) Let  $x = 1 + t$ , so that when  $x \rightarrow 1$ ,  $t \rightarrow 0$ .

$$\therefore \lim_{x \rightarrow 1} x^{\frac{1}{x-1}} = \lim_{t \rightarrow 0} (1+t)^{\frac{1}{t}}.$$

## ANSWERS

### EXERCISE 7.1

- |                          |                           |   |                             |           |                       |
|--------------------------|---------------------------|---|-----------------------------|-----------|-----------------------|
| 1. (i) 108               | (ii) $n$ .                | 2. (i) $\frac{a}{b}$  | (ii) $e^2$ .                | 3. (i) -1 | (ii) $\frac{1}{2}$ .  |
| 4. (i) $\frac{1}{2}$     | (ii) 2.                   | 5. (i) $\frac{1}{4}$  | (ii) $\frac{1}{2} \log 2$ . | 6. (i) -2 | (ii) 1.               |
| 7. (i) 4                 | (ii) $-\frac{3}{2}$ .     | 8. (i) $\frac{1}{6}$  | (ii) 1.                     | 9. (i) 0  | (ii) $\frac{1}{12}$ . |
| 10. (i) $-\frac{2}{\pi}$ | (ii) 1.                   | 11. (i) 2   | (ii) 1.                     | 12. (i) 2 | (ii) $\frac{1}{3}$ .  |
| 13. (i) 0                | (ii) $\frac{n(n-1)}{2}$ . | 14. $\frac{3x^2 - 2x - 1}{3x^2 - 6x + 3}$ is not of the form $\frac{0}{0}$ as $x \rightarrow 2$ . |                             |           |                       |

### EXERCISE 7.2

- |                      |                       |                         |                        |                         |                         |
|----------------------|-----------------------|-------------------------|------------------------|-------------------------|-------------------------|
| 1. (i) 0             | (ii) $-\infty$ .      | 2. (i) 0                | (ii) 0.                | 3. (i) 0                | (ii) 1.                 |
| 4. (i) 0             | (ii) 5.               | 5. (i) 0                | (ii) 2.                | 6. (i) $-\frac{1}{\pi}$ | (ii) 0.                 |
| 7. (i) $\frac{1}{2}$ | (ii) $\frac{1}{2}$ .  | 8. (i) $\frac{1}{2}$    | (ii) $\frac{1}{2}$ .   | 9. (i) $\frac{1}{2}$    | (ii) $\frac{\pi}{4}$ .  |
| 10. (i) 0            | (ii) $-\frac{2}{3}$ . | 11. (i) 0               | (ii) 0.                | 12. (i) 0               | (ii) 0.                 |
| 13. (i) 0            | (ii) 0.               | 14. (i) $\frac{2}{\pi}$ | (ii) $\frac{2}{\pi}$ . | 15. (i) 1               | (ii) $\frac{2c}{\pi}$ . |

### EXERCISE 7.3

- |          |            |          |                      |                      |              |
|----------|------------|----------|----------------------|----------------------|--------------|
| 1. (i) 1 | (ii) $e$ . | 2. (i) 1 | (ii) 1.              | 3. (i) $\frac{1}{e}$ | (ii) 1.      |
| 4. (i) 1 | (ii) 1.    | 5. (i) 1 | (ii) $\frac{1}{e}$ . | 6. (i) $e$           | (ii) $e^5$ . |

### EXERCISE 7.4

- |                            |                 |               |  |                         |
|----------------------------|-----------------|---------------|--|-------------------------|
| 1. (i) 2                   | (ii) 1.         | 2. $-2; -1$ . | 3. $a = \frac{1}{2}, b = -\frac{1}{2}$ . |                         |
| 4. $a = 1, b = 2, c = 1$ . |                 | 5. (i) 0      | (ii) $a$ .                               |                         |
| 7. (i) $e^{1/3}$           | (ii) $e^{10}$ . | 8. $e^{-1}$ . | 6. (i) 1                                 | (ii) $-\frac{2}{\pi}$ . |

### CHAPTER TEST

- |                        |                              |                       |                       |                       |                       |
|------------------------|------------------------------|-----------------------|-----------------------|-----------------------|-----------------------|
| 1. (i) $\frac{2}{\pi}$ | (ii) $-\frac{1}{\sqrt{2}}$ . | 2. (i) $-\frac{2}{3}$ | (ii) -2.              | 3. (i) $-\frac{1}{4}$ | (ii) $-\frac{1}{3}$ . |
| 4. (i) 0               | (ii) $\frac{1}{3}$ .         | 5. (i) $-\frac{1}{3}$ | (ii) $-\frac{1}{6}$ . | 6. (i) $\frac{1}{3}$  | (ii) 0.               |
| 7. (i) $e^{2/\pi}$     | (ii) $e$ .                   |                       |                       |                       |                       |