

# 10 Definite Integrals

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## 10.1 FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS

Let  $f$  be a continuous function on the closed interval  $[a, b]$  and  $\phi$  be an anti-derivative of  $f$ , then

$$\int_a^b f(x) dx = \phi(b) - \phi(a). \quad (\text{We assume it without proof})$$

In words, the above theorem tells us that

$$\int_a^b f(x) dx = (\text{value of an anti-derivative at } b, \text{ the upper limit}) - (\text{value of the same anti-derivative at } a, \text{ the lower limit}).$$

### Remarks

1. We often write  $\phi(b) - \phi(a)$  as  $[\phi(x)]_a^b$ .
2. No matter which anti-derivative we take as  $\phi$ , the value of the definite integral comes out to be the same.

#### 10.1.1 Evaluation of Definite Integrals

The fundamental theorem enables us to evaluate the definite integrals by making use of *anti-derivatives*.

### ILLUSTRATIVE EXAMPLES

**Example 1.** Evaluate the following :

$$(i) \int_0^1 (2x^3 + 3)^2 dx \qquad (ii) \int_3^5 \frac{dt}{1+3t}.$$

**Solution.** (i) 
$$\begin{aligned} \int_0^1 (2x^3 + 3)^2 dx &= \int_0^1 (4x^6 + 12x^3 + 9) dx \\ &= \left[ 4 \cdot \frac{x^7}{7} + 12 \cdot \frac{x^4}{4} + 9x \right]_0^1 = \left[ \frac{4}{7}x^7 + 3x^4 + 9x \right]_0^1 \\ &= \left( \frac{4}{7} + 3 + 9 \right) - (0 + 0 + 0) = \frac{88}{7}. \end{aligned}$$

(ii) 
$$\begin{aligned} \int_3^5 \frac{dt}{1+3t} &= \left[ \frac{\log|1+3t|}{3} \right]_3^5 = \frac{1}{3} (\log 16 - \log 10) \\ &= \frac{1}{3} \log \frac{16}{10} = \frac{1}{3} \log \frac{8}{5}. \end{aligned}$$

**Example 2.** Evaluate the following :

$$(i) \int_0^{\pi/2} \cos^2 x \, dx \quad (ii) \int_0^{\pi/2} \cos^4 x \, dx.$$

$$\begin{aligned}\text{Solution. } (i) \int_0^{\pi/2} \cos^2 x \, dx &= \int_0^{\pi/2} \frac{1 + \cos 2x}{2} \, dx = \frac{1}{2} \left[ x + \frac{\sin 2x}{2} \right]_0^{\pi/2} \\ &= \frac{1}{2} \left[ \left( \frac{\pi}{2} + \frac{1}{2} \sin \pi \right) - \left( 0 + \frac{1}{2} \sin 0 \right) \right] \\ &= \frac{1}{2} \left[ \left( \frac{\pi}{2} + \frac{1}{2} \cdot 0 \right) - \frac{1}{2} \cdot 0 \right] = \frac{\pi}{4}.\end{aligned}$$

$$\begin{aligned}(ii) \int_0^{\pi/2} \cos^4 x \, dx &= \int_0^{\pi/2} (\cos^2 x)^2 \, dx = \int_0^{\pi/2} \left( \frac{1 + \cos 2x}{2} \right)^2 \, dx \\ &= \frac{1}{4} \int_0^{\pi/2} (1 + 2 \cos 2x + \cos^2 2x) \, dx = \frac{1}{4} \int_0^{\pi/2} \left( 1 + 2 \cos 2x + \frac{1 + \cos 4x}{2} \right) \, dx \\ &= \frac{1}{8} \int_0^{\pi/2} (3 + 4 \cos 2x + \cos 4x) \, dx = \frac{1}{8} \left[ 3x + 4 \cdot \frac{\sin 2x}{2} + \frac{\sin 4x}{4} \right]_0^{\pi/2} \\ &= \frac{1}{8} \left[ 3 \left( \frac{\pi}{2} - 0 \right) + 2(\sin \pi - \sin 0) + \frac{1}{4} (\sin 2\pi - \sin 0) \right] \\ &= \frac{1}{8} \left[ \frac{3\pi}{2} + 2(0 - 0) + \frac{1}{4}(0 - 0) \right] = \frac{3\pi}{16}.\end{aligned}$$

**Example 3.** Evaluate the following :

$$(i) \int_0^{\pi/2} \sqrt{1 + \cos 2x} \, dx \quad (ii) \int_0^{\pi/2} \sqrt{1 + \sin 2x} \, dx.$$

$$\begin{aligned}\text{Solution. } (i) \int_0^{\pi/2} \sqrt{1 + \cos 2x} \, dx &= \int_0^{\pi/2} \sqrt{2 \cos^2 x} \, dx = \sqrt{2} \int_0^{\pi/2} |\cos x| \, dx \\ &= \sqrt{2} \int_0^{\pi/2} \cos x \, dx \\ &\quad (\text{As } 0 \leq x \leq \frac{\pi}{2} \Rightarrow \cos x \geq 0 \Rightarrow |\cos x| = \cos x) \\ &= \sqrt{2} \left[ \sin x \right]_0^{\pi/2} = \sqrt{2} \left[ \sin \frac{\pi}{2} - \sin 0 \right] \\ &= \sqrt{2} (1 - 0) = \sqrt{2}.\end{aligned}$$

$$\begin{aligned}(ii) \int_0^{\pi/2} \sqrt{1 + \sin 2x} \, dx &= \int_0^{\pi/2} \sqrt{1 + \cos \left( \frac{\pi}{2} - 2x \right)} \, dx = \int_0^{\pi/2} \sqrt{2 \cos^2 \left( \frac{\pi}{4} - x \right)} \, dx \\ &= \sqrt{2} \int_0^{\pi/2} \left| \cos \left( \frac{\pi}{4} - x \right) \right| \, dx = \sqrt{2} \int_0^{\pi/2} \cos \left( \frac{\pi}{4} - x \right) \, dx\end{aligned}$$

$$\begin{aligned}&\left[ \text{As } 0 \leq x \leq \frac{\pi}{2} \Rightarrow 0 \geq -x \geq -\frac{\pi}{2} \Rightarrow \frac{\pi}{4} \geq \frac{\pi}{4} - x \geq -\frac{\pi}{4} \right. \\ &\quad \left. \Rightarrow -\frac{\pi}{4} \leq \frac{\pi}{4} - x \leq \frac{\pi}{4} \Rightarrow \cos \left( \frac{\pi}{4} - x \right) > 0 \Rightarrow \left| \cos \left( \frac{\pi}{4} - x \right) \right| = \cos \left( \frac{\pi}{4} - x \right) \right]\end{aligned}$$

$$\begin{aligned}
&= \sqrt{2} \left[ \frac{\sin\left(\frac{\pi}{4} - x\right)}{-1} \right]_0^{\pi/2} = -\sqrt{2} \left[ \sin\left(-\frac{\pi}{4}\right) - \sin\frac{\pi}{4} \right] \\
&= -\sqrt{2} \left[ -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \right] = \sqrt{2} \cdot \frac{2}{\sqrt{2}} = 2.
\end{aligned}$$

**Example 4.** Evaluate the following integrals :

$$(i) \int_0^{\pi/4} \sqrt{1 - \sin 2x} \, dx \quad (ii) \int_{\pi}^{3\pi/2} \sqrt{1 - \cos 2x} \, dx.$$

$$\begin{aligned}
\text{Solution. } (i) \int_0^{\pi/4} \sqrt{1 - \sin 2x} \, dx &= \int_0^{\pi/4} \sqrt{1 - \cos\left(\frac{\pi}{2} - 2x\right)} \, dx = \int_0^{\pi/4} \sqrt{2 \sin^2\left(\frac{\pi}{4} - x\right)} \, dx \\
&= \sqrt{2} \int_0^{\pi/4} \left| \sin\left(\frac{\pi}{4} - x\right) \right| \, dx = \sqrt{2} \int_0^{\pi/4} \sin\left(\frac{\pi}{4} - x\right) \, dx
\end{aligned}$$

$$\begin{aligned}
&\left[ \text{As } 0 \leq x \leq \frac{\pi}{4} \Rightarrow 0 \geq -x \geq -\frac{\pi}{4} \Rightarrow \frac{\pi}{4} \geq \frac{\pi}{4} - x \geq 0 \right. \\
&\Rightarrow 0 \leq \frac{\pi}{4} - x \leq \frac{\pi}{4} \Rightarrow \sin\left(\frac{\pi}{4} - x\right) \geq 0 \Rightarrow \left| \sin\left(\frac{\pi}{4} - x\right) \right| = \sin\left(\frac{\pi}{4} - x\right) \Big] \\
&= \sqrt{2} \left[ \frac{-\cos\left(\frac{\pi}{4} - x\right)}{-1} \right]_0^{\pi/4} = \sqrt{2} \left[ \cos 0 - \cos \frac{\pi}{4} \right] \\
&= \sqrt{2} \left( 1 - \frac{1}{\sqrt{2}} \right) = \sqrt{2} - 1.
\end{aligned}$$

$$\begin{aligned}
(ii) \int_{\pi}^{3\pi/2} \sqrt{1 - \cos 2x} \, dx &= \int_{\pi}^{3\pi/2} \sqrt{2 \sin^2 x} \, dx = \sqrt{2} \int_{\pi}^{3\pi/2} |\sin x| \, dx \\
&= \sqrt{2} \int_{\pi}^{3\pi/2} (-\sin x) \, dx \\
&\left[ \text{As } \pi \leq x \leq \frac{3\pi}{2} \Rightarrow \sin x \leq 0 \Rightarrow |\sin x| = -\sin x \right] \\
&= -\sqrt{2} \left[ -\cos x \right]_{\pi}^{3\pi/2} = \sqrt{2} (\cos \frac{3\pi}{2} - \cos \pi) \\
&= \sqrt{2} (0 - (-1)) = \sqrt{2}.
\end{aligned}$$

**Example 5.** Evaluate the following integrals :

$$(i) \int_{\pi/4}^{\pi/2} \sqrt{1 - \sin 2x} \, dx \quad (ii) \int_0^{2\pi} \sqrt{1 + \sin \frac{x}{2}} \, dx.$$

$$\begin{aligned}
\text{Solution. } (i) \int_{\pi/4}^{\pi/2} \sqrt{1 - \sin 2x} \, dx &= \int_{\pi/4}^{\pi/2} \sqrt{1 - \cos\left(\frac{\pi}{2} - 2x\right)} \, dx = \int_{\pi/4}^{\pi/2} \sqrt{2 \sin^2\left(\frac{\pi}{4} - x\right)} \, dx \\
&= \sqrt{2} \int_{\pi/4}^{\pi/2} \left| \sin\left(\frac{\pi}{4} - x\right) \right| \, dx = \sqrt{2} \int_{\pi/4}^{\pi/2} \left( -\sin\left(\frac{\pi}{4} - x\right) \right) \, dx
\end{aligned}$$

$$\begin{aligned}
&\left[ \text{As } \frac{\pi}{4} \leq x \leq \frac{\pi}{2} \Rightarrow -\frac{\pi}{4} \geq -x \geq -\frac{\pi}{2} \Rightarrow 0 \geq \frac{\pi}{4} - x \geq -\frac{\pi}{4} \right. \\
&\Rightarrow -\frac{\pi}{4} \leq \frac{\pi}{4} - x \leq 0 \Rightarrow \sin\left(\frac{\pi}{4} - x\right) \leq 0 \Rightarrow \left| \sin\left(\frac{\pi}{4} - x\right) \right| = -\sin\left(\frac{\pi}{4} - x\right) \Big]
\end{aligned}$$

$$= -\sqrt{2} \left[ \frac{-\cos\left(\frac{\pi}{4} - x\right)}{-1} \right]_{\pi/4}^{\pi/2} = -\sqrt{2} \left( \cos\left(-\frac{\pi}{4}\right) - \cos 0 \right)$$

$$= -\sqrt{2} \left( \frac{1}{\sqrt{2}} - 1 \right) = \sqrt{2} - 1.$$

$$(ii) \int_0^{2\pi} \sqrt{1 + \sin \frac{x}{2}} dx = \int_0^{2\pi} \sqrt{1 + \cos\left(\frac{\pi}{2} - \frac{x}{2}\right)} dx = \int_0^{2\pi} \sqrt{2 \cos^2\left(\frac{\pi}{4} - \frac{x}{4}\right)} dx$$

$$= \sqrt{2} \int_0^{2\pi} \left| \cos\left(\frac{\pi}{4} - \frac{x}{4}\right) \right| dx = \sqrt{2} \int_0^{2\pi} \cos\left(\frac{\pi}{4} - \frac{x}{4}\right) dx$$

$$\begin{aligned} & \left[ \text{As } 0 \leq x \leq 2\pi \Rightarrow 0 \leq \frac{x}{4} \leq \frac{\pi}{2} \Rightarrow 0 \geq -\frac{x}{4} \geq -\frac{\pi}{2} \Rightarrow \frac{\pi}{4} \geq \frac{\pi}{4} - \frac{x}{4} \geq -\frac{\pi}{4} \right. \\ & \quad \Rightarrow -\frac{\pi}{4} \leq \frac{\pi}{4} - \frac{x}{4} \leq \frac{\pi}{4} \Rightarrow \cos\left(\frac{\pi}{4} - \frac{x}{4}\right) > 0 \Rightarrow \left| \cos\left(\frac{\pi}{4} - \frac{x}{4}\right) \right| = \cos\left(\frac{\pi}{4} - \frac{x}{4}\right) \end{aligned}$$

$$\begin{aligned} & = \sqrt{2} \left[ \frac{\sin\left(\frac{\pi}{4} - \frac{x}{4}\right)}{-\frac{1}{4}} \right]_0^{2\pi} = -4\sqrt{2} \left( \sin\left(-\frac{\pi}{4}\right) - \sin\frac{\pi}{4} \right) \\ & = -4\sqrt{2} \left( -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \right) = 4\sqrt{2} \cdot \frac{2}{\sqrt{2}} = 8. \end{aligned}$$

**Example 6.** Evaluate the following integrals :

$$(i) \int_0^{\pi/4} \sec x \sqrt{\frac{1-\sin x}{1+\sin x}} dx$$

$$(ii) \int_0^{\pi/4} (\tan x + \cot x)^{-2} dx.$$

$$\begin{aligned} \text{Solution. } (i) \int_0^{\pi/4} \sec x \sqrt{\frac{1-\sin x}{1+\sin x}} dx &= \int_0^{\pi/4} \sec x \sqrt{\frac{1-\sin x}{1+\sin x} \times \frac{1-\sin x}{1-\sin x}} dx \\ &= \int_0^{\pi/4} \sec x \sqrt{\frac{(1-\sin x)^2}{\cos^2 x}} dx = \int_0^{\pi/4} \sec x \left| \frac{1-\sin x}{\cos x} \right| dx \\ &\quad \left[ \text{As } 0 \leq x \leq \frac{\pi}{4} \Rightarrow \cos x > 0, 1 - \sin x > 0 \Rightarrow \frac{1-\sin x}{\cos x} > 0 \right] \\ &= \int_0^{\pi/4} \sec x \cdot \frac{1-\sin x}{\cos x} dx = \int_0^{\pi/4} \sec x (\sec x - \tan x) dx \\ &= \int_0^{\pi/4} (\sec^2 x - \sec x \tan x) dx = [\tan x - \sec x]_0^{\pi/4} \\ &= \left( \tan \frac{\pi}{4} - \sec \frac{\pi}{4} \right) - (\tan 0 - \sec 0) \\ &= (1 - \sqrt{2}) - (0 - 1) = 2 - \sqrt{2}. \end{aligned}$$

$$\begin{aligned} (ii) \int_0^{\pi/4} (\tan x + \cot x)^{-2} dx &= \int_0^{\pi/4} \left( \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} \right)^{-2} dx = \int_0^{\pi/4} \left( \frac{\sin^2 x + \cos^2 x}{\sin x \cos x} \right)^{-2} dx \\ &= \int_0^{\pi/4} \left( \frac{1}{\sin x \cos x} \right)^{-2} dx = \int_0^{\pi/4} (\sin x \cos x)^2 dx \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4} \int_0^{\pi/4} (2 \sin x \cos x)^2 dx = \frac{1}{4} \int_0^{\pi/4} \sin^2 2x dx \\
&= \frac{1}{4} \int_0^{\pi/4} \frac{1 - \cos 4x}{2} dx = \frac{1}{8} \left[ x - \frac{\sin 4x}{4} \right]_0^{\pi/4} \\
&= \frac{1}{8} \left[ \left( \frac{\pi}{4} - \frac{\sin \pi}{4} \right) - \left( 0 - \frac{\sin 0}{4} \right) \right] = \frac{1}{8} \left[ \left( \frac{\pi}{4} - 0 \right) - 0 \right] = \frac{\pi}{32}.
\end{aligned}$$

**Example 7.** Evaluate the following integrals :

$$(i) \int_0^1 \frac{(\sin^{-1} x)^2}{\sqrt{1-x^2}} dx \quad (ii) \int_0^1 \frac{x+1}{(x^2+2x+3)^2} dx.$$

**Solution.** (i)  $\int_0^1 \frac{(\sin^{-1} x)^2}{\sqrt{1-x^2}} dx = \int_0^1 (\sin^{-1} x)^2 \cdot \frac{1}{\sqrt{1-x^2}} dx$  |  $\int (f(x))^n f'(x) dx$  form

$$\begin{aligned}
&= \left[ \frac{(\sin^{-1} x)^3}{3} \right]_0^1 = \frac{1}{3} [(\sin^{-1} 1)^3 - (\sin^{-1} 0)^3] \\
&= \frac{1}{3} \left[ \left( \frac{\pi}{2} \right)^3 - 0^3 \right] = \frac{\pi^3}{24}.
\end{aligned}$$

$$\begin{aligned}
(ii) \int_0^1 \frac{x+1}{(x^2+2x+3)^2} dx &= \frac{1}{2} \int_0^1 (x^2+2x+3)^{-2} (2x+2) dx \\
&= \frac{1}{2} \cdot \left[ \frac{(x^2+2x+3)^{-1}}{-1} \right]_0^1 = -\frac{1}{2} \left[ \frac{1}{x^2+2x+3} \right]_0^1 \\
&= -\frac{1}{2} \left[ \frac{1}{1+2+3} - \frac{1}{0+0+3} \right] = -\frac{1}{2} \left( \frac{1}{6} - \frac{1}{3} \right) = -\frac{1}{2} \left( -\frac{1}{6} \right) = \frac{1}{12}.
\end{aligned}$$

**Example 8.** Evaluate the following integrals :

$$(i) \int_a^b \frac{\log x}{x} dx \quad (I.S.C. 2000) \quad (ii) \int_0^{\pi/4} \frac{2 \cos 2x}{1 + \sin 2x} dx . \quad (I.S.C. 2002)$$

**Solution.** (i)  $\int_a^b \frac{\log x}{x} dx = \int_a^b (\log x)^1 \cdot \frac{1}{x} dx$  |  $\int (f(x))^n f'(x) dx$  form

$$\begin{aligned}
&= \left[ \frac{(\log x)^2}{2} \right]_a^b = \frac{1}{2} [(\log b)^2 - (\log a)^2] \\
&= \frac{1}{2} (\log b + \log a) (\log b - \log a) \\
&= \frac{1}{2} \log ab \log \left( \frac{b}{a} \right).
\end{aligned}$$

$$(ii) \text{ As } \frac{d}{dx} (1 + \sin 2x) = 0 + \cos 2x \cdot 2 = 2 \cos 2x,$$

$$\begin{aligned}
\therefore \int_0^{\pi/4} \frac{2 \cos 2x}{1 + \sin 2x} dx &= \left[ \log |1 + \sin 2x| \right]_0^{\pi/4} \quad | \int \frac{f'(x)}{f(x)} dx \text{ form} \\
&= \log |1 + \sin \frac{\pi}{2}| - \log |1 + \sin 0| \\
&= \log (1+1) - \log (1+0) = \log 2 - \log 1 \\
&= \log 2 - 0 = \log 2.
\end{aligned}$$

**Example 9.** Evaluate the following :

$$(i) \int_0^1 \frac{x^9}{5+x^{10}} dx \quad (ii) \int_0^{\pi/2} x \cos 2x dx.$$

**Solution.** (i)  $\int_0^1 \frac{x^9}{5+x^{10}} dx = \frac{1}{10} \int_0^1 \frac{10x^9}{5+x^{10}} dx$   $\left| \int \frac{f'(x)}{f(x)} dx \text{ form} \right.$   
 $= \frac{1}{10} [\log |5+x^{10}|]_0^1 = \frac{1}{10} [\log 6 - \log 5] = \frac{1}{10} \log \frac{6}{5}.$

$$(ii) \int_0^{\pi/2} x \cos 2x dx = \left[ x \cdot \frac{\sin 2x}{2} \right]_0^{\pi/2} - \int_0^{\pi/2} 1 \cdot \frac{\sin 2x}{2} dx \quad (\text{using integration by parts})$$
 $= \frac{1}{2} \left( \frac{\pi}{2} \sin \pi - 0 \right) - \frac{1}{2} \int_0^{\pi/2} \sin 2x dx = \frac{1}{2} \left( \frac{\pi}{2} \cdot 0 \right) - \frac{1}{2} \left[ -\frac{\cos 2x}{2} \right]_0^{\pi/2}$ 
 $= \frac{1}{4} (\cos \pi - \cos 0) = \frac{1}{4} (-1 - 1) = -\frac{1}{2}.$

**Example 10.** Evaluate the following :

$$(i) \int_0^{\pi/2} x \sin^2 x dx \quad (ii) \int_0^1 x^2 e^x dx.$$

**Solution.** (i)  $\int_0^{\pi/2} x \sin^2 x dx = \int_0^{\pi/2} x \cdot \frac{1-\cos 2x}{2} dx = \frac{1}{2} \int_0^{\pi/2} x dx - \frac{1}{2} \int_0^{\pi/2} x \cos 2x dx$   
 $= \frac{1}{2} \left[ \frac{x^2}{2} \right]_0^{\pi/2} - \frac{1}{2} \left( \left[ x \cdot \frac{\sin 2x}{2} \right]_0^{\pi/2} - \int_0^{\pi/2} 1 \cdot \frac{\sin 2x}{2} dx \right)$   
 $= \frac{1}{4} \left( \frac{\pi^2}{4} - 0 \right) - \frac{1}{4} \left( \frac{\pi}{2} \sin \pi - 0 \right) + \frac{1}{4} \int_0^{\pi/2} \sin 2x dx$   
 $= \frac{\pi^2}{16} - \frac{1}{4} \left( \frac{\pi}{2} \cdot 0 \right) + \frac{1}{4} \left[ -\frac{\cos 2x}{2} \right]_0^{\pi/2} = \frac{\pi^2}{16} - \frac{1}{8} [\cos 2x]_0^{\pi/2}$   
 $= \frac{\pi^2}{16} - \frac{1}{8} (\cos \pi - \cos 0) = \frac{\pi^2}{16} - \frac{1}{8} (-1 - 1) = \frac{\pi^2}{16} + \frac{1}{4}.$

$$(ii) \int_0^1 x^2 e^x dx = [x^2 \cdot e^x]_0^1 - \int_0^1 2x e^x dx$$
 $= (1 \cdot e^1 - 0) - 2 \int_0^1 x e^x dx = e - 2 \left( [x e^x]_0^1 - \int_0^1 1 \cdot e^x dx \right)$ 
 $= e - 2 (1 \cdot e^1 - 0) + 2 \int_0^1 e^x dx = e - 2e + 2 [e^x]_0^1$ 
 $= -e + 2(e^1 - e^0) = -e + 2(e - 1) = e - 2.$

**Example 11.** Evaluate the following integrals :

$$(i) \int_{\pi/4}^{\pi/2} \cos 2x \log \sin x dx \quad (ii) \int_0^1 x \tan^{-1} x dx. \quad (\text{I.S.C. 2002})$$

**Solution.** (i)  $\int_{\pi/4}^{\pi/2} \cos 2x \log \sin x dx = \int_{\pi/4}^{\pi/2} \log \sin x \cdot \cos 2x dx \quad (\text{integrate by parts})$   
 $= \left[ \log \sin x \cdot \frac{\sin 2x}{2} \right]_{\pi/4}^{\pi/2} - \int_{\pi/4}^{\pi/2} \frac{1}{\sin x} \cdot \cos x \cdot \frac{\sin 2x}{2} dx$

$$\begin{aligned}
&= \frac{1}{2} \left[ 0 - \log \frac{1}{\sqrt{2}} \right] - \int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin x} \cdot \frac{2 \sin x \cos x}{2} dx \\
&= -\frac{1}{2} (\log 1 - \frac{1}{2} \log 2) - \int_{\pi/4}^{\pi/2} \cos^2 x dx \\
&= \frac{1}{4} \log 2 - \int_{\pi/4}^{\pi/2} \frac{1 + \cos 2x}{2} dx \\
&= \frac{1}{4} \log 2 - \frac{1}{2} \left[ x + \frac{\sin 2x}{2} \right]_{\pi/4}^{\pi/2} \\
&= \frac{1}{4} \log 2 - \frac{1}{2} \left[ \left( \frac{\pi}{2} - \frac{\pi}{4} \right) + \frac{1}{2}(0 - 1) \right] = \frac{1}{4} \log 2 - \frac{\pi}{8} + \frac{1}{4}.
\end{aligned}$$

(ii)  $\int_0^1 x \tan^{-1} x dx = \int_0^1 \tan^{-1} x \cdot x dx$  (integrate by parts)

$$\begin{aligned}
&= \left[ \tan^{-1} x \cdot \frac{x^2}{2} \right]_0^1 - \int_0^1 \frac{1}{1+x^2} \cdot \frac{x^2}{2} dx \\
&= \frac{1}{2} [\tan^{-1} 1 - 0] - \frac{1}{2} \int_0^1 \frac{x^2}{1+x^2} dx \\
&= \frac{1}{2} \cdot \frac{\pi}{4} - \frac{1}{2} \int_0^1 \left( 1 - \frac{1}{1+x^2} \right) dx \\
&= \frac{\pi}{8} - \frac{1}{2} [x - \tan^{-1} x]_0^1 = \frac{\pi}{8} - \frac{1}{2} [(1 - \tan^{-1} 1) - (0 - \tan^{-1} 0)] \\
&= \frac{\pi}{8} - \frac{1}{2} \left[ \left( 1 - \frac{\pi}{4} \right) - 0 \right] = \frac{\pi}{4} - \frac{1}{2}.
\end{aligned}$$

**Example 12.** Evaluate the following :

$$(i) \int_0^{\pi/2} \frac{\cos x}{1 + \cos x + \sin x} dx \quad (ii) \int_1^5 \frac{\log x}{(x+1)^2} dx.$$

**Solution.** (i)  $\int_0^{\pi/2} \frac{\cos x}{1 + \cos x + \sin x} dx = \int_0^{\pi/2} \frac{\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}}{2 \cos^2 \frac{x}{2} + 2 \sin \frac{x}{2} \cos \frac{x}{2}} dx$

$$\begin{aligned}
&= \int_0^{\pi/2} \frac{\cos \frac{x}{2} - \sin \frac{x}{2}}{2 \cos \frac{x}{2}} dx = \frac{1}{2} \int_0^{\pi/2} \left( 1 - \tan \frac{x}{2} \right) dx \\
&= \frac{1}{2} \left[ x + \frac{\log \left| \cos \frac{x}{2} \right|}{\frac{1}{2}} \right]_0^{\pi/2} \\
&= \frac{1}{2} \left[ \left( \frac{\pi}{2} - 0 \right) + 2 \left( \log \frac{1}{\sqrt{2}} - \log 1 \right) \right] \\
&= \frac{\pi}{4} + \log 1 - \log \sqrt{2} = \frac{\pi}{4} - \frac{1}{2} \log 2.
\end{aligned}$$

$$\begin{aligned}
 (ii) \int_1^5 \frac{\log x}{(x+1)^2} dx &= \int_1^5 \log x \cdot (x+1)^{-2} dx && \text{(integrate by parts)} \\
 &= \left[ \log x \cdot \frac{(x+1)^{-1}}{-1} \right]_1^5 - \int_1^5 \frac{1}{x} \cdot \frac{(x+1)^{-1}}{-1} dx \\
 &= - \left[ \frac{\log x}{x+1} \right]_1^5 + \int_1^5 \frac{dx}{x(x+1)} \\
 &= - \left( \frac{\log 5}{6} - \frac{\log 1}{2} \right) + \int_1^5 \left( \frac{1}{x} - \frac{1}{x+1} \right) dx && \text{(by partial fractions)} \\
 &= - \left( \frac{\log 5}{6} - 0 \right) + [\log |x| - \log |x+1|]_1^5 \\
 &= - \frac{1}{6} \log 5 + (\log 5 - \log 6) - (\log 1 - \log 2) \\
 &= \frac{5}{6} \log 5 - (\log 6 - \log 2) = \frac{5}{6} \log 5 - \log 3.
 \end{aligned}$$

**Example 13.** Evaluate the following :

$$(i) \int_0^{\sqrt{2}} \sqrt{2-x^2} dx \quad (ii) \int_1^2 \frac{2}{4x^2-1} dx.$$

$$\begin{aligned}
 \textbf{Solution.} (i) \int_0^{\sqrt{2}} \sqrt{2-x^2} dx &= \int_0^{\sqrt{2}} \sqrt{(\sqrt{2})^2 - x^2} dx \\
 &= \left[ \frac{x\sqrt{2-x^2}}{2} + \frac{(\sqrt{2})^2}{2} \sin^{-1} \frac{x}{\sqrt{2}} \right]_0^{\sqrt{2}} \\
 &= \left( \frac{\sqrt{2} \cdot 0}{2} + \sin^{-1} 1 \right) - \left( \frac{0 \cdot \sqrt{2}}{2} + \sin^{-1} 0 \right) \\
 &= \left( 0 + \frac{\pi}{2} \right) - (0 + 0) = \frac{\pi}{2}.
 \end{aligned}$$

$$\begin{aligned}
 (ii) \int_1^2 \frac{2}{4x^2-1} dx &= 2 \cdot \frac{1}{4} \int_1^2 \frac{dx}{x^2 - \left(\frac{1}{2}\right)^2} = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \left[ \log \left| \frac{x - \frac{1}{2}}{x + \frac{1}{2}} \right| \right]_1^2 = \frac{1}{2} \left[ \log \left| \frac{2x-1}{2x+1} \right| \right]_1^2 \\
 &= \frac{1}{2} \left( \log \frac{3}{5} - \log \frac{1}{3} \right) = \frac{1}{2} \log \left( \frac{3}{5} \times \frac{3}{1} \right) = \frac{1}{2} \log \frac{9}{5}.
 \end{aligned}$$

**Example 14.** Evaluate the following integrals :

$$(i) \int_0^1 \frac{x e^x}{(x+1)^2} dx \quad (\text{I.S.C. 2011}) \quad (ii) \int_1^5 \frac{\log x}{(x+1)^2} dx \quad (iii) \int_0^1 \sin^{-1} \left( \frac{2x}{1+x^2} \right) dx.$$

$$\begin{aligned}
 \textbf{Solution.} (i) \int_0^1 \frac{x e^x}{(x+1)^2} dx &= \int_0^1 \frac{(x+1)-1}{(x+1)^2} e^x dx \\
 &= \int_0^1 \frac{1}{x+1} \cdot e^x dx - \int_0^1 \frac{1}{(x+1)^2} e^x dx
 \end{aligned}$$

(evaluate the first integral by parts, taking  $\frac{1}{x+1}$  as the first function)

$$\begin{aligned}
&= \left[ \frac{1}{x+1} \cdot e^x \right]_0^1 - \int_0^1 (-1)(x+1)^{-2} \cdot e^x dx - \int_0^1 \frac{1}{(x+1)^2} e^x dx \\
&= \left( \frac{1}{2} \cdot e^1 - 1 \cdot e^0 \right) + \int_0^1 \frac{1}{(x+1)^2} e^x dx - \int_0^1 \frac{1}{(x+1)^2} e^x dx = \frac{1}{2} e - 1.
\end{aligned}$$

$$\begin{aligned}
(ii) \quad \int_1^5 \frac{\log x}{(x+1)^2} dx &= \int_1^5 \log x \cdot (x+1)^{-2} dx \quad (\text{by parts}) \\
&= \left[ \log x \cdot \frac{(x+1)^{-1}}{-1} \right]_1^5 - \int_1^5 \frac{1}{x} \cdot \frac{(x+1)^{-1}}{-1} dx \\
&= - \left[ \frac{\log x}{x+1} \right]_1^5 + \int_1^5 \frac{1}{x(x+1)} dx \\
&= - \left[ \frac{\log 5}{6} - \frac{\log 1}{2} \right] + \int_1^5 \left( \frac{1}{x} - \frac{1}{x+1} \right) dx \\
&= - \left( \frac{\log 5}{6} - 0 \right) + \left[ \log |x| - \log |x+1| \right]_1^5 \\
&= - \frac{\log 5}{6} + (\log 5 - \log 6) - (\log 1 - \log 2) \\
&= \frac{5}{6} \log 5 - (\log 6 - \log 2) = \frac{5}{6} \log 5 - \log 3.
\end{aligned}$$

$$\begin{aligned}
(iii) \quad \int_0^1 \sin^{-1} \left( \frac{2x}{1+x^2} \right) dx &= \int_0^1 2 \tan^{-1} x dx = 2 \int_0^1 \tan^{-1} x \cdot 1 dx \quad (\text{integrate by parts}) \\
&= 2 \left[ \left[ \tan^{-1} x \cdot x \right]_0^1 - \int_0^1 \frac{1}{1+x^2} \cdot x dx \right] \\
&= 2(\tan^{-1} 1 - 0) - \int_0^1 \frac{2x}{1+x^2} dx \\
&= 2 \left( \frac{\pi}{4} - 0 \right) - \left[ \log(1+x^2) \right]_0^1 \\
&= \frac{\pi}{2} - (\log 2 - \log 1) = \frac{\pi}{2} - (\log 2 - 0) \\
&= \frac{\pi}{2} - \log 2.
\end{aligned}$$

**Example 15.** Evaluate the following :

$$(i) \int_{1/4}^{1/2} \frac{dx}{\sqrt{x-x^2}} \qquad (ii) \int_2^3 \frac{x^3+1}{x(x-1)} dx.$$

$$\text{Solution. } (i) \quad \int_{1/4}^{1/2} \frac{dx}{\sqrt{x-x^2}} = \int_{1/4}^{1/2} \frac{dx}{\sqrt{\frac{1}{4} - (x^2 - x + \frac{1}{4})}} = \int_{1/4}^{1/2} \frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2}}$$

$$\begin{aligned}
&= \left[ \sin^{-1} \left( \frac{x - \frac{1}{2}}{\frac{1}{2}} \right) \right]_{1/4}^{1/2} = \left[ \sin^{-1} (2x - 1) \right]_{1/4}^{1/2} \\
&= \sin^{-1} 0 - \sin^{-1} \left( -\frac{1}{2} \right) = 0 - \left( -\frac{\pi}{6} \right) = \frac{\pi}{6}.
\end{aligned}$$

$$(ii) \int_2^3 \frac{x^3 + 1}{x(x-1)} dx = \int_2^3 \left( x + 1 + \frac{x+1}{x(x-1)} \right) dx \quad (\text{by division})$$

$$\text{Let } \frac{x+1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1} \Rightarrow x+1 = A(x-1) + Bx.$$

On putting  $x = 0$  and  $x = 1$ , we get

$$1 = -A \text{ and } 2 = B \Rightarrow A = -1 \text{ and } B = 2.$$

$$\begin{aligned} \therefore \int_2^3 \frac{x^3 + 1}{x(x-1)} dx &= \int_2^3 \left( x + 1 - \frac{1}{x} + \frac{2}{x-1} \right) dx \\ &= \left[ \frac{x^2}{2} + x - \log|x| + 2\log|x-1| \right]_2^3 \\ &= \left( \frac{9}{2} + 3 - \log 3 + 2\log 2 \right) - (2 + 2 - \log 2 - 2\log 1) \\ &= \frac{15}{2} - \log 3 + 2\log 2 - 4 + \log 2 - 2.0 = \frac{7}{2} + 3\log 2 - \log 3. \end{aligned}$$

$$\text{Example 16. Prove that : } \int_0^{\pi/2} \frac{3\sin\theta + 4\cos\theta}{\sin\theta + \cos\theta} d\theta = \frac{7\pi}{4}. \quad (\text{I.S.C. 2005})$$

**Solution.** Let  $3\sin\theta + 4\cos\theta = l(\sin\theta + \cos\theta) + m(\cos\theta - \sin\theta)$ .

Equating coefficients of  $\sin\theta$  and  $\cos\theta$  on both sides, we get

$$3 = l - m \text{ and } 4 = l + m.$$

Solving these for  $l$  and  $m$ , we get  $l = \frac{7}{2}$  and  $m = \frac{1}{2}$ .

$$\begin{aligned} \therefore \int_0^{\pi/2} \frac{3\sin\theta + 4\cos\theta}{\sin\theta + \cos\theta} d\theta &= \int_0^{\pi/2} \frac{\frac{7}{2}(\sin\theta + \cos\theta) + \frac{1}{2}(\cos\theta - \sin\theta)}{\sin\theta + \cos\theta} d\theta \\ &= \int_0^{\pi/2} \left( \frac{7}{2} + \frac{1}{2} \cdot \frac{\cos\theta - \sin\theta}{\sin\theta + \cos\theta} \right) d\theta \\ &= \frac{7}{2} [\theta]_0^{\pi/2} + \frac{1}{2} [\log|\sin\theta + \cos\theta|]_0^{\pi/2} \\ &= \frac{7}{2} \left[ \frac{\pi}{2} - 0 \right] + \frac{1}{2} [\log 1 - \log 1] = \frac{7\pi}{4}. \end{aligned}$$

## EXERCISE 10.1

Evaluate the following (1 to 21) definite integrals :

$$1. (i) \int_0^8 \left( \sqrt{8x} - \frac{x^2}{8} \right) dx \quad (ii) \int_0^1 \frac{1}{2x-3} dx.$$

$$2. (i) \int_{-4}^{-1} \frac{1}{x} dx \quad (ii) \int_{\pi/6}^{\pi/4} \operatorname{cosec} x dx.$$

$$3. (i) \int_0^1 \sqrt{5x+4} dx \quad (ii) \int_0^1 \frac{dx}{\sqrt{1+x+\sqrt{x}}}.$$

$$4. (i) \int_0^{\pi} \frac{dx}{1+\sin x} \quad (ii) \int_0^1 \frac{dx}{\sqrt{1-x^2}}.$$

5. (i)  $\int_0^1 \frac{1-x}{1+x} dx$  (ii)  $\int_0^{\pi/2} \frac{\cos x}{5+4\sin x} dx.$

6. (i)  $\int_1^3 (x^2 + e^x)(x^3 + 3e^x + 4) dx$  (ii)  $\int_1^2 \frac{3x}{9x^2 - 1} dx.$

7. (i)  $\int_0^{\pi/4} \tan^3 x \sec^2 x dx$  (ii)  $\int_0^{\pi/2} \sin^4 x dx.$

8. (i)  $\int_0^{\pi/2} \sqrt{1 - \cos 2x} dx$  (ii)  $\int_0^{\pi/4} \sqrt{1 + \sin 2x} dx.$

9. (i)  $\int_0^{\pi/2} \frac{\sin^2 x}{(1 + \cos x)^2} dx$  (ii)  $\int_0^{\pi/2} \frac{\sin \theta}{\sqrt{1 + \cos \theta}} d\theta.$

10. (i)  $\int_0^{\pi/4} \sin 2x \sin 3x dx$  (ii)  $\int_0^{\pi/2} (a^2 \cos^2 x + b^2 \sin^2 x) dx.$

11. (i)  $\int_0^{\pi/4} 2 \tan^3 x dx$  (ii)  $\int_0^{\pi/4} \frac{\tan^3 x}{1 + \cos 2x} dx.$

12. (i)  $\int_0^{\pi/4} \frac{\sin x}{\cos 3x + 3 \cos x} dx$  (ii)  $\int_{\pi/3}^{\pi/4} (\tan x + \cot x)^2 dx$

(iii)  $\int_0^{\pi/4} (\tan x + \cot x)^{-1} dx.$  (I.S.C. 2003)

13. (i)  $\int_0^4 \frac{dx}{\sqrt{x^2 + 2x + 3}}$  (ii)  $\int_0^a \frac{dx}{\sqrt{ax - x^2}}.$

14. (i)  $\int_0^1 \frac{\tan^{-1} x}{1+x^2} dx$  (ii)  $\int_0^1 \frac{x^5}{1+x^6} dx.$

15. (i)  $\int_0^1 x e^x dx$  (ii)  $\int_1^2 \frac{x+3}{x(x+2)} dx.$

16. (i)  $\int_0^{\pi/2} x^2 \cos 2x dx$  (ii)  $\int_{-a}^a \sqrt{\frac{a-x}{a+x}} dx.$

17. (i)  $\int_1^3 \frac{dx}{x^2(x+1)}$  (ii)  $\int_1^2 \frac{dx}{(x+1)(x^2 - 7x + 12)}.$

18. (i)  $\int_0^2 \frac{1}{\sqrt{3+2x-x^2}} dx$  (ii)  $\int_0^2 \frac{dx}{4+x-x^2}.$

19. (i)  $\int_0^1 \sin^{-1} x dx$  (ii)  $\int_0^1 \tan^{-1} x dx.$

20. (i)  $\int_1^2 e^x \left( \frac{1}{x} - \frac{1}{x^2} \right) dx$  (ii)  $\int_0^{\pi/2} \frac{x + \sin x}{1 + \cos x} dx.$

$$21. \text{ (i)} \int_{\pi/4}^{\pi/2} e^x (\log(\sin x) + \cot x) dx \quad \text{(ii)} \int_{\pi/2}^{\pi} e^x \left( \frac{1 - \sin x}{1 - \cos x} \right) dx.$$

$$22. \text{ If } \int_0^a 3x^2 dx = 8, \text{ find the value of } a.$$

### 10.1.2 Evaluation of Definite Integrals by Substitution

#### ILLUSTRATIVE EXAMPLES

**Example 1.** Evaluate the following :

$$\text{(i)} \int_0^1 x^2 e^{x^3} dx \quad \text{(ii)} \int_0^1 \frac{x}{1+x^4} dx.$$

**Solution.** (i) Put  $x^3 = t \Rightarrow 3x^2 dx = dt \Rightarrow x^2 dx = \frac{1}{3} dt$ .

When  $x = 0, t = 0$  and when  $x = 1, t = 1^3 = 1$ .

$$\begin{aligned} \therefore \int_0^1 x^2 e^{x^3} dx &= \int_0^1 e^t \cdot \frac{1}{3} dt = \frac{1}{3} [e^t]_0^1 \\ &= \frac{1}{3} [e^1 - e^0] = \frac{1}{3} (e - 1). \end{aligned}$$

(ii) Put  $x^2 = t \Rightarrow 2x dx = dt \Rightarrow x dx = \frac{1}{2} dt$ .

When  $x = 0, t = 0$  and when  $x = 1, t = 1$ .

$$\begin{aligned} \therefore \int_0^1 \frac{x}{1+x^4} dx &= \int_0^1 \frac{1}{1+t^2} \cdot \frac{1}{2} dt = \frac{1}{2} \cdot \frac{1}{2} \left[ \tan^{-1} \frac{t}{1} \right]_0^1 \\ &= \frac{1}{2} [\tan^{-1} 1 - \tan^{-1} 0] = \frac{1}{2} \left[ \frac{\pi}{4} - 0 \right] = \frac{\pi}{8}. \end{aligned}$$

**Example 2.** Evaluate the following :

$$\text{(i)} \int_0^1 \frac{e^x}{1+e^{2x}} dx \quad \text{(ii)} \int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x} dx \quad \text{(iii)} \int_0^{\pi/4} \frac{\sin^2 x \cos^2 x}{(\sin^3 x + \cos^3 x)^2} dx.$$

**Solution.** (i) Put  $e^x = t \Rightarrow e^x dx = dt$ .

When  $x = 0, t = e^0 = 1$  and when  $x = 1, t = e^1 = e$ .

$$\begin{aligned} \therefore \int_0^1 \frac{e^x}{1+e^{2x}} dx &= \int_1^e \frac{dt}{1+t^2} = \frac{1}{2} \left[ \tan^{-1} \frac{t}{1} \right]_1^e \\ &= \tan^{-1} e - \tan^{-1} 1 = \tan^{-1} e - \frac{\pi}{4}. \end{aligned}$$

(ii) Put  $\cos x = t \Rightarrow -\sin x dx = dt \Rightarrow \sin x dx = -dt$ .

When  $x = 0, t = \cos 0 = 1$  and when  $x = \frac{\pi}{2}, t = \cos \frac{\pi}{2} = 0$ .

$$\begin{aligned} \therefore \int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x} dx &= \int_1^0 \frac{-dt}{1+t^2} = -\frac{1}{2} \left[ \tan^{-1} \frac{t}{1} \right]_1^0 \\ &= -[\tan^{-1} 0 - \tan^{-1} 1] = -\left(0 - \frac{\pi}{4}\right) = \frac{\pi}{4}. \end{aligned}$$

## EXERCISE 10.2

1. (i)  $\frac{1}{2}(e - 1)$       (ii)  $\frac{1}{2}\tan^{-1}e^2 - \frac{\pi}{2}$ .      2. (i)  $2(\sqrt{2} - 1)$       (ii)  $\frac{8}{21}$ .

3. (i)  $\tan^{-1}\left(\frac{1}{3}\right)$       (ii)  $\tan^{-1}\left(\frac{1}{3}\right)$ .      4. (i)  $\sin(\log 3)$       (ii)  $\frac{\log 2}{1 + \log 2}$ .

5. (i)  $\frac{16}{15}(2 + \sqrt{2})$       (ii)  $\frac{132}{7}\sqrt[3]{4}$ .      6. (i)  $1 - \log 2$       (ii)  $\frac{\pi}{2} - 1$ .

7. (i)  $\log \frac{4}{3}$       (ii)  $\log \frac{4}{3}$ .      8. (i)  $\frac{\pi}{8}$       (ii)  $\log \frac{2e - 1}{e}$ .

9. (i)  $\frac{3}{16}\pi a^4$       (ii)  $\frac{\pi}{2ab}$ .      10.  $\frac{\pi}{2} - \log 2$       (ii)  $\frac{\pi}{2} - \log 2$ .

11. (i)  $\frac{\pi^2}{16} - \frac{\pi}{4} + \frac{1}{2}\log 2$       (ii)  $e - \frac{2}{\log 2}$ .

12. (i)  $\frac{2}{3}\tan^{-1}\frac{1}{3}$       (ii)  $\frac{1}{\sqrt{5}}\log\left(\frac{\sqrt{5} + 1}{\sqrt{5} - 1}\right)$ .

13. (i)  $\frac{\pi}{3}$       (ii)  $\frac{\pi}{\sqrt{35}}$ .      14. (i)  $\frac{1}{2}\log 3$       (ii)  $\frac{1}{24}$ .

15. (i)  $\frac{\pi}{4}$       (ii)  $\frac{\pi}{2}$ .

### EXERCISE 10.3

1. (i)  $\frac{13}{2}$       (ii) 34      (iii)  $\frac{25}{2}$ .

2. (i)  $2 - \sqrt{2}$       (ii) 2      (iii) 1.

3. (i) 4      (ii)  $2(e - 1)$       (iii)  $2 - \frac{2}{e}$ .

4. (i) 0      (ii) 4.      5. 37.      6. 47.

7. (i) 0      (ii) 0      (iii) 0.

8. (i)  $\frac{\pi}{2}$       (ii)  $\frac{3\pi}{8}$       (iii) 0      (iv) 0.

9. (i) 0      (ii) 0      (iii) 0      (iv) 0.

10. (i)  $\frac{1}{42}$       (ii)  $\frac{4}{63}$       (iii)  $\frac{144}{35}\sqrt{3}$       (iv)  $\frac{16}{315}a^{9/2}$ .

11. (i)  $\frac{\pi}{4}$       (ii)  $\frac{\pi}{4}$       (iii)  $\frac{\pi}{4}$       (iv)  $\frac{\pi}{4}$ .

12. (i)  $\frac{\pi}{4}$       (ii)  $\frac{\pi}{4}$       (iii)  $\frac{\pi}{4}$       (iv)  $\frac{\pi}{4}$ .

13. (i)  $\frac{\pi^2}{4}$       (ii)  $\frac{2\pi}{3}$ .      14. (i)  $\frac{\pi}{5}$       (ii)  $\frac{\pi}{2\sqrt{2}} \log(\sqrt{2} + 1)$ .