

# 14

# Complex Numbers

---

## 14.1 COMPLEX NUMBERS

A **complex number**  $z$  is defined as an ordered pair of real numbers and is written as  $z = (a, b)$  or  $z = a + ib$ , where  $a, b$  are real numbers and  $i = \sqrt{-1}$ .

Thus, a number of the form  $a + ib$  where  $a, b$  are real numbers and  $i = \sqrt{-1}$  is called a complex number.

For example,  $3 + 5i$ ,  $-2 + 3i$ ,  $-2 + i\sqrt{5}$ ,  $7 + i\left(-\frac{2}{3}\right)$  are all complex numbers.

The system of numbers  $\mathbf{C} = \{z; z = a + ib; a, b \in \mathbf{R}\}$  is called the set of **complex numbers**.

### Real and imaginary parts of a complex number

If  $z = a + ib$  ( $a, b \in \mathbf{R}$ ) is a complex number, then  $a$  is called the **real part**, denoted by  $\text{Re}(z)$  and  $b$  is called **imaginary part**, denoted by  $\text{Im}(z)$ .

For example :

(i) If  $z = 2 + 3i$ , then  $\text{Re}(z) = 2$  and  $\text{Im}(z) = 3$ .

(ii) If  $z = -3 + \sqrt{5}i$ , then  $\text{Re}(z) = -3$  and  $\text{Im}(z) = \sqrt{5}$ .

(iii) If  $z = 7$ , then  $z = 7 + 0i$ , so that  $\text{Re}(z) = 7$  and  $\text{Im}(z) = 0$ .

(iv) If  $z = -5i$ , then  $z = 0 + (-5)i$ , so that  $\text{Re}(z) = 0$  and  $\text{Im}(z) = -5$ .

Note that imaginary part is a real number.

In  $z = a + ib$  ( $a, b \in \mathbf{R}$ ), if  $b = 0$  then  $z = a$ , which is a **real number**. If  $a = 0$  and  $b \neq 0$ , then  $z = ib$ , which is called **purely imaginary number**. If  $b \neq 0$ , then  $z = a + ib$  is **non-real** complex number. Since every real number  $a$  can be written as  $a + 0i$ , we see that  $\mathbf{R} \subset \mathbf{C}$  i.e. the set of real numbers  $\mathbf{R}$  is a **proper subset** of  $\mathbf{C}$ , the set of complex numbers.

Note that  $\sqrt{3}, 0, 2, \pi$  are real numbers;  $3 + 2i, 3 - 2i$  etc. are non-real complex numbers;  $2i, -\sqrt{2}i$  etc. are purely imaginary numbers.

### Equality of two complex numbers

Two complex numbers  $z_1 = a + ib$  and  $z_2 = c + id$  are called **equal**, written as  $z_1 = z_2$ , if and only if  $a = c$  and  $b = d$ .

For example, if the complex numbers  $z_1 = a + ib$  and  $z_2 = -3 + 5i$  are equal, then  $a = -3$  and  $b = 5$ .

## 14.2 ALGEBRA OF COMPLEX NUMBERS

In this section, we shall define the usual mathematical operations—addition, subtraction, multiplication, division, square, power etc. on complex numbers.

### 14.2.1 Addition of two complex numbers

Let  $z_1 = a + ib$  and  $z_2 = c + id$  be any two complex numbers, then their sum  $z_1 + z_2$  is defined as

$$z_1 + z_2 = (a + c) + i(b + d).$$

For example, let  $z_1 = 2 + 3i$  and  $z_2 = -5 + 4i$ , then

$$z_1 + z_2 = (2 + (-5)) + (3 + 4)i = -3 + 7i.$$

If  $z_1, z_2, z_3$  are any complex numbers, then it is easy to see that

$$(i) z_1 + z_2 = z_2 + z_1 \quad (\text{Commutative law})$$

$$(ii) (z_1 + z_2) + z_3 = z_1 + (z_2 + z_3) \quad (\text{Associative law})$$

(iii)  $z + 0 = z = 0 + z$ , so 0 acts as the *additive identity* for the set of complex numbers.

### Negative of a complex number

For a complex number,  $z = a + ib$ , the negative is defined as  $-z = (-a) + i(-b) = -a - ib$ .

Note that  $z + (-z) = (a - a) + i(b - b) = 0 + i0 = 0$ .

Thus,  $-z$  acts as the *additive inverse* of  $z$ .

### 14.2.2 Subtraction of complex numbers

Let  $z_1 = a + ib$  and  $z_2 = c + id$  be any two complex numbers, then the difference of  $z_2$  from  $z_1$  is defined as

$$\begin{aligned} z_1 - z_2 &= z_1 + (-z_2) \\ &= (a + ib) + (-c - id) \\ &= (a - c) + i(b - d). \end{aligned}$$

For example, let  $z_1 = 2 + 3i$  and  $z_2 = -1 + 4i$ , then

$$\begin{aligned} z_1 - z_2 &= (2 + 3i) - (-1 + 4i) \\ &= (2 + 3i) + (1 - 4i) \\ &= (2 + 1) + (3 - 4)i = 3 - i \end{aligned}$$

$$\begin{aligned} \text{and } z_2 - z_1 &= (-1 + 4i) - (2 + 3i) \\ &= (-1 + 4i) + (-2 - 3i) \\ &= (-1 - 2) + (4 - 3)i = -3 + i. \end{aligned}$$

### 14.2.3 Multiplication of two complex numbers

Let  $z_1 = a + ib$  and  $z_2 = c + id$  be any two complex numbers, then their product  $z_1 z_2$  is defined as

$$z_1 z_2 = (ac - bd) + i(ad + bc).$$

Note that intuitively,

$$(a + ib)(c + id) = ac + ibc + iad + i^2bd ; \text{ now put } i^2 = -1.$$

For example, let  $z_1 = 3 + 7i$  and  $z_2 = -2 + 5i$ , then

$$\begin{aligned} z_1 z_2 &= (3 + 7i)(-2 + 5i) \\ &= (3 \times (-2) - 7 \times 5) + i(3 \times 5 + 7 \times (-2)) \\ &= -41 + i. \end{aligned}$$

If  $z_1, z_2, z_3$  are any complex numbers, then it is easy to see that

$$(i) z_1 z_2 = z_2 z_1 \quad (\text{Commutative law})$$

$$(ii) (z_1 z_2) z_3 = z_1(z_2 z_3) \quad (\text{Associative law})$$

(iii)  $z \cdot 1 = z = 1 \cdot z$ , so 1 acts as the *multiplicative identity* for the set of complex numbers.

(iv) *Multiplicative inverse of a non-zero complex number*

For every non-zero complex number  $z = a + ib$ , we have the complex number

$$\frac{a}{a^2 + b^2} - i \frac{b}{a^2 + b^2} \text{ (denoted by } z^{-1} \text{ or } \frac{1}{z} \text{ ) such that}$$

$$z \cdot \frac{1}{z} = 1 = \frac{1}{z} \cdot z \quad (\text{check it})$$

$\frac{1}{z}$  is called the multiplicative inverse of  $z$ .

$$\text{Note that intuitively, } \frac{1}{a+ib} = \frac{1}{a+ib} \times \frac{a-ib}{a-ib} = \frac{a-ib}{a^2+b^2} = \frac{a}{a^2+b^2} - i \frac{b}{a^2+b^2}.$$

(v) *Multiplication of complex numbers is distributive over addition of complex numbers*

If  $z_1$ ,  $z_2$  and  $z_3$  are any three complex numbers, then

$$z_1(z_2 + z_3) = z_1z_2 + z_1z_3$$

$$\text{and } (z_1 + z_2)z_3 = z_1z_3 + z_2z_3.$$

These results are known as *distributive laws*.

#### 14.2.4 Division of complex numbers

Division of a complex number  $z_1 = a + ib$  by  $z_2 = c + id \neq 0$  is defined as

$$\frac{z_1}{z_2} = z_1 \cdot \frac{1}{z_2} = z_1 \cdot z_2^{-1} = (a+ib) \cdot \frac{c-id}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + i \frac{bc-ad}{c^2+d^2}.$$

Note that intuitively,

$$\frac{z_1}{z_2} = \frac{a+ib}{c+id} = \frac{a+ib}{c+id} \times \frac{c-id}{c-id} = \frac{(ac+bd)+i(bc-ad)}{c^2+d^2}.$$

For example, if  $z_1 = 3 + 4i$  and  $z_2 = 5 - 6i$ , then

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{3+4i}{5-6i} = \frac{3+4i}{5-6i} \times \frac{5+6i}{5+6i} = \frac{(3 \times 5 - 4 \times 6) + (3 \times 6 + 4 \times 5)i}{5^2 - 6^2 \times i^2} \\ &= \frac{-9 + 38i}{25 + 36} = -\frac{9}{61} + \frac{38}{61}i. \end{aligned}$$

#### 14.2.5 Integral powers of a complex number

If  $z$  is any complex number, then positive integral powers of  $z$  are defined as

$$z^1 = z, z^2 = z.z, z^3 = z^2.z, z^4 = z^3.z \text{ and so on.}$$

If  $z$  is any non-zero complex number, then negative integral powers of  $z$  are defined as :

$$z^{-1} = \frac{1}{z}, z^{-2} = \frac{1}{z^2}, z^{-3} = \frac{1}{z^3} \text{ etc.}$$

If  $z \neq 0$ , then  $z^0 = 1$ .

#### 14.2.6 Powers of $i$

Integral power of  $i$  are defined as :

$$i^0 = 1, i^1 = i, i^2 = -1, i^3 = i^2.i = (-1).i = -i,$$

$$i^4 = (i^2)^2 = (-1)^2 = 1, i^5 = i^4.i = 1.i = i,$$

$$i^6 = i^4.i^2 = 1.(-1) = -1, \text{ and so on.}$$

$$i^{-1} = \frac{1}{i} = \frac{1}{i} \times \frac{i}{i} = \frac{i}{-1} = -i$$

*Remember that  $\frac{1}{i} = -i$*

$$i^{-2} = \frac{1}{i^2} = \frac{1}{-1} = -1, i^{-3} = \frac{1}{i^3} = \frac{1}{i^3} \times \frac{i}{i} = \frac{i}{i^4} = \frac{i}{1} = i,$$

$$i^{-4} = \frac{1}{i^4} = \frac{1}{1} = 1, \text{ and so on.}$$

*Note that  $i^4 = 1$  and  $i^{-4} = 1$ . It follows that for any integer  $k$ ,*

$$i^{4k} = 1, i^{4k+1} = i, i^{4k+2} = i^2 = -1, i^{4k+3} = i^3 = -i.$$

#### 14.2.7 Modulus of a complex number

Modulus of a complex number  $z = a + ib$ , denoted by  $\text{mod}(z)$  or  $|z|$ , is defined as

$$|z| = \sqrt{a^2 + b^2}, \text{ where } a = \text{Re}(z), b = \text{Im}(z).$$

Sometimes,  $|z|$  is called **absolute value** of  $z$ . Note that  $|z| \geq 0$ .

For example,

$$(i) \text{ If } z = -3 + 5i, \text{ then } |z| = \sqrt{(-3)^2 + 5^2} = \sqrt{34}.$$

$$(ii) \text{ If } z = 3 - \sqrt{7}i, \text{ then } |z| = \sqrt{3^2 + (-\sqrt{7})^2} = \sqrt{9 + 7} = 4.$$

#### Properties of modulus of a complex number

If  $z, z_1$  and  $z_2$  are complex numbers, then

$$(i) |-z| = |z|$$

$$(ii) |z| = 0 \text{ if and only if } z = 0$$

$$(iii) |z_1 z_2| = |z_1| |z_2|$$

$$(iv) \left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}, \text{ provided } z_2 \neq 0.$$

**Proof.** (i) Let  $z = a + ib$ , then  $-z = -a - ib$ .

$$\therefore |-z| = \sqrt{(-a)^2 + (-b)^2} = \sqrt{a^2 + b^2} = |z|.$$

$$(ii) \text{ Let } z = a + ib, \text{ then } |z| = \sqrt{a^2 + b^2}.$$

$$\text{Now } |z| = 0 \text{ iff } \sqrt{a^2 + b^2} = 0$$

$$\text{i.e. iff } a^2 + b^2 = 0 \text{ i.e. iff } a^2 = 0 \text{ and } b^2 = 0$$

$$\text{i.e. iff } a = 0 \text{ and } b = 0 \text{ i.e. iff } z = 0 + i0$$

$$\text{i.e. iff } z = 0.$$

$$(iii) \text{ Let } z_1 = a + ib, \text{ and } z_2 = c + id, \text{ then}$$

$$z_1 z_2 = (ac - bd) + i(ad + bc).$$

$$\begin{aligned} \therefore |z_1 z_2| &= \sqrt{(ac - bd)^2 + (ad + bc)^2} \\ &= \sqrt{a^2 c^2 + b^2 d^2 - 2abcd + a^2 d^2 + b^2 c^2 + 2abcd} \\ &= \sqrt{(a^2 + b^2)(c^2 + d^2)} \\ &= \sqrt{a^2 + b^2} \sqrt{c^2 + d^2} \quad (\because a^2 + b^2 \geq 0, c^2 + d^2 \geq 0) \\ &= |z_1| |z_2|. \end{aligned}$$

$$(iv) \text{ Here } z_2 \neq 0 \Rightarrow |z_2| \neq 0.$$

$$\text{Let } \frac{z_1}{z_2} = z_3 \Rightarrow z_1 = z_2 z_3 \Rightarrow |z_1| = |z_2 z_3|$$

$$\Rightarrow |z_1| = |z_2| |z_3|$$

(using part (iii))

$$\Rightarrow \frac{|z_1|}{|z_2|} = |z_3| \Rightarrow \frac{|z_1|}{|z_2|} = \left| \frac{z_1}{z_2} \right|$$

$$\left( \because z_3 = \frac{z_1}{z_2} \right)$$

**Remark.** From (iii), on replacing both  $z_1$  and  $z_2$  by  $z$ , we get

$$|zz| = |z||z| \text{ i.e. } |z^2| = |z|^2.$$

$$\text{Similarly, } |z^3| = |z^2z| = |z^2||z| = |z|^2|z| = |z|^3 \text{ etc.}$$

### 14.2.8 Conjugate of a complex number

Conjugate of a complex number  $z = a + ib$ , denoted by  $\bar{z}$ , is defined as

$$\bar{z} = a - ib \text{ i.e. } \overline{a+ib} = a - ib.$$

For example,

$$(i) \overline{2+5i} = 2 - 5i, \quad \overline{2-5i} = 2 + 5i$$

$$(ii) \overline{-3-7i} = -3 + 7i, \quad \overline{-3+7i} = -3 - 7i.$$

#### Properties of conjugate of a complex number

If  $z, z_1$  and  $z_2$  are complex numbers, then

$$(i) \overline{\overline{z}} = z$$

$$(ii) \overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$$

$$(iii) \overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$$

$$(iv) \overline{z_1 z_2} = \overline{z_1} \overline{z_2}$$

$$(v) \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}, \text{ provided } z_2 \neq 0$$

$$(vi) |\bar{z}| = |z|$$

$$(vii) z \bar{z} = |z|^2$$

$$(viii) z^{-1} = \frac{\bar{z}}{|z|^2}, \text{ provided } z \neq 0.$$

**Proof.** (i) Let  $z = a + ib$ , where  $a, b \in \mathbb{R}$ , so that  $\bar{z} = a - ib$ .

$$\therefore \overline{(\bar{z})} = \overline{a-ib} = a + ib = z.$$

(ii) Let  $z_1 = a + ib$  and  $z_2 = c + id$ , then

$$\overline{z_1 + z_2} = \overline{(a+ib)+(c+id)} = \overline{(a+c)+i(b+d)}$$

$$= (a+c) - i(b+d) = (a - ib) + (c - id) = \overline{z_1} + \overline{z_2}.$$

(iii) Let  $z_1 = a + ib$  and  $z_2 = c + id$ , then

$$\overline{z_1 - z_2} = \overline{(a+ib)-(c+id)} = \overline{(a-c)+i(b-d)}$$

$$= (a-c) - i(b-d) = (a - ib) - (c - id) = \overline{z_1} - \overline{z_2}.$$

(iv) Let  $z_1 = a + ib$  and  $z_2 = c + id$ , then

$$\overline{z_1 z_2} = \overline{(a+ib)(c+id)} = \overline{(ac-bd)+i(ad+bc)}$$

$$= (ac - bd) - i(ad + bc).$$

$$\text{Also } \overline{z_1} \overline{z_2} = (a - ib)(c - id) = (ac - bd) - i(ad + bc).$$

$$\text{Hence } \overline{z_1 z_2} = \overline{z_1} \overline{z_2}.$$

(v) Here  $z_2 \neq 0 \Rightarrow \overline{z_2} \neq 0$ .

$$\text{Let } \frac{z_1}{z_2} = z_3 \Rightarrow z_1 = z_2 z_3 \Rightarrow \overline{z_1} = \overline{z_2 z_3}$$

$$\Rightarrow \overline{z_1} = \overline{z_2} \overline{z_3}$$

(using part (iv))

$$\Rightarrow \overline{\frac{z_1}{z_2}} = \overline{z_3} \Rightarrow \overline{\frac{z_1}{z_2}} = \overline{\left(\frac{z_1}{z_2}\right)}$$

$$\left( \because z_3 = \frac{z_1}{z_2} \right)$$

(vi) Let  $z = a + ib$ , then  $\bar{z} = a - ib$ .

$$\therefore |\bar{z}| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2} = |z|.$$

(vii) Let  $z = a + ib$ , then  $\bar{z} = a - ib$ .

$$\begin{aligned}\therefore z \bar{z} &= (a+ib)(a-ib) \\ &= (aa - b(-b)) + i(a(-b) + ba) \\ &= (a^2 + b^2) + i \cdot 0 \\ &= a^2 + b^2 = \left(\sqrt{a^2 + b^2}\right)^2 = |z|^2.\end{aligned}$$

**Remember that**  $(a+ib)(a-ib) = a^2 + b^2$ .

(viii) Let  $z = a + ib \neq 0$ , then  $|z| \neq 0$ .

$$\therefore z \bar{z} = (a+ib)(a-ib) = a^2 + b^2 = |z|^2$$

$$\Rightarrow \frac{z \bar{z}}{|z|^2} = 1 \Rightarrow \frac{\bar{z}}{|z|^2} = \frac{1}{z} = z^{-1}$$

$$\text{Thus, } z^{-1} = \frac{\bar{z}}{|z|^2}, \text{ provided } z \neq 0.$$

**Remark** From (iv), on replacing both  $z_1$  and  $z_2$  by  $z$ , we get

$$\overline{z z} = \bar{z} \bar{z} \text{ i.e. } \overline{z^2} = (\bar{z})^2.$$

$$\text{Similarly, } (\overline{z^3}) = (\overline{z^2} z) = (\overline{z^2}) \bar{z} = (\bar{z})^2 \bar{z} = (\bar{z})^3 \text{ etc.}$$

## ILLUSTRATIVE EXAMPLES

**Example 1.** Evaluate the following :

$$(i) i^{50}$$

$$(ii) i^{-57}$$

$$(iii) (-\sqrt{-1})^{31}$$

$$(iv) i + i^2 + i^3 + i^4$$

$$(v) \frac{i + i^2 + i^4}{1 + i^2 + i^4}$$

$$(vi) \frac{i^2 + i^4 + i^6 + i^7}{1 + i^2 + i^3}.$$

**Solution.** (i)  $i^{50} = i^{4 \cdot 12 + 2} = i^2 = -1$ .

$$(ii) i^{-57} = i^{4 \cdot (-15) + 3} = i^3 = -i.$$

$$(iii) (-\sqrt{-1})^{31} = (-i)^{31} = (-1)^{31} \cdot i^{31} = -i^{4 \cdot 7 + 3} = -i^3 = -(-i) = i.$$

$$(iv) i + i^2 + i^3 + i^4 = i + (-1) + (-i) + (1) = 0.$$

$$(v) \frac{i + i^2 + i^4}{1 + i^2 + i^4} = \frac{i + (-1) + (1)}{1 + (-1) + (1)} = \frac{i}{1} = i.$$

$$(vi) \frac{i^2 + i^4 + i^6 + i^7}{1 + i^2 + i^3} = \frac{(-1) + (1) + (-1) + (-i)}{1 + (-1) + (-i)} = \frac{-1 - i}{-i} = \frac{1}{i} + 1 = -i + 1 = 1 - i.$$

**Example 2.** (i) Show that  $i^n + i^{n+1} + i^{n+2} + i^{n+3} = 0$  for all  $n \in \mathbb{N}$ .

$$(ii) \text{ If } z = \frac{3}{2} - 2i, \text{ find } \operatorname{Re}(z), \operatorname{Im}(z), \bar{z}, |z|, z + \bar{z}, z - \bar{z}.$$

$$(iii) \text{ If } \operatorname{Re}(z) = \sqrt{2} \text{ and } \operatorname{Im}(z) = -\sqrt{3}, \text{ find } z, \bar{z}, |z|, |\bar{z}|.$$

$$(iv) \text{ Find values of } x \text{ and } y \text{ if } (3x - 1) + (\sqrt{3} + 2y)i = 5.$$

$$\text{Solution. (i) } i^n + i^{n+1} + i^{n+2} + i^{n+3} = i^n(1 + i + i^2 + i^3)$$

$$= i^n(1 + i - 1 - i) = i^n \times 0 = 0.$$

$$(ii) \text{ Given } z = \frac{3}{2} - 2i \Rightarrow \operatorname{Re}(z) = \frac{3}{2}, \operatorname{Im}(z) = -2,$$

$$\therefore \bar{z} = \frac{3}{2} + 2i,$$

$$\begin{aligned} |z| &= \sqrt{\left(\frac{3}{2}\right)^2 + (-2)^2} = \sqrt{\frac{9}{4} + 4} = \sqrt{\frac{9+16}{4}} = \sqrt{\frac{25}{4}} = \frac{5}{2}, \\ z + \bar{z} &= \left(\frac{3}{2} - 2i\right) + \left(\frac{3}{2} + 2i\right) = \frac{3}{2} + \frac{3}{2} = 3 \text{ and} \\ z - \bar{z} &= \left(\frac{3}{2} - 2i\right) - \left(\frac{3}{2} + 2i\right) = -4i. \end{aligned}$$

(iii)  $z = \operatorname{Re}(z) + i \cdot \operatorname{Im}(z) = \sqrt{2} - \sqrt{3}i,$

$$\bar{z} = \sqrt{2} + \sqrt{3}i,$$

$$|z| = \sqrt{(\sqrt{2})^2 + (-\sqrt{3})^2} = \sqrt{2+3} = \sqrt{5} \text{ and}$$

$$|\bar{z}| = \sqrt{(\sqrt{2})^2 + (\sqrt{3})^2} = \sqrt{2+3} = \sqrt{5}.$$

(iv) Equating real and imaginary parts on both sides, we get

$$3x - 1 = 5, \sqrt{3} + 2y = 0 \Rightarrow x = 2, y = -\frac{\sqrt{3}}{2}.$$

**Example 3.** (i) Simplify  $(-2 + 3i) + 3\left(-\frac{1}{2}i + 1\right) - (2i).$

(ii) Simplify  $(-3 + \sqrt{-5})(3 + 4\sqrt{-5}).$

(iii) Find the conjugate and modulus of  $i^7.$

**Solution.** (i) Given expression =  $(-2 + 3) + (3 - \frac{3}{2} - 2)i$   
 $= 1 - \frac{1}{2}i.$

(ii)  $(-3 + \sqrt{-5})(3 + 4\sqrt{-5}) = (-3 + \sqrt{5}i)(3 + 4\sqrt{5}i)$   
 $= -9 - 12\sqrt{5}i + 3\sqrt{5}i + \sqrt{5} \cdot 4\sqrt{5}i^2$   
 $= -9 - 9\sqrt{5}i + 20(-1)$   
 $= -29 - 9\sqrt{5}i.$

(iii) Let  $z = i^7 = i^4 \cdot i^3 = 1 \cdot (-i) = -i,$

so  $\bar{z} = -i = i \text{ and } |z| = |-i| = \sqrt{0^2 + (-1)^2} = 1.$

**Example 4.** If  $i = \sqrt{-1}$ , prove the following :

$$(x+1+i)(x+1-i)(x-1+i)(x-1-i) = x^4 + 4. \quad (\text{I.S.C. 2004})$$

**Solution.** L.H.S. =  $[(\overline{x+1} + i)(\overline{x+1} - i)][(\overline{x-1} + i)(\overline{x-1} - i)]$   
 $= [(x+1)^2 - i^2][(\overline{x-1})^2 - i^2]$   
 $= [x^2 + 2x + 1 - (-1)][x^2 - 2x + 1 - (-1)]$   
 $= (\overline{x^2 + 2} + 2x)(\overline{x^2 + 2} - 2x)$   
 $= (x^2 + 2)^2 - (2x)^2$   
 $= x^4 + 4x^2 + 4 - 4x^2 = x^4 + 4 = \text{R.H.S.}$

**Example 5.** Express each of the following in the standard form  $a + ib$ .

(i)  $\frac{i}{1+i}$

(ii)  $(-1 + \sqrt{3}i)^{-1}$

(iii)  $(2i - i^2)^2 + (1 - 3i)^3 \quad (\text{iv}) \quad \frac{(1+i)(3+i)}{3-i} - \frac{(1-i)(3-i)}{3+i}.$

**Solution.** (i)  $\frac{i}{1+i} = \frac{i(1-i)}{(1+i)(1-i)} = \frac{i - i^2}{1 - i^2} = \frac{i - (-1)}{1 - (-1)} = \frac{1+i}{2} = \frac{1}{2} + \frac{1}{2}i.$

(ii)  $(-1 + \sqrt{3}i)^{-1} = \frac{1}{-1 + \sqrt{3}i} = \frac{1}{-1 + \sqrt{3}i} \cdot \frac{-1 - \sqrt{3}i}{-1 - \sqrt{3}i}$   
 $= \frac{-1 - \sqrt{3}i}{(-1)^2 - (\sqrt{3}i)^2} = \frac{-1 - \sqrt{3}i}{1 - (-3)} = -\frac{1}{4} - \frac{\sqrt{3}}{4}i.$

$$\begin{aligned}
 (iii) \quad & (2i - i^2)^2 + (1 - 3i)^3 = (2i + 1)^2 + (1 - 3i)^3 \\
 & = (4i^2 + 4i + 1) + (1 - 9i + 27i^2 - 27i^3) \\
 & = -4 + 4i + 1 + 1 - 9i - 27 + 27i = -29 + 22i.
 \end{aligned}$$

$$\begin{aligned}
 (iv) \quad & \frac{(1+i)(3+i)}{3-i} - \frac{(1-i)(3-i)}{3+i} = \frac{(1+i)(3+i)(3+i) - (1-i)(3-i)(3-i)}{(3-i)(3+i)} \\
 & = \frac{(1+i)(8+6i) - (1-i)(8-6i)}{9-i^2} = \frac{(2+14i) - (2-14i)}{9+1} = \frac{28i}{10} = 0 + \frac{14}{5}i.
 \end{aligned}$$

**Example 6.** (i) Find the conjugate of  $\frac{(3-2i)(2+3i)}{(1+2i)(2-i)}$ .

(ii) Find the modulus of  $\frac{1+i}{1-i} - \frac{1-i}{1+i}$ .

(iii) Find the modulus of  $\frac{(2-3i)^2}{-1+5i}$ .

$$\begin{aligned}
 \text{Solution. (i) Let } z &= \frac{(3-2i)(2+3i)}{(1+2i)(2-i)} = \frac{6+9i-4i+6}{2-i+4i+2} \\
 &= \frac{12+5i}{4+3i} = \frac{12+5i}{4+3i} \times \frac{4-3i}{4-3i} \\
 &= \frac{48-36i+20i+15}{4^2-(3i)^2} = \frac{63-16i}{16-9(-1)} \\
 &= \frac{63-16i}{25} = \frac{63}{25} - \frac{16}{25}i.
 \end{aligned}$$

$\therefore$  Conjugate of  $z = \frac{63}{25} + \frac{16}{25}i$ .

$$\begin{aligned}
 (ii) \quad z &= \frac{1+i}{1-i} - \frac{1-i}{1+i} = \frac{(1+i)^2 - (1-i)^2}{(1-i)(1+i)} \\
 &= \frac{(1+i^2+2i) - (1+i^2-2i)}{1^2-i^2} = \frac{4i}{1+1} = 2i \\
 &= 0 + 2i.
 \end{aligned}$$

$\therefore$  Modulus of  $z = \sqrt{0^2 + 2^2} = \sqrt{4} = 2$ .

(iii) Let  $z = \frac{(2-3i)^2}{-1+5i}$ , then

$$\begin{aligned}
 |z| &= \left| \frac{(2-3i)^2}{-1+5i} \right| = \frac{|(2-3i)^2|}{|-1+5i|} \quad \left( \because \left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|} \right) \\
 &= \frac{|2-3i|^2}{|-1+5i|} \quad (\because |z^2| = |z|^2) \\
 &= \frac{\left( \sqrt{(2)^2 + (-3)^2} \right)^2}{\sqrt{(-1)^2 + 5^2}} = \frac{(\sqrt{13})^2}{\sqrt{26}} = \frac{13}{\sqrt{26}} = \sqrt{\frac{13}{2}}.
 \end{aligned}$$

**Example 7.** If  $(-2 + \sqrt{-3})(-3 + 2\sqrt{-3}) = a + ib$ , find the real numbers  $a$  and  $b$ . With these values of  $a$  and  $b$ , also find the modulus of  $a + ib$ . (I.S.C. 2009)

**Solution.** Given  $(-2 + \sqrt{-3})(-3 + 2\sqrt{-3}) = a + ib$

$$\Rightarrow (-2 + \sqrt{3}i)(-3 + 2\sqrt{3}i) = a + ib$$

$$\begin{aligned}\Rightarrow \quad & 6 - 4\sqrt{3}i - 3\sqrt{3}i + \sqrt{3}.2\sqrt{3}i^2 = a + ib \\ \Rightarrow \quad & 6 - 7\sqrt{3}i + 6(-1) = a + ib \\ \Rightarrow \quad & 0 - 7\sqrt{3}i = a + ib.\end{aligned}$$

Equating real and imaginary parts on both sides, we get

$$a = 0 \text{ and } b = -7\sqrt{3}.$$

$$\begin{aligned}|a + ib| &= \sqrt{a^2 + b^2} = \sqrt{0^2 + (-7\sqrt{3})^2} = \sqrt{0 + 147} \\ &= \sqrt{147} = 7\sqrt{3}.\end{aligned}$$

**Example 8.** Find the values of  $x$  and  $y$  given that  $(x + iy)(2 - 3i) = 4 + i$ . (I.S.C. 2008)

**Solution.** Given  $(x + iy)(2 - 3i) = 4 + i$

$$\begin{aligned}\Rightarrow \quad & x + iy = \frac{4+i}{2-3i} = \frac{4+i}{2-3i} \times \frac{2+3i}{2+3i} \\ \Rightarrow \quad & x + iy = \frac{8 + 12i + 2i + 3i^2}{2^2 - 9i^2} = \frac{8 + 14i + 3(-1)}{4 - 9(-1)} \\ \Rightarrow \quad & x + iy = \frac{5}{13} + \frac{14}{13}i.\end{aligned}$$

Equating real and imaginary parts on both sides, we get

$$x = \frac{5}{13} \text{ and } y = \frac{14}{13}.$$

**Example 9.** Given that  $\frac{2\sqrt{3}\cos 30^\circ - 2i\sin 30^\circ}{\sqrt{2}(\cos 45^\circ + i\sin 45^\circ)} = A + iB$ , find the values of  $A$  and  $B$ .

(I.S.C. 2002)

**Solution.** Given  $\frac{2\sqrt{3}\cos 30^\circ - 2i\sin 30^\circ}{\sqrt{2}(\cos 45^\circ + i\sin 45^\circ)} = A + iB$

$$\begin{aligned}\Rightarrow \quad & A + iB = \frac{2\sqrt{3} \cdot \frac{\sqrt{3}}{2} - 2i \cdot \frac{1}{2}}{\sqrt{2} \left( \frac{1}{\sqrt{2}} + i \cdot \frac{1}{\sqrt{2}} \right)} = \frac{3-i}{1+i} \\ & = \frac{3-i}{1+i} \times \frac{1-i}{1-i} = \frac{3-3i-i+i^2}{1^2-i^2} \\ & = \frac{3+(-1)-4i}{1-(-1)} = \frac{2-4i}{2} = 1-2i.\end{aligned}$$

Equating real and imaginary parts on both sides, we get

$$A = 1 \text{ and } B = -2.$$

**Example 10.** Find the real values of  $x$  and  $y$  satisfying the equality

$$\frac{x-2+(y-3)i}{1+i} = 1-3i. \quad (\text{I.S.C. 2003})$$

**Solution.** Given  $\frac{x-2+(y-3)i}{1+i} = 1-3i$

$$\begin{aligned}\Rightarrow \quad & (x-2)+(y-3)i = (1-3i)(1+i) \\ \Rightarrow \quad & (x-2)+(y-3)i = (1+3)+(-3+1)i \\ \Rightarrow \quad & (x-2)+(y-3)i = 4-2i.\end{aligned}$$

Equating real and imaginary parts on both sides, we get

$$x-2=4 \text{ and } y-3=-2$$

$$\Rightarrow \quad x=6 \text{ and } y=1.$$

**Example 11.** Find the real values of  $x$  and  $y$  if  $(x - iy)(3 + 5i)$  is the conjugate of  $-6 - 24i$ .

**Solution.** According to given,  $(x - iy)(3 + 5i) = \overline{-6 - 24i}$

$$\Rightarrow 3x + 5xi - 3yi - 5y(-1) = -6 + 24i$$

$$\Rightarrow (3x + 5y) + (5x - 3y)i = -6 + 24i.$$

Equating real and imaginary parts on both sides, we get

$$3x + 5y = -6 \text{ and } 5x - 3y = 24.$$

Solving these equations simultaneously, we get

$$x = 3 \text{ and } y = -3.$$

**Example 12.** For what real values of  $x$  and  $y$  are the following numbers equal

$$(i) (1 + i)y^2 + (6 + i)x \text{ and } (2 + i)x$$

$$(ii) x^2 - 7x + 9yi \text{ and } y^2i + 20i - 12 ?$$

**Solution.** (i)  $(1 + i)y^2 + (6 + i)x = (2 + i)x$

$$\Rightarrow (y^2 + 6) + i(y^2 + 1) = 2x + ix$$

$$\Rightarrow y^2 + 6 = 2x \text{ and } y^2 + 1 = x$$

$$\Rightarrow x = 5 \text{ and } y^2 = 4 \Rightarrow x = 5 \text{ and } y = \pm 2.$$

Hence,  $x = 5, y = 2; x = 5, y = -2$ .

$$(ii) x^2 - 7x + 9yi = y^2i + 20i - 12$$

$$\Rightarrow (x^2 - 7x) + i(9y) = (-12) + i(y^2 + 20)$$

$$\Rightarrow x^2 - 7x = -12 \text{ and } 9y = y^2 + 20$$

$$\Rightarrow x^2 - 7x + 12 = 0 \text{ and } y^2 - 9y + 20 = 0$$

$$\Rightarrow (x - 4)(x - 3) = 0 \text{ and } (y - 5)(y - 4) = 0$$

$$\Rightarrow x = 4, 3 \text{ and } y = 5, 4.$$

Hence,  $x = 4, y = 5; x = 4, y = 4; x = 3, y = 5; x = 3, y = 4$ .

**Example 13.** If  $z$  is a complex number and  $iz^3 + z^2 - z + i = 0$ , then prove that  $|z| = 1$ .

**Solution.** Given  $iz^3 + z^2 - z + i = 0$

$$\Rightarrow iz^3 + z^2 + i^2z + i = 0 \quad (\because i^2 = -1)$$

$$\Rightarrow z^2(iz + 1) + i(iz + 1) = 0$$

$$\Rightarrow (z^2 + i)(iz + 1) = 0$$

$$\Rightarrow z^2 = -i \text{ or } iz = -1$$

$$\Rightarrow z^2 = -i \text{ or } z = -\frac{1}{i} = -(-i) = i.$$

$$\text{Now } z = i \Rightarrow |z| = |i| = \sqrt{0^2 + 1^2} = 1$$

$$\text{and } z^2 = -i \Rightarrow |z^2| = |-i| \Rightarrow |z|^2 = \sqrt{0 + (-1)^2} = 1 \Rightarrow |z| = 1.$$

Hence,  $|z| = 1$  in both cases.

**Example 14.** Solve the equation  $2z = |z| + 2i$ .

**Solution.** Let  $z = x + iy$  where  $x, y \in \mathbb{R}$  so that  $|z| = \sqrt{x^2 + y^2}$ .

$$\text{Given } 2z = |z| + 2i \Rightarrow 2(x + iy) = \sqrt{x^2 + y^2} + 2i$$

$$\Rightarrow 2x = \sqrt{x^2 + y^2} \text{ and } 2y = 2$$

$$\Rightarrow 4x^2 = x^2 + y^2 \text{ and } y = 1$$

$$\Rightarrow 3x^2 = 1 \text{ and } y = 1 \Rightarrow x = \pm \frac{1}{\sqrt{3}} \text{ and } y = 1.$$

$$\begin{aligned}
 &= \frac{|\beta - \alpha| |\bar{\beta}|}{|\beta - \alpha|} \quad (\because |z_1 z_2| = |z_1| |z_2| \text{ and } \bar{z}_1 - \bar{z}_2 = \overline{z_1 - z_2}) \\
 &= \frac{|\beta - \alpha| |\beta|}{|\beta - \alpha|} \quad (\because |\bar{z}| = |z|) \\
 &= |\beta| = 1 \quad (\because |\beta| = 1 \text{ given})
 \end{aligned}$$

Hence,  $\left| \frac{\beta - \alpha}{1 - \bar{\alpha}\beta} \right| = 1$ .

**Example 26.** If  $z_1, z_2$  are complex numbers such that  $\left| \frac{z_1 - 3z_2}{3 - z_1 \bar{z}_2} \right| = 1$  and  $|z_2| \neq 1$ , then

find  $|z_1|$ .

**Solution.** Given  $\left| \frac{z_1 - 3z_2}{3 - z_1 \bar{z}_2} \right| = 1 \Rightarrow \left| \frac{z_1 - 3z_2}{3 - z_1 \bar{z}_2} \right|^2 = 1$

$$\begin{aligned}
 &\Rightarrow \left( \frac{z_1 - 3z_2}{3 - z_1 \bar{z}_2} \right) \left( \frac{\overline{z_1 - 3z_2}}{\overline{3 - z_1 \bar{z}_2}} \right) = 1 \quad (\because |z|^2 = z \bar{z}) \\
 &\Rightarrow \frac{z_1 - 3z_2}{3 - z_1 \bar{z}_2} \cdot \frac{\bar{z}_1 - 3\bar{z}_2}{3 - \bar{z}_1 z_2} = 1 \\
 &\Rightarrow (z_1 - 3z_2)(\bar{z}_1 - 3\bar{z}_2) = (3 - z_1 \bar{z}_2)(3 - \bar{z}_1 z_2) \\
 &\Rightarrow z_1 \bar{z}_1 - 3z_1 \bar{z}_2 - 3z_2 \bar{z}_1 + 9z_2 \bar{z}_2 = 9 - 3\bar{z}_1 z_2 - 3z_1 \bar{z}_2 + z_1 \bar{z}_1 z_2 \bar{z}_2 \\
 &\Rightarrow |z_1|^2 + 9|z_2|^2 = 9 + |z_1|^2 |z_2|^2 \\
 &\Rightarrow |z_1|^2 |z_2|^2 - |z_1|^2 - 9|z_2|^2 + 9 = 0 \\
 &\Rightarrow |z_1|^2 (|z_2|^2 - 1) - 9(|z_2|^2 - 1) = 0 \\
 &\Rightarrow (|z_2|^2 - 1) (|z_1|^2 - 9) = 0 \\
 &\Rightarrow |z_1|^2 = 9 \quad (\because |z_2| \neq 1 \text{ given}) \\
 &\Rightarrow |z_1| = 3 \quad (\because |z_1| \geq 0)
 \end{aligned}$$

**Example 27.** If  $|z_1| = |z_2| = \dots = |z_n| = 1$ , prove that

$$|z_1 + z_2 + \dots + z_n| = \left| \frac{1}{z_1} + \frac{1}{z_2} + \dots + \frac{1}{z_n} \right|.$$

**Solution.** Given  $|z_1| = |z_2| = \dots = |z_n| = 1$

$$\begin{aligned}
 &\Rightarrow |z_1|^2 = |z_2|^2 = \dots = |z_n|^2 = 1 \\
 &\Rightarrow z_1 \bar{z}_1 = 1, z_2 \bar{z}_2 = 1, \dots, z_n \bar{z}_n = 1 \\
 &\Rightarrow \frac{1}{z_1} = \bar{z}_1, \frac{1}{z_2} = \bar{z}_2, \dots, \frac{1}{z_n} = \bar{z}_n.
 \end{aligned}$$

$$\begin{aligned}
 \therefore \left| \frac{1}{z_1} + \frac{1}{z_2} + \dots + \frac{1}{z_n} \right| &= \left| \bar{z}_1 + \bar{z}_2 + \dots + \bar{z}_n \right| \\
 &= \left| \overline{z_1 + z_2 + \dots + z_n} \right| = |z_1 + z_2 + \dots + z_n|. \quad (\because |\bar{z}| = |z|)
 \end{aligned}$$

## EXERCISE 14.1

1. Evaluate

(i) $\sqrt{-9} \sqrt{-4}$	(ii) $\sqrt{(-9)(-4)}$	(iii) $i + i^3 + i^5 + \dots + i^{15}$
(iv) $\frac{1}{1+i+i^2}$	(v) $(-\sqrt{-4})^3$	(vi) $\sum_{k=0}^{100} i^k$ .

2. If  $z = -3 - i$ , find  $\operatorname{Re}(z)$ ,  $\operatorname{Im}(z)$ ,  $\overline{z}$ ,  $|z|$ ,  $z^{-1}$ .

3. Express each of the following in the standard form  $a + ib$ :

$$(i) \frac{1}{3-4i} \quad (ii) \frac{1}{-2-\sqrt{-3}} \quad (iii) \frac{13i}{2-3i} \quad (\text{I.S.C. 2000})$$

$$(iv) \frac{(1+i)^2}{3-i} \quad (v) \frac{3-4i}{(4-2i)(1+i)} \quad (vi) \frac{1-2i}{2+i} + \frac{3+i}{2-i}. \quad (\text{I.S.C. 2006})$$

4. Find  $x$  and  $y$  if

$$(i) 2y + (3x - y)i = 5 - 2i$$

(ii)  $3 + ix^2y$  and  $x^2 + y + 4i$  are conjugate complexes of each other.

$$(iii) (x - yi)(2 + 3i) = \frac{x+2i}{1-i}$$

(iv)  $(x + iy)(2 - 3i)$  is the conjugate of  $4 - i$ .

5. If  $z_1 = 1 - i$ ,  $z_2 = -2 + 4i$ , calculate the values of  $a$  and  $b$  if  $a + bi = \frac{z_1 z_2}{z_1}$ .

6. Solve the equation :

$$\frac{(1+i)x - 2i}{3+i} + \frac{(2-3i)y + i}{3-i} = i, \quad x, y \in \mathbb{R}, \quad i = \sqrt{-1}.$$

7. (i) If  $|z_1| = |z_2|$ , does it follow that  $z_1 = z_2$ ?

(ii) Show that  $z = -\overline{z}$  iff  $z$  is either zero or purely imaginary.

8. If  $z_1, z_2, z_3, \dots, z_n$  are any complex numbers, prove that

$$(i) |z_1 z_2 z_3 \dots z_n| = |z_1| |z_2| |z_3| \dots |z_n|$$

$$(ii) \overline{z_1 + z_2 + z_3 + \dots + z_n} = \overline{z_1} + \overline{z_2} + \overline{z_3} + \dots + \overline{z_n}$$

$$(iii) \overline{z_1 z_2 \dots z_n} = \overline{z_1} \overline{z_2} \overline{z_3} \dots \overline{z_n}.$$

**Hint.** Use induction.

9. (i) Simplify  $(3 + 2i)^3$ . Also find its conjugate.

(ii) Simplify  $(1 - 2i)^{-3}$ . Also find its conjugate.

(iii) Separate into real and imaginary parts  $\frac{3 + \sqrt{-1}}{2 - \sqrt{-1}}$ . Also find its modulus.

(iv) Simplify  $(\sqrt{5} + 7i)(\sqrt{5} - 7i)^2 + (-2 + 7i)^2$ .

(v) Express the square of  $\frac{i}{1+i}$  in the form  $x + iy$ .

(vi) Show that  $\frac{\sqrt{7} + \sqrt{3}i}{\sqrt{7} - \sqrt{3}i} + \frac{\sqrt{7} - \sqrt{3}i}{\sqrt{7} + \sqrt{3}i}$  is real.

(vii) If  $z = 1 + i$ , evaluate  $z^3 - 2z^2 + 3z - 4$

(viii) If  $z = 3 + 5i$ , evaluate  $z^3 + \overline{z} + 198$ .

10. Find the modulus of the following numbers :

$$(i) \frac{(1+3i)(2-5i)}{(2-\sqrt{6}i)(-3+2\sqrt{5}i)} \quad (ii) \frac{(3+4i)(4-5i)}{(4-3i)(6+7i)}.$$

11. If  $z$  is a complex number and  $|2z - 1| = |z - 2|$ , show that  $|z| = 1$ .

12. Let  $z = x + iy$  and  $w = \frac{1-iz}{z-i}$ , show that if  $|w| = 1$ , then  $z$  is purely real.

13. If  $x$  is real and  $\frac{1-ix}{1+ix} = a - ib$ , prove that  $a^2 + b^2 = 1$ .

**Hint.**  $\left| \frac{1-ix}{1+ix} \right| = |a - ib|$ .

14. Using  $z_1 = 2 - 3i$ ,  $z_2 = 4 + 5i$ ,  $z_3 = -1 - i$ , verify that complex number multiplication is left as well as right distributive over addition.

15. If  $\left( \frac{1-i}{1+i} \right)^{500} = a + ib$ , find the values of  $a$  and  $b$ .

**Hint.** First simplify  $\frac{1-i}{1+i} = -i$ .

16. Find the least positive integral value of  $n$  such that  $\left( \frac{1+i}{1-i} \right)^n$  is real.

**Hint.** First simplify  $\frac{1+i}{1-i} = i$ .

17. Solve the equation  $|1 - i|^x = 2^x$ .

18. Solve the equation  $|z| + z = 2 + i$ .

19. Show that if  $\left| \frac{z-5i}{z+5i} \right| = 1$ , then  $z$  is a real number.

20. If  $(1+i)(1+2i)(1+3i)\dots(1+ni) = x+iy$ , show that  
 $2.5.10 \dots (1+n^2) = x^2 + y^2$ .

**Hint.** Take modulus of both sides and use the fact that

$$|z_1 z_2 \dots z_n| = |z_1| |z_2| \dots |z_n|.$$

21. If  $(x+iy)^{1/3} = a+ib$ , prove that  $\frac{x}{a} + \frac{y}{b} = 4(a^2 - b^2)$ .

**Hint.** Cube both sides and find  $x, y$  in terms of  $a$  and  $b$ .

22. If  $f(z) = z^4 + 9z^3 + 35z^2 - z + 4$ , find  $f(-5 + 2\sqrt{-4})$ .

23. Find the complex numbers satisfying the equations

$$\left| \frac{z-4}{z-8} \right| = 1 \text{ and } \left| \frac{z-12}{z-8i} \right| = \frac{5}{3}.$$

24. Prove that

(i) if  $x+iy = \frac{a+ib}{a-ib}$ , then  $x^2 + y^2 = 1$

(ii) if  $x+iy = \sqrt{\frac{1+i}{1-i}}$ , then  $x^2 + y^2 = 1$

(iii) if  $x-iy = \sqrt{\frac{a-ib}{c-id}}$ , then  $(x^2 + y^2)^2 = \frac{a^2 + b^2}{c^2 + d^2}$ .

25. If  $a+ib = \frac{c+i}{c-i}$ , where  $c$  is real, then prove that

$$a^2 + b^2 = 1 \text{ and } \frac{b}{a} = \frac{2c}{c^2 - 1}.$$

26. If  $z_1, z_2$  are complex numbers such that  $\left| \frac{z_1 - 2z_2}{2 - \bar{z}_1 z_2} \right| = 1$  and  $|z_2| \neq 1$ , then find  $|z_1|$ .

## 14.2 REPRESENTATIONS OF COMPLEX NUMBERS

A complex number can be represented geometrically or trigonometrically.

### 14.2.1 Geometric representation

We know that corresponding to every real number there exists a unique real number on the number line (called real axis) and conversely, corresponding to every point on the line there exists a unique real number *i.e.* there is a one-one correspondence between the set  $\mathbf{R}$  of real numbers and the points on the real axis.

In a similar way, corresponding to every ordered pair  $(x, y)$  of real numbers there exists a unique point  $P$  in the co-ordinate plane with  $x$  as **abscissa** and  $y$  as **ordinate** of the point  $P$  and conversely, corresponding to every point  $P$  in the plane there exists a unique ordered pair of real numbers. Thus, there is a one-one correspondence between the set of ordered pairs  $\{(x, y) ; x, y \in \mathbf{R}\}$  and the points in the co-ordinate plane.

The point  $P$  with co-ordinates  $(x, y)$  is said to represent the complex number  $z = x + iy$ .

It follows that the complex number  $z = x + iy$  can be uniquely represented by the point  $P(x, y)$  in the co-ordinate plane and conversely, corresponding to the point  $P(x, y)$  in the plane there exists a unique complex number  $z = x + iy$ . The plane is called the **complex plane** and the representation of complex numbers as points in the plane is called **Argand diagram**.

Notice that length  $OP = \sqrt{x^2 + y^2} = |z|$ .

Figure 14.2 represents the complex numbers  $0, 2, -2, 3i, -3i, 2 + 3i, -2 + 3i, -2 - 3i, 2 - 3i$ .

Note that every real number  $x = x + 0i$  is represented by point  $(x, 0)$  lying on  $x$ -axis, and every purely imaginary number  $iy$  is represented by point  $(0, y)$  lying on  $y$ -axis. Consequently,  $x$ -axis is called the **real axis** and  $y$ -axis is called the **imaginary axis**.

Since any complex number  $z = x + iy$  is represented by point  $P(x, y)$  in complex plane, many geometric ideas arise from this — distance between two points ; equation of line ; equation of a circle ; describing some region in complex plane *e.g.* all points lying inside a circle, and so on.

- If  $z_1 = x_1 + iy_1$  and  $z_2 = x_2 + iy_2$  are two complex numbers, then the distance between two corresponding points  $P_1(x_1, y_1)$  and  $P_2(x_2, y_2)$  is

$$|P_1 P_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = |z_1 - z_2| = |z_2 - z_1|.$$

- If  $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$  are two complex numbers and  $P_1(x_1, y_1), P_2(x_2, y_2)$  are two corresponding points, then the point  $Q$  dividing  $[P_1 P_2]$  *i.e.* the segment  $P_1 P_2$  in the ratio  $m : n$  is given by

$$Q\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right).$$

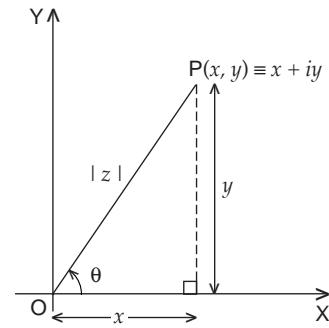


Fig. 14.1.

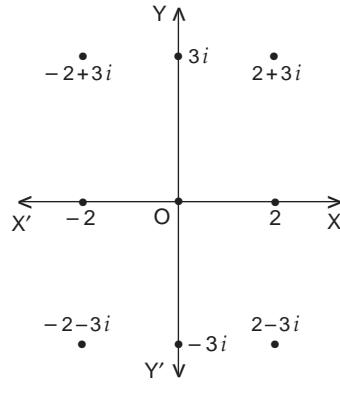


Fig. 14.2.

So corresponding complex number is

$$\begin{aligned} z_3 &= \frac{mx_2 + nx_1}{m+n} + i \frac{my_2 + ny_1}{m+n} \\ &= \frac{m(x_2 + iy_2) + n(x_1 + iy_1)}{m+n} \\ &= \frac{mz_2 + nz_1}{m+n}. \end{aligned}$$

In particular, putting  $m = n = 1$ , the **mid-point**

of  $[P_1P_2]$  is given by  $\frac{z_1 + z_2}{2}$ .

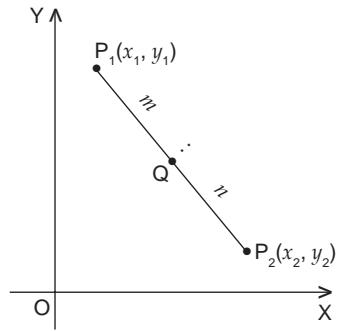


Fig. 14.3.

3. If  $z = x + iy$ ,  $x, y \in \mathbb{R}$ , is represented by the point  $P(x, y)$  in the complex plane, then the complex numbers  $-z$ ,  $\bar{z}$ ,  $-\bar{z}$  are represented by the points  $P'(-x, -y)$ ,  $Q(x, -y)$  and  $Q'(-x, y)$  respectively in the complex plane (see fig. 14.4.).

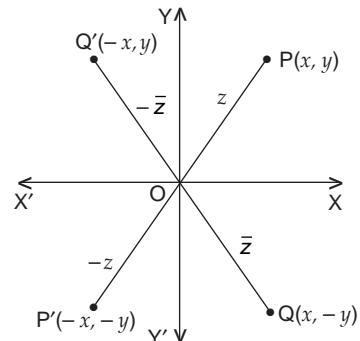


Fig. 14.4.

#### 4. Geometric representation of sum of two complex numbers.

Let the complex numbers  $z_1$  and  $z_2$  be represented by the points  $P$  and  $Q$  respectively in the complex plane. Complete the parallelogram  $OPRQ$  and let its diagonals meet at  $M$ . Since diagonals of a parallelogram bisect each other,  $M$  is mid-point of  $[PQ]$  and so it

represents the complex number  $\frac{z_1 + z_2}{2}$ .

Let  $R$  represent the complex number  $z$ . As  $M$  is also mid-point of  $[OR]$ , it will represent the complex number  $\frac{0+z}{2}$

$$\Rightarrow \frac{z}{2} = \frac{z_1 + z_2}{2} \Rightarrow z = z_1 + z_2.$$

Hence, the point  $R$  represents the complex number  $z_1 + z_2$ .

This is known as **parallelogram law**.

#### 5. Geometric representation of difference of two complex numbers.

Let the complex numbers  $z_1$  and  $z_2$  be represented by the points  $P$  and  $Q$  respectively in the complex plane. Join  $QO$  and produce it to  $Q'$  such that  $O$  is mid-point of  $[QQ']$  then  $Q'$  represents the complex number  $-z_2$ . Complete the parallelogram  $OPRQ'$  then the point  $R$  represents the complex number  $z_1 + (-z_2)$  i.e.  $z_1 - z_2$ .

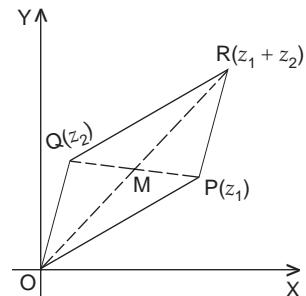


Fig. 14.5.

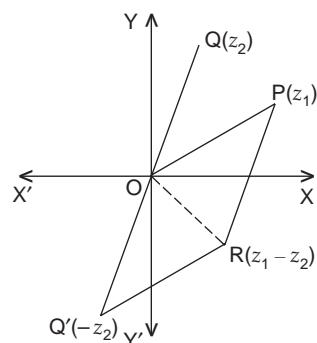


Fig. 14.6.

### 14.2.2 Trigonometric (polar) representation

We have already seen that a complex number  $x + iy$  is uniquely represented by a point  $P(x, y)$  in the complex plane. The angle  $XOP$  is called **amplitude** or **argument** of  $z$  and is written as  $\text{amp}(z)$  or  $\arg(z)$ .

Let  $OP = |z| = r > 0$ , and  $\text{amp}(z) = \theta$ .

From fig. 14.7, we see that

$$x = r \cos \theta \text{ and } y = r \sin \theta$$

$$\therefore z = x + iy = r \cos \theta + ir \sin \theta = r(\cos \theta + i \sin \theta).$$

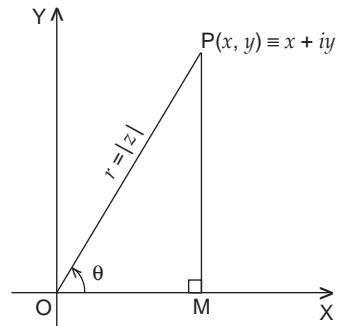


Fig. 14.7.

This form of  $z$  is called **trigonometric form** or **polar form**. The number  $(\cos \theta + i \sin \theta)$  is usually abbreviated as  $\text{cis } \theta$ . Thus, if modulus of  $z$  is  $r$  and  $\text{amp}(z) = \theta$ , then

$$z = r(\cos \theta + i \sin \theta) = r \text{ cis } \theta.$$

The unique value of  $\theta$  such that  $-\pi < \theta \leq \pi$  is called **principal value of amplitude or argument**.

Note that the number zero cannot be put into the form  $r(\cos \theta + i \sin \theta)$ . Therefore, if  $z = 0$  then the  $\arg(z)$  is not defined.

For example :

$$(i) \text{ Let } z = 1 + i, \text{ then } r = \sqrt{1^2 + 1^2} = 2$$

$$\text{and } r \cos \theta = 1, r \sin \theta = 1$$

$$\Rightarrow \cos \theta = \frac{1}{\sqrt{2}}, \sin \theta = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \theta = \frac{\pi}{4}.$$

So the principal value of amplitude is  $\frac{\pi}{4}$ .

$$\therefore z = \sqrt{2} \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right).$$

Conversely, if  $r = \sqrt{2}$  and  $\theta = \frac{\pi}{4}$

$$\text{then } z = \sqrt{2} \left( \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)$$

$$= \sqrt{2} \left( \frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right) = 1 + i.$$

$$(ii) \text{ Let } z = 1 - i, \text{ then } r = \sqrt{1^2 + (-1)^2} = 2$$

$$\text{and } r \cos \theta = 1, r \sin \theta = -1$$

$$\Rightarrow \cos \theta = \frac{1}{\sqrt{2}}, \sin \theta = -\frac{1}{\sqrt{2}}$$

$$\Rightarrow \theta = -\frac{\pi}{4}.$$

So the principal value of amplitude is  $-\frac{\pi}{4}$ .

$$\therefore z = \sqrt{2} \left( \cos \left( -\frac{\pi}{4} \right) + i \sin \left( -\frac{\pi}{4} \right) \right).$$

Conversely, if  $r = \sqrt{2}$  and  $\theta = -\frac{\pi}{4}$

$$\text{then } z = \sqrt{2} \left( \cos \left( -\frac{\pi}{4} \right) + i \sin \left( -\frac{\pi}{4} \right) \right)$$

$$= \sqrt{2} \left( \frac{1}{\sqrt{2}} - i \frac{1}{\sqrt{2}} \right) = 1 - i.$$

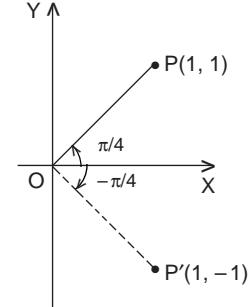


Fig. 14.8.

26. If complex numbers  $z_1, z_2, z_3$  are the vertices A, B, C respectively of an isosceles right angled triangle at C, then show that

$$(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2).$$

**Hint.** Take C as origin. Let BC = CA =  $k$ , then

$$z_1 = k + i0 = k, z_2 = 0 + ik = ik \text{ and } z_3 = 0 + i0 = 0.$$

27. If  $\omega (\neq 1)$  is a cube root of unity, then find the conjugate of  $2\omega - 3i$ .

28. If  $z$  is a complex number, prove that  $(z - 1)(\bar{z} - 1) = |z - 1|^2$ .

## ANSWERS

### EXERCISE 14.1

1. (i)  $-6$       (ii)  $6$       (iii)  $0$       (iv)  $-i$       (v)  $8i$       (vi)  $1$ .

2.  $-3, -1, -3+i, \sqrt{10}, -\frac{3}{10} + \frac{1}{10}i$ .

3. (i)  $\frac{3}{25} + \frac{4}{25}i$       (ii)  $-\frac{2}{7} + \frac{\sqrt{3}}{7}i$       (iii)  $-3 + 2i$   
 (iv)  $-\frac{1}{5} + \frac{3}{5}i$       (v)  $\frac{1}{4} - \frac{3}{4}i$       (vi)  $1 + 0i$ .

4. (i)  $x = \frac{1}{6}, y = \frac{5}{2}$       (ii)  $x = \pm 2, y = -1$       (iii)  $x = \frac{2}{21}, y = -\frac{8}{21}$   
 (iv)  $x = \frac{5}{13}, y = \frac{14}{13}$ .

5.  $a = 4, b = 2$ .      6.  $x = 3, y = -1$ .

7. (ii) No; e.g. let  $z_1 = 1 + i, z_2 = 1 - i$ .

9. (i)  $-9 + 46i; -9 - 46i$       (ii)  $-\frac{11}{125} - \frac{2}{125}i; -\frac{11}{125} + \frac{2}{125}i$       (iii)  $1 + i, \sqrt{2}$   
 (iv)  $(54\sqrt{5} - 45) - 406i$       (v)  $0 + \frac{1}{2}i$       (vii)  $-3 + i$       (viii)  $3 + 5i$ .

10. (i)  $1$       (ii)  $\sqrt{\frac{41}{85}}$ .      15.  $a = 1, b = 0$ .      16.  $n = 2$ .

17.  $x = 0$  is the only solution.      18.  $z = \frac{3}{4} + i$  is the only solution.

22.  $-160$ .      23. Solutions are  $6 + 8i, 6 + 17i$ .      26.  $2$ .

### EXERCISE 14.2

1. (i)  $2 \operatorname{cis} \frac{\pi}{6}$       (ii)  $\sqrt{2} \operatorname{cis} \left(-\frac{3\pi}{4}\right)$       (iii)  $\pi \operatorname{cis} 0$       (iv)  $\operatorname{cis} 0$

(v)  $\sqrt{2} \operatorname{cis} \left(\frac{3\pi}{4}\right)$       (vi)  $2 \operatorname{cis} \frac{\pi}{3}$       (vii)  $\sqrt{2} \operatorname{cis} \frac{\pi}{4}$ .

2. (i)  $\sqrt{2}; \frac{3\pi}{4}$       (ii)  $1; \pi - \tan^{-1} \frac{4}{3}$       (iii)  $\sqrt{2}; -\frac{3\pi}{4}$ .      3.  $\frac{\pi}{10}$ .

4. (i)  $\frac{1}{2} + \frac{\sqrt{3}}{2}i$       (ii)  $1 - i$ .      14. Maximum value =  $\sqrt{2} + 1$ , minimum value =  $\sqrt{2} - 1$ .

### EXERCISE 14.3

1. (i)  $|z| = 2$       (ii)  $|z + 1 + i| > \sqrt{2}$

(iii)  $|z - 1 - i| = \sqrt{2}$

(v)  $\operatorname{Re}(z) = 0$

(iv)  $\operatorname{Re}(z) > 0$

(vi)  $\operatorname{Re}(z) > 0$  and  $\operatorname{Im}(z) > 0$ .