20

TRIGONOMETRICAL RATIOS OF STANDARD ANGLES

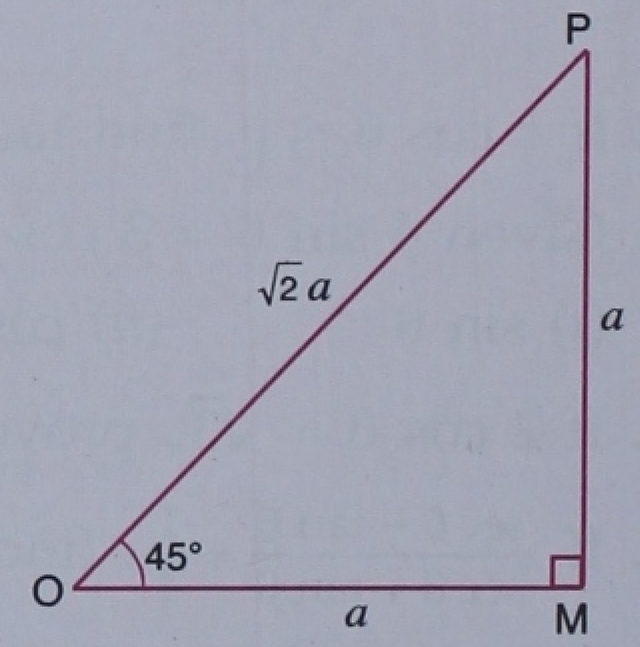
20.1 T-RATIOS OF 45°

Let OMP be an isosceles triangle right-angled at M with sides OM = MP = a (say), then $\angle MOP = 45°$.

$(\because \angle MOP = \angle MPO, \text{ and } \angle MOP + \angle MPO = 180° - \angle M = 180° - 90° = 90°)$

From right-angled ΔOMP, by Pythagoras theorem, we get

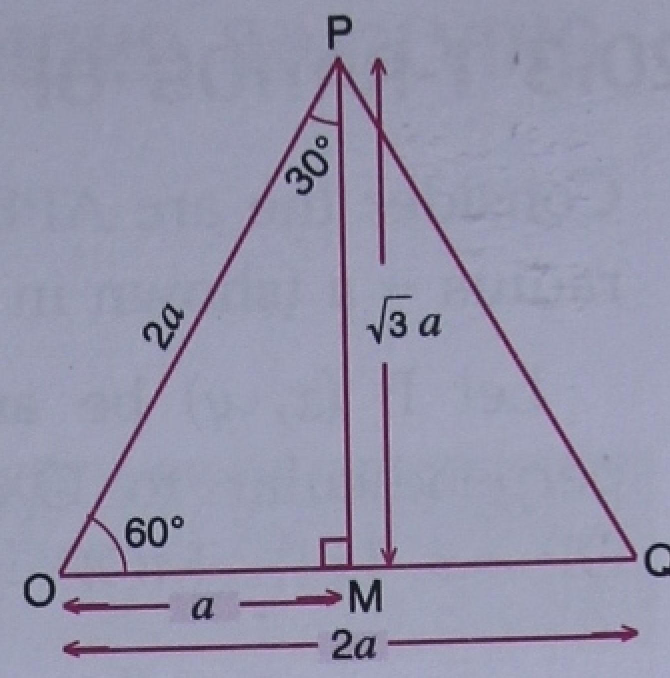
$$OP^2 = OM^2 + MP^2$$
$$= a^2 + a^2 = 2a^2$$
$$\Rightarrow \quad OP = \sqrt{2}\,a.$$



$$\therefore \quad \sin 45° = \frac{MP}{OP} = \frac{a}{\sqrt{2}a} = \frac{1}{\sqrt{2}},$$
$$\cos 45° = \frac{OM}{OP} = \frac{a}{\sqrt{2}a} = \frac{1}{\sqrt{2}},$$
$$\tan 45° = \frac{MP}{OM} = \frac{a}{a} = 1,$$
$$\cot 45° = \frac{OM}{MP} = \frac{a}{a} = 1,$$
$$\sec 45° = \frac{OP}{OM} = \frac{\sqrt{2}a}{a} = \sqrt{2} \text{ and}$$
$$\text{cosec } 45° = \frac{OP}{MP} = \frac{\sqrt{2}a}{a} = \sqrt{2}.$$

20.2 T-RATIOS OF 30° AND 60°

Let OQP be an equilateral triangle with each side = $2a$ (say), and let MP be perpendicular to OQ, then M is mid-point of OQ,



$\therefore \quad OM = \frac{1}{2}\, OQ = \frac{1}{2} \times 2a = a.$

Since each angle of an equilateral triangle is 60°,

$\angle MOP = 60°$ and

$\angle OPM = 180° - (60° + 90°) = 30°.$

From right-angled ΔOMP, by Pythagoras theorem, we get

$$OP^2 = OM^2 + MP^2$$

$$\Rightarrow \quad MP^2 = OP^2 - OM^2 = (2a)^2 - a^2 = 4a^2 - a^2 = 3a^2$$

$$\Rightarrow \quad MP = \sqrt{3}\,a.$$

(i) T-ratios of 30°.

In ΔOMP, $\angle OMP = 90°$, $\angle MPO = 30°$, height = OM = a, base = MP = $\sqrt{3}\,a$ and hypotenuse = OP = $2a$.

$$\therefore \quad \sin 30° = \frac{\text{height}}{\text{hypotenuse}} = \frac{OM}{OP} = \frac{a}{2a} = \frac{1}{2},$$

$$\cos 30° = \frac{\text{base}}{\text{hypotenuse}} = \frac{MP}{OP} = \frac{\sqrt{3}\,a}{2a} = \frac{\sqrt{3}}{2},$$

$$\tan 30° = \frac{\text{height}}{\text{base}} = \frac{OM}{MP} = \frac{a}{\sqrt{3}\,a} = \frac{1}{\sqrt{3}},$$

$$\cot 30° = \frac{\text{base}}{\text{height}} = \frac{MP}{OM} = \frac{\sqrt{3}\,a}{a} = \sqrt{3},$$

$$\sec 30° = \frac{\text{hypotenuse}}{\text{base}} = \frac{OP}{MP} = \frac{2a}{\sqrt{3}\,a} = \frac{2}{\sqrt{3}} \text{ and}$$

$$\text{cosec } 30° = \frac{\text{hypotenuse}}{\text{height}} = \frac{OP}{OM} = \frac{2a}{a} = 2.$$

(ii) T-ratios of 60°.

In ΔOMP, $\angle OMP = 90°$, $\angle MOP = 60°$, height = MP = $\sqrt{3}a$, base = OM = a and hypotenuse = OP = $2a$.

$$\therefore \quad \sin 60° = \frac{\text{height}}{\text{hypotenuse}} = \frac{MP}{OP} = \frac{\sqrt{3}\,a}{2a} = \frac{\sqrt{3}}{2},$$

$$\cos 60° = \frac{\text{base}}{\text{hypotenuse}} = \frac{OM}{OP} = \frac{a}{2a} = \frac{1}{2},$$

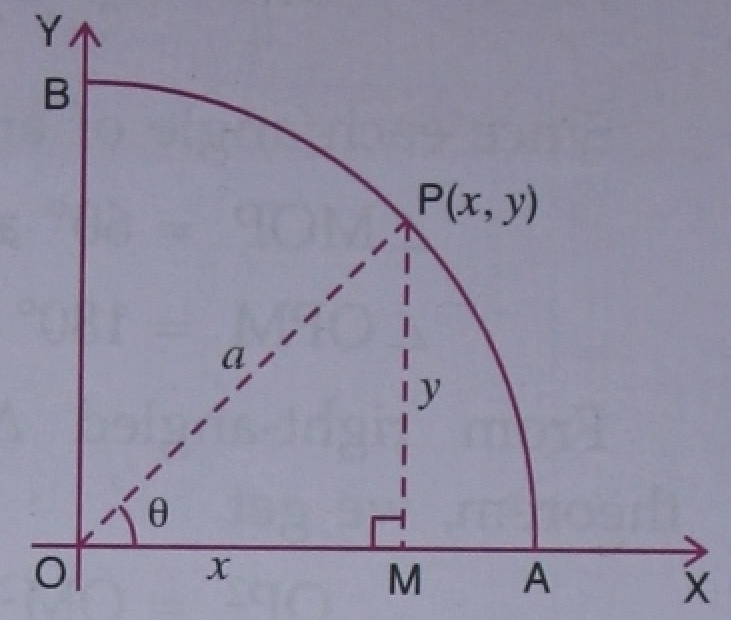
$$\tan 60° = \frac{\text{height}}{\text{base}} = \frac{MP}{OM} = \frac{\sqrt{3}\,a}{a} = \sqrt{3},$$

$$\cot 60° = \frac{\text{base}}{\text{height}} = \frac{OM}{MP} = \frac{a}{\sqrt{3}\,a} = \frac{1}{\sqrt{3}},$$

$$\sec 60° = \frac{\text{hypotenuse}}{\text{base}} = \frac{OP}{OM} = \frac{2a}{a} = 2 \text{ and}$$

$$\text{cosec } 60° = \frac{\text{hypotenuse}}{\text{height}} = \frac{OP}{MP} = \frac{2a}{\sqrt{3}\,a} = \frac{2}{\sqrt{3}}.$$

20.3 T-RATIOS OF 0° AND 90°



Consider the arc APB of a circle with centre at O and radius = a (shown in the adjoining figure).

Let P (x, y) be any point on this arc. Draw MP perpendicular to OX, then OM = x, MP = y and OP = a. Let ∠MOP = θ, then

$$\sin\theta = \frac{y}{a},\ \cos\theta = \frac{x}{a},\ \tan\theta = \frac{y}{x},$$

$$\cot\theta = \frac{x}{y},\ \sec\theta = \frac{a}{x},\ \text{cosec}\ \theta = \frac{a}{y}.$$

(*i*) *T-ratios of 0°.*

When θ = 0°, P coincides with A so that $x = a$ and $y = 0$.

∴ $\sin 0° = \frac{0}{a} = 0$, $\cos 0° = \frac{a}{a} = 1$,

$\tan 0° = \frac{0}{a} = 0$, $\cot 0° = \frac{a}{0}$, which is not defined,

$\sec 0° = \frac{a}{a} = 1$, $\text{cosec}\ 0° = \frac{a}{0}$, which is not defined.

(*ii*) *T-ratios of 90°.*

When θ = 90°, P coincides with B so that $x = 0$ and $y = a$.

∴ $\sin 90° = \frac{a}{a} = 1$, $\cos 90° = \frac{0}{a} = 0$,

$\tan 90° = \frac{a}{0}$, which is not defined, $\cot 90° = \frac{0}{a} = 0$,

$\sec 90° = \frac{a}{0}$, which is not defined, $\text{cosec}\ 90° = \frac{a}{a} = 1$.

20.4 AID TO MEMORY

The trigonometrical ratios of the above standard angles can be easily remembered with the help of the following table :

Angle → *Ratio ↓*	*0°*	*30°*	*45°*	*60°*	*90°*
sin	$\sqrt{\frac{0}{4}}$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{2}{4}}$	$\sqrt{\frac{3}{4}}$	$\sqrt{\frac{4}{4}}$
cos	$\sqrt{\frac{4}{4}}$	$\sqrt{\frac{3}{4}}$	$\sqrt{\frac{2}{4}}$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{0}{4}}$
tan	$\sqrt{\frac{0}{4-0}}$	$\sqrt{\frac{1}{4-1}}$	$\sqrt{\frac{2}{4-2}}$	$\sqrt{\frac{3}{4-3}}$	not defined

Notes

1. The values of cot θ, sec θ and cosec θ have not been written in the above table, for, these are the reciprocals of the values of tan θ, cos θ and sin θ respectively.
2. *As θ increases from 0° to 90°, the values of sin θ and tan θ go on increasing while the values of cos θ go on decreasing.*

20.5 EVALUATION OF TRIGONOMETRICAL EXPRESSIONS INVOLVING T-RATIOS OF THE STANDARD ANGLES

ILLUSTRATIVE EXAMPLES

Example 1. *Find the value of*
(cos 0° + sin 45° + sin 30°) (sin 90° – cos 45° + cos 60°).

Solution. (cos 0° + sin 45° + sin 30°) (sin 90° – cos 45° + cos 60°)

$$= \left(1+\frac{1}{\sqrt{2}}+\frac{1}{2}\right)\left(1-\frac{1}{\sqrt{2}}+\frac{1}{2}\right) = \left(\frac{3}{2}+\frac{1}{\sqrt{2}}\right)\left(\frac{3}{2}-\frac{1}{\sqrt{2}}\right)$$

$$= \left(\frac{3}{2}\right)^2 - \left(\frac{1}{\sqrt{2}}\right)^2 \qquad [\because (a+b)(a-b) = a^2 - b^2]$$

$$= \frac{9}{4} - \frac{1}{2} = \frac{9-2}{4} = \frac{7}{4} = 1\frac{3}{4}.$$

Example 2. *Find the value of*
$\sin^2 30° \cos^2 45° + 4 \tan^2 30° + \frac{1}{2} \sin^2 90° - 2 \cos 90° + \frac{1}{24}$.

Solution. $\sin^2 30° \cos^2 45° + 4 \tan^2 30° + \frac{1}{2} \sin^2 90° - 2 \cos 90° + \frac{1}{24}$

$$= \left(\frac{1}{2}\right)^2 \times \left(\frac{1}{\sqrt{2}}\right)^2 + 4 \times \left(\frac{1}{\sqrt{3}}\right)^2 + \frac{1}{2} \times (1)^2 - 2 \times 0 + \frac{1}{24}$$

$$= \frac{1}{4} \times \frac{1}{2} + 4 \times \frac{1}{3} + \frac{1}{2} \times 1 - 0 + \frac{1}{24} = \frac{1}{8} + \frac{4}{3} + \frac{1}{2} + \frac{1}{24}$$

$$= \frac{3+32+12+1}{24} = \frac{48}{24} = 2.$$

Example 3. *If* $4 \sin^2 \theta - 1 = 0$ *and θ is acute angle, find the value of θ and hence find the values of (i)* $\cos^2 \theta + \tan^2 \theta$ *(ii)* $\cos 2\theta$ *(iii)* $\sin 3\theta$.

Solution. Given $4 \sin^2 \theta - 1 = 0$

$\Rightarrow \quad 4 \sin^2 \theta = 1 \quad \Rightarrow \quad \sin^2 \theta = \frac{1}{4}$

$\Rightarrow \quad \sin \theta = \frac{1}{2}$ (∵ θ is acute angle, sin θ is positive)

$\Rightarrow \quad \sin \theta = \sin 30°$ $\left(\because \sin 30° = \frac{1}{2}\right)$

$\Rightarrow \quad \theta = 30°.$

(*i*) $\cos^2 \theta + \tan^2 \theta = \cos^2 30° + \tan^2 30°$

$$= \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 = \frac{3}{4} + \frac{1}{3} = \frac{9+4}{12} = \frac{13}{12} = 1\frac{1}{12}.$$

(*ii*) $2\theta = (2 \times 30)° = 60°.$

$\therefore \quad \cos 2\theta = \cos 60° = \frac{1}{2}.$

(*iii*) $3\theta = (3 \times 30)° = 90°.$

$\therefore \quad \sin 3\theta = \sin 90° = 1.$

Example 4. *If θ is an acute angle and sin θ = cos θ, find the value of*

$2\sin^2\theta - 3\cos^2\theta + \frac{1}{2}\cot^2\theta.$

Solution. Given, sin θ = cos θ

$\Rightarrow \quad \frac{\sin\theta}{\cos\theta} = 1 \Rightarrow \tan\theta = 1$

$\Rightarrow \quad \tan\theta = \tan 45°$ $\qquad (\because \tan 45° = 1)$

$\Rightarrow \quad \theta = 45°.$

$\therefore \quad 2\sin^2\theta - 3\cos^2\theta + \frac{1}{2}\cot^2\theta$

$$= 2\sin^2 45° - 3\cos^2 45 + \frac{1}{2}\cot^2 45°$$

$$= 2\left(\frac{1}{\sqrt{2}}\right)^2 - 3\left(\frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}(1)^2$$

$$= 2\times\frac{1}{2} - 3\times\frac{1}{2} + \frac{1}{2}\times 1 = 1 - \frac{3}{2} + \frac{1}{2} = 0.$$

Example 5. *If θ is an acute angle and* $\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} = \frac{1+\sqrt{3}}{1-\sqrt{3}}$, *find the value of*

$2\sec^2\theta - 3\,\mathrm{cosec}^2\theta.$

Solution. Given, $\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} = \frac{1+\sqrt{3}}{1-\sqrt{3}}.$

Applying componendo and dividendo, we get

$$\frac{(\cos\theta + \sin\theta) + (\cos\theta - \sin\theta)}{(\cos\theta + \sin\theta) - (\cos\theta - \sin\theta)} = \frac{(1+\sqrt{3}) + (1-\sqrt{3})}{(1+\sqrt{3}) - (1-\sqrt{3})}$$

$\Rightarrow \quad \frac{2\cos\theta}{2\sin\theta} = \frac{2}{2\sqrt{3}} \Rightarrow \frac{\cos\theta}{\sin\theta} = \frac{1}{\sqrt{3}}$

$\Rightarrow \quad \frac{\sin\theta}{\cos\theta} = \sqrt{3} \Rightarrow \tan\theta = \sqrt{3}$

$\Rightarrow \quad \tan\theta = \tan 60°$ $\qquad (\because \tan 60° = \sqrt{3})$

$\Rightarrow \quad \theta = 60°.$

$\therefore \quad 2\sec^2\theta - 3\,\mathrm{cosec}^2\theta = 2\sec^2 60° - 3\,\mathrm{cosec}^2 60°$

$$= 2(2)^2 - 3\left(\frac{2}{\sqrt{3}}\right)^2 = 2\times 4 - 3\times\frac{4}{3}$$

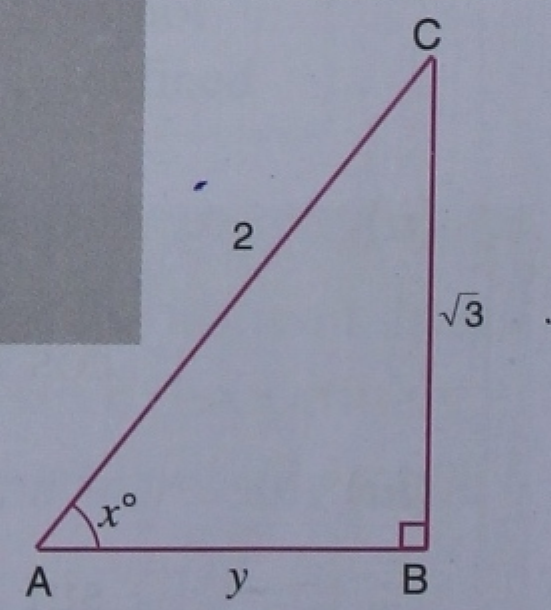
$$= 8 - 4 = 4.$$

Example 6. *In the figure given along side, Δ ABC is right-angled at B. Given that AB = y units, BC = $\sqrt{3}$ units, AC = 2 units and ∠A = x°, find*

(i) sin x° *(ii) x*

(iii) tan x°

(iv) use cos x° to find the value of y.



Solution. (*i*) From right-angled Δ ABC,

$$\sin x° = \frac{\text{height}}{\text{hypotenuse}} = \frac{\sqrt{3}}{2}.$$

(*ii*) From (*i*), $\sin x° = \frac{\sqrt{3}}{2}$

$\Rightarrow \sin x° = \sin 60°$ $\left(\because \sin 60° = \frac{\sqrt{3}}{2}\right)$

$\Rightarrow x° = 60° \quad \Rightarrow \quad x = 60.$

(*iii*) From (*ii*), $x = 60$,

$\therefore \tan x° = \tan 60° = \sqrt{3}$.

(*iv*) From right-angled Δ ABC, $\cos x° = \frac{\text{base}}{\text{hypotenuse}} = \frac{y}{2}$

$\cos 60° = \frac{y}{2}$ $[\because x = 60 \text{ from } (ii)]$

$\Rightarrow \frac{1}{2} = \frac{y}{2}$ $\left(\because \cos 60° = \frac{1}{2}\right)$

$\Rightarrow y = 1.$

Example 7. *If $0° < \theta < 90°$, solve the equation $(\text{cosec } \theta - 2)(2 \cos 3\theta - 1) = 0$.*

Solution. Given $(\text{cosec } \theta - 2)(2 \cos 3\theta - 1) = 0$

$\Rightarrow \text{cosec } \theta - 2 = 0$ or $2 \cos 3\theta - 1 = 0$

$\Rightarrow \text{cosec } \theta = 2$ or $2 \cos 3\theta = 1$

$\Rightarrow \frac{1}{\sin \theta} = 2$ or $\cos 3\theta = \frac{1}{2}$

$\Rightarrow \sin \theta = \frac{1}{2}$ or $\cos 3\theta = \cos 60°$

$\Rightarrow \sin \theta = \sin 30°$ or $3\theta = 60°$

$\Rightarrow \theta = 30°$ or $\theta = 20°$.

Example 8. *If $A = 60°$ and $B = 30°$, verify that*
$\cos (A - B) = \cos A \cos B + \sin A \sin B$.

Solution. L.H.S. $= \cos (A - B)$

$= \cos (60° - 30°) = \cos 30° = \frac{\sqrt{3}}{2}$ and

R.H.S. $= \cos A \cos B + \sin A \sin B$

$= \cos 60° \cos 30° + \sin 60° \sin 30°$

$= \frac{1}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{1}{2}$

$= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} = \frac{\sqrt{3} + \sqrt{3}}{4} = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$.

$\therefore$ L.H.S. = R.H.S. Hence the result.

Example 9. *If $\cos (A + B) = \frac{1}{2} = \sin (A - B)$ and A, B $(A > B)$ are acute angles, find the values of A and B.*

Solution. Given $\cos (A + B) = \frac{1}{2}$

$\Rightarrow \cos (A + B) = \cos 60°$ $\left(\because \cos 60° = \frac{1}{2}\right)$

$\Rightarrow \quad A + B = 60°$ ...(*i*)

Also $\quad \sin(A - B) = \frac{1}{2}$ (given)

$\Rightarrow \quad \sin(A - B) = \sin 30°$ $\left(\because \sin 30° = \frac{1}{2}\right)$

$\Rightarrow \quad A - B = 30°$ ...(*ii*)

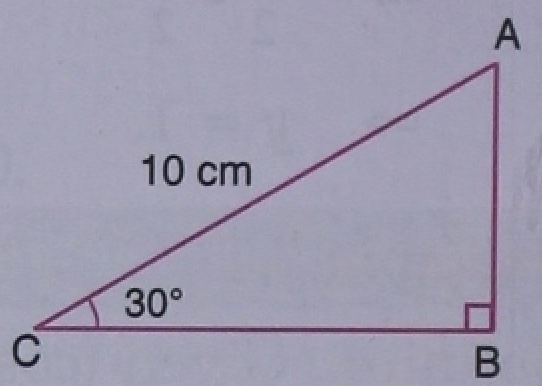
On adding (*i*) and (*ii*), we get $2A = 90° \Rightarrow A = 45°$.

On subtracting (*ii*) from (*i*), we get $2B = 30° \Rightarrow B = 15°$.

Hence A= 45° and B = 15°.

Example 10. *In the right-angled triangle ABC, ∠ B = 90° and ∠ C = 30°. If AC = 10 cm, find the lengths of the sides AB and BC.*

Solution. In the right-angled Δ ABC,



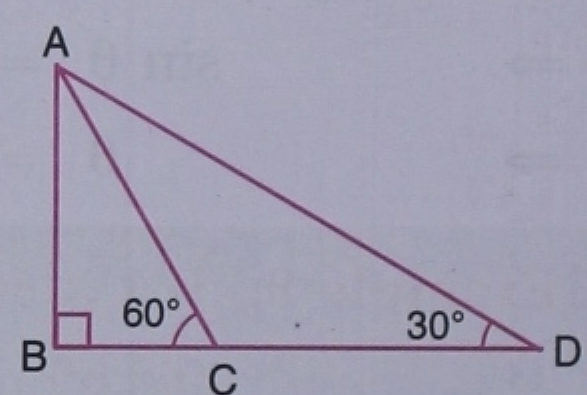
$$\sin 30° = \frac{AB}{AC} \Rightarrow \frac{1}{2} = \frac{AB}{10 \text{ cm}}$$

$$\Rightarrow \quad AB = \left(\frac{1}{2} \times 10\right) \text{ cm} = 5 \text{ cm}.$$

$$\cos 30° = \frac{BC}{AC} \Rightarrow \frac{\sqrt{3}}{2} = \frac{BC}{10 \text{ cm}}$$

$$\Rightarrow \quad BC = \left(\frac{\sqrt{3}}{2} \times 10\right) \text{ cm} = 5\sqrt{3} \text{ cm}.$$

Example 11. *In the adjoining figure, ABC is a right-angled triangle in which ∠ABC = 90° and ∠ACB = 60°. BC is produced to D such that ∠ADB = 30°. If CD = 4 cm, find the lengths of AB and BC.*



Solution. In the right-angled Δ ABC,

$\tan 60° = \frac{AB}{BC} \Rightarrow \sqrt{3} = \frac{AB}{BC} \Rightarrow AB = \sqrt{3}\,BC$ ...(*i*)

In the right-angled Δ ABD,

$\tan 30° = \frac{AB}{BD} \Rightarrow \frac{1}{\sqrt{3}} = \frac{AB}{BD} \Rightarrow BD = \sqrt{3}\,AB$

$\Rightarrow \quad BD = \sqrt{3}\,(\sqrt{3}\,BC)$ [using (*i*)]

$\Rightarrow \quad BD = 3\,BC$

$\Rightarrow \quad BC + CD = 3\,BC$ (from figure)

$\Rightarrow \quad CD = 3\,BC - BC$

$\Rightarrow \quad 4 \text{ cm} = 2\,BC$ (CD = 4 cm, given)

$\Rightarrow \quad BC = 2$ cm.

From (*i*), $AB = \sqrt{3}\,BC$

$\Rightarrow \quad AB = (\sqrt{3} \times 2) \text{ cm} = 2\sqrt{3}$ cm.

Hence, $AB = 2\sqrt{3}$ cm and BC = 2 cm.

Exercise 20.1

1. Find the values of

   (*i*) $7 \sin 30° \cos 60°$

   (*ii*) $3 \sin^2 45° + 2 \cos^2 60°$

   (*iii*) $\cos^2 45° + \sin^2 60° + \sin^2 30°$

   (*iv*) $\cos 90° + \cos^2 45° \sin 30° \tan 45°$.

2. Find the values of

   (*i*) $\dfrac{\sin^2 45° + \cos^2 45°}{\tan^2 60°}$

   (*ii*) $\dfrac{\sin 30° - \sin 90° + 2 \cos 0°}{\tan 30° \times \tan 60°}$

   (*iii*) $\frac{4}{3} \tan^2 30° + \sin^2 60° - 3 \cos^2 60° + \frac{3}{4} \tan^2 60° - 2 \tan^2 45°$.

3. Find the values of

   (*i*) $\dfrac{\sin 60°}{\cos^2 45°} - 3 \tan 30° + 5 \cos 90°$

   (*ii*) $2\sqrt{2} \cos 45° \cos 60° + 2\sqrt{3} \sin 30° \tan 60° - \cos 0°$

   (*iii*) $\frac{4}{5} \tan^2 60° - \dfrac{2}{\sin^2 30°} - \frac{3}{4} \tan^2 30°$.

4. Prove that

   (*i*) $\cos^2 30° + \sin 30° + \tan^2 45° = 2\frac{1}{4}$

   (*ii*) $4 (\sin^4 30° + \cos^4 60°) - 3 (\cos^2 45° - \sin^2 90°) = 2$

   (*iii*) $\cos 60° = \cos^2 30° - \sin^2 30°$.

5. (*i*) If $x = 30°$, verify that $\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}$.

   (*ii*) If $x = 15°$, verify that $4 \sin 2x \cos 4x \sin 6x = 1$.

6. Find the values of

   (*i*) $\sqrt{\dfrac{1 - \cos^2 30°}{1 - \sin^2 30°}}$

   (*ii*) $\dfrac{\sin 45° \cos 45° \cos 60°}{\sin 60° \cos 30° \tan 45°}$.

7. If $\theta = 30°$, verify that

   (*i*) $\sin 2\theta = 2 \sin \theta \cos \theta$

   (*ii*) $\cos 2\theta = 2 \cos^2 \theta - 1$

   (*iii*) $\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$

   (*iv*) $\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$.

8. If $\theta = 30°$, find the ratio $2 \sin \theta : \sin 2\theta$.

9. By means of an example, show that $\sin (A + B) \neq \sin A + \sin B$.

Hint

Take $A = 30°$ and $B = 60°$.

10. If $A = 60°$ and $B = 30°$, verify that

    (*i*) $\sin (A + B) = \sin A \cos B + \cos A \sin B$

    (*ii*) $\cos (A + B) = \cos A \cos B - \sin A \sin B$

(*iii*) $\sin (A - B) = \sin A \cos B - \cos A \sin B$

(*iv*) $\tan (A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$.

11. If $3 \sin^2 \theta = 2\frac{1}{4}$ and θ is less than 90°, find the value of θ.

12. If θ is an acute angle and $\sin \theta = \cos \theta$, find the vlaue of θ and hence find the value of $2 \tan^2 \theta + \sin^2 \theta - 1$.

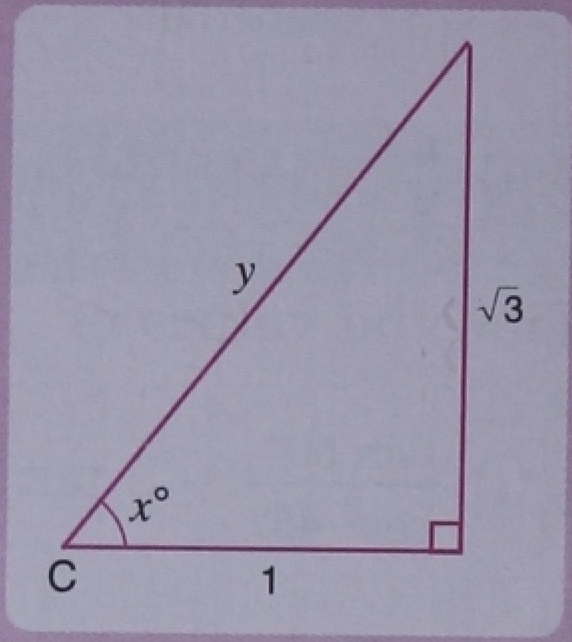
13. From the adjoining figure, find

(*i*) $\tan x°$

(*ii*) x

(*iii*) $\cos x°$

(*iv*) use $\sin x°$ to find y.



14. If θ is an acute angle, solve the following equations for θ :

(*i*) $2 \sin \theta = 1$ (*ii*) $2 \sin 3\theta = \sqrt{3}$

(*iii*) $\tan 3\theta = 1$ (*iv*) $\sqrt{3} \cot 2\theta = 1$.

15. If $\tan 3x = \sin 45° \cos 45° + \sin 30°$, find the value of x.

16. If $4 \cos^2 x° - 1 = 0$ and $0 \le x \le 90$, find

(*i*) x (*ii*) $\sin^2 x° + \cos^2 x°$ (*iii*) $\cos^2 x° - \sin^2 x°$.

17. (*i*) If $\sec \theta = \text{cosec } \theta$ and $0° \le \theta \le 90°$, find the value of θ.

(*ii*) If $\tan \theta = \cot \theta$ and $0° \le \theta \le 90°$, find the value of θ.

18. If $\sin 3x = 1$ and $0° \le x \le 90°$, find the values of

(*i*) $\sin x$ (*ii*) $\cos 2x$ (*iii*) $\tan^2 x - \sec^2 x$.

19. If $3 \tan^2 \theta - 1 = 0$, find $\cos 2\theta$, given that θ is acute.

20. If $\sin x + \cos y = 1$, $x = 30°$ and y is acute angle, find the value of y.

21. If $\sin (A + B) = \dfrac{\sqrt{3}}{2} = \cos (A - B)$ and A, B (A > B) are acute angles, find the values of A and B.

22. If $0° \le \theta \le 90°$, solve the following equations for θ :

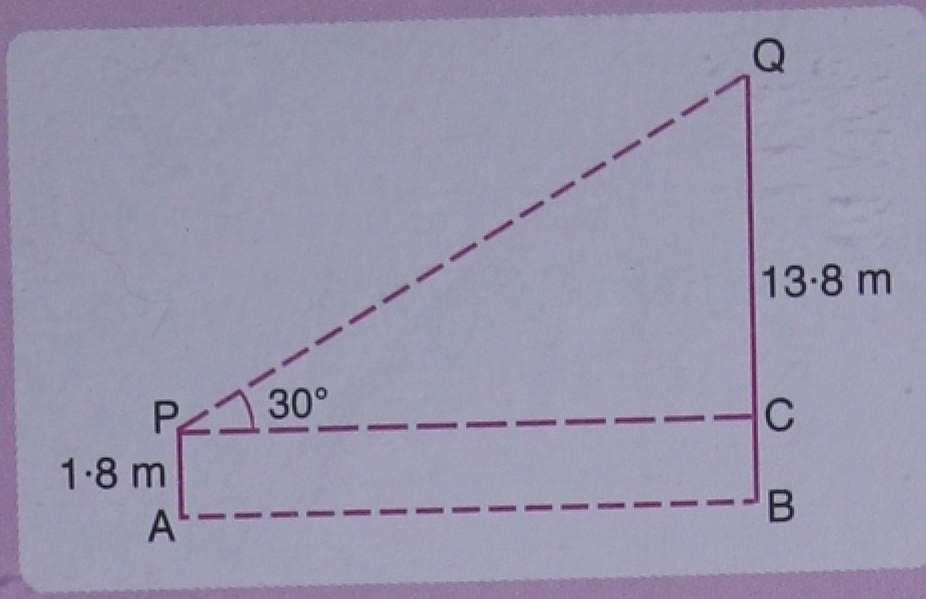
(*i*) $(\sin \theta - 1)(2 \cos \theta - 1) = 0$ (*ii*) $\cos 3\theta (2 \sin 2\theta - 1) = 0$

(*iii*) $(\sec \theta - 2)(\tan 3\theta - 1) = 0$.

23. If the length of each side of a rhombus is 8 cm and its one angle is 60°, then find the lengths of the diagonals of the rhombus.

24. In the right-angled triangle ABC, $\angle C = 90°$ and $\angle B = 60°$. If AC = 6 cm, find the lengths of the sides BC and AB.

25. In the adjoining figure, AP is a man of height 1·8 m and BQ is a building 13·8 m high. If the man sees the top of the building by focussing his binoculars at an angle of 30° to the horizontal, find the distance of the man from the building.



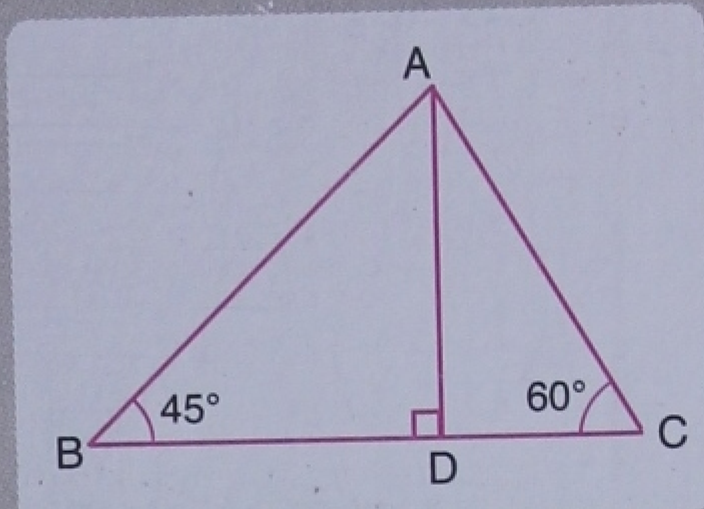
Hint

Let AB = d metres, then PC = d metres. From right-angled ΔPCQ,

$$\tan 30° = \frac{CQ}{PC} \Rightarrow \frac{1}{\sqrt{3}} = \frac{13{\cdot}8 - 1{\cdot}8}{d}.$$

26. In the adjoining figure, ABC is a triangle in which $\angle B = 45°$ and $\angle C = 60°$. If AD $\perp$ BC and BC = 8 m, find the length of the altitude AD.

Hint



In Δ ABD, $\tan 45° = \frac{AD}{BD}$

$\Rightarrow 1 = \frac{AD}{BD} \Rightarrow BD = AD.$

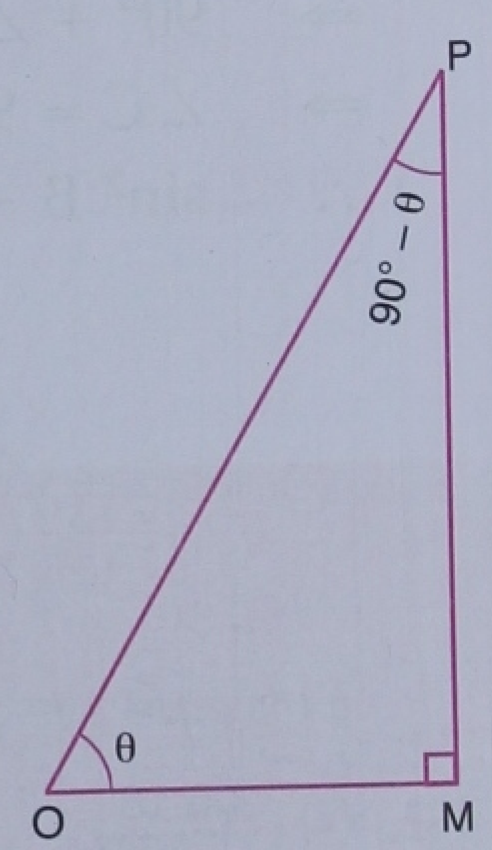
In Δ ADC, $\tan 60° = \frac{AD}{DC} \Rightarrow \sqrt{3} = \frac{AD}{DC} \Rightarrow DC = \frac{AD}{\sqrt{3}}.$

But BD + DC = BC = 8 m $\Rightarrow AD + \frac{AD}{\sqrt{3}} = 8.$

20.5 COMPLEMENTS OF SINE AND COSINE

Let OMP be a right angled triangle at M and $\angle MOP = \theta$, then

$$\angle OPM = 180° - (\angle OMP + \angle MOP)$$
$$= 180° - (90° + \theta) = 90° - \theta.$$



From right angled ΔOMP, we get

$$\sin (90° - \theta) = \frac{OM}{OP} = \cos \theta \text{ and}$$

$$\cos (90° - \theta) = \frac{MP}{OP} = \sin \theta.$$

Hence sine and cosine are complements of each other.

ILLUSTRATIVE EXAMPLES

Example 1. *Evaluate the following :*

(*i*) $\frac{\sin 55°}{\cos 35°}$ (*ii*) $\cos 67° - \sin 23°$ (*iii*) $\sin 24° \sin 66° - \cos 24° \cos 66°$.

Solution. (*i*) $\frac{\sin 55°}{\cos 35°} = \frac{\sin (90° - 35°)}{\cos 35°} = \frac{\cos 35°}{\cos 35°}$ [$\because \sin (90° - \theta) = \cos \theta$]

$= 1.$

(*ii*) $\cos 67° - \sin 23° = \cos (90° - 23°) - \sin 23°$

$= \sin 23° - \sin 23°$ $\quad [\because \cos (90° - \theta) = \sin \theta]$

$= 0.$

(*iii*) $\sin 24° \sin 66° - \cos 24° \cos 66°$

$= \sin 24° \sin (90° - 24°) - \cos 24° \cos (90° - 24°)$

$= \sin 24° \cos 24° - \cos 24° \sin 24°$

$= 0.$

Example 2. *Evaluate the following :*

(*i*) $\sin^2 34° + \sin^2 56°$ $\quad$ (*ii*) $\left(\frac{\sin 31°}{\cos 59°}\right)^2 + \left(\frac{\cos 59°}{\sin 31°}\right)^2 - 4 \sin^2 45°.$

Solution. (*i*) $\sin^2 34° + \sin^2 56° = \sin^2 34° + \sin^2 (90° - 34°)$

$= \sin^2 34° + \cos^2 34°$

$= 1.$ $\quad (\because \sin^2 \theta + \cos^2 \theta = 1)$

(*ii*) $\left(\frac{\sin 31°}{\cos 59°}\right)^2 + \left(\frac{\cos 59°}{\sin 31°}\right)^2 - 4 \sin^2 45°$

$= \left(\frac{\sin 31°}{\cos (90° - 31°)}\right)^2 + \left(\frac{\cos (90° - 31°)}{\sin 31°}\right)^2 - 4\left(\frac{1}{\sqrt{2}}\right)^2$

$\left(\because \sin 45° = \frac{1}{\sqrt{2}}\right)$

$= \left(\frac{\sin 31°}{\sin 31°}\right)^2 + \left(\frac{\sin 31°}{\sin 31°}\right)^2 - 4 . \frac{1}{2}$

$= 1^2 + 1^2 - 2 = 1 + 1 - 2 = 0.$

Example 3. *In ΔABC, if $\angle A = 90°$ then find the value of $\sin^2 B + \sin^2 C$.*

Solution. In Δ ABC, $\angle A = 90°$ (given)

We know that sum of angles in a triangle is 180°

$\Rightarrow \angle A + \angle B + \angle C = 180°$

$\Rightarrow 90° + \angle B + \angle C = 180° \Rightarrow \angle B + \angle C = 90°$

$\Rightarrow \angle C = 90° - \angle B.$

$\therefore \sin^2 B + \sin^2 C = \sin^2 B + \sin^2 (90° - B)$

$= \sin^2 B + \cos^2 B$ $\quad (\because \sin (90° - \theta) = \cos \theta)$

$= 1$ $\quad (\because \sin^2 \theta + \cos^2 \theta = 1)$

Exercise 20.2

Evaluate the following (1 – 5) :

1. (*i*) $\frac{\sin 18°}{\cos 72°}$ $\quad$ (*ii*) $\frac{\sin 21°}{\cos 69°} - \frac{1}{2}\left(\frac{\cos 63°}{\sin 27°}\right).$
2. (*i*) $\cos 58° - \sin 32°$ $\quad$ (*ii*) $\sin 65° - \cos 25°.$
3. (*i*) $\sin^2 37° - \cos^2 53°$ $\quad$ (*ii*) $\sin^2 82° + \sin^2 8°.$
4. (*i*) $\cos 40° \cos 50° - \sin 40° \sin 50°$ $\quad$ (*ii*) $\sin 35° \cos 55° + \cos 35° \sin 55°.$
5. (*i*) $\sin^2 15° + \sin^2 75° - \tan^2 30°$ $\quad$ (*ii*) $\left(\frac{\sin 20°}{\cos 70°}\right)^2 + \left(\frac{\cos 62°}{\sin 28°}\right)^2 - \sin^2 30°.$
6. If $\sin A = \frac{2}{3}$ and $A + B = 90°$, find the value of $\cos B$.
7. If $\cos B = \frac{3}{\sqrt{13}}$ and $A + B = 90°$, find the value of $\sin A$.

CHAPTER TEST

1. Find the values of :

(i) $\sin^2 60° - \cos^2 45° + 3\tan^2 30°$ (ii) $\frac{2\cos^2 45° + 3\tan^2 30°}{\sqrt{3}\cos 30° + \sin 30°}$

(iii) $\sec 30° \tan 60° + \sin 45° \operatorname{cosec} 45° + \cos 30° \cot 60°$.

2. Taking $A = 30°$, verify that

(i) $\cos^4 A - \sin^4 A = \cos 2A$

(ii) $4\cos A \cos(60° - A)\cos(60° + A) = \cos 3A$.

3. If $A = 45°$ and $B = 30°$, verify that $\frac{\sin A}{\cos A + \sin A \sin B} = \frac{2}{3}$.

4. Taking $A = 60°$ and $B = 30°$, verify that

(i) $\frac{\sin(A + B)}{\cos A \cos B} = \tan A + \tan B$ (ii) $\frac{\sin(A - B)}{\sin A \sin B} = \cot B - \cot A$.

5. If $\sqrt{2}\tan 2\theta = \sqrt{6}$ and $0° < \theta < 90°$, find the value of $\sin\theta + \sqrt{3}\cos\theta - 2\tan^2\theta$.

6. If θ is an acute angle, solve the following equations for θ :

(i) $(\tan\theta - 1)(\operatorname{cosec} 3\theta - 1) = 0$ (ii) $(\operatorname{cosec} 3\theta - 2)(\cot 2\theta - 1) = 0$.

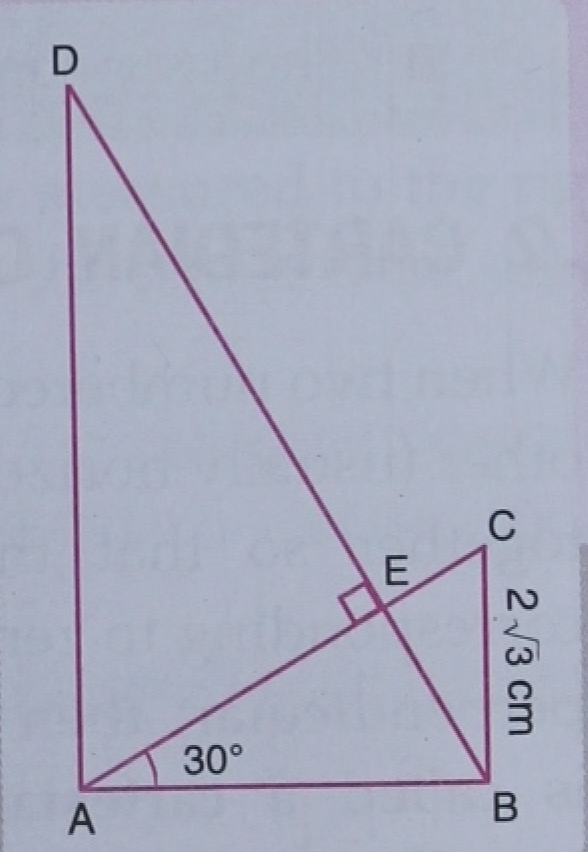
7. If $\tan(A + B) = \sqrt{3}$, $\tan(A - B) = 1$ and A, B $(B < A)$ are acute angles, find the values of A and B.

8. Find the values of :

(i) $\frac{5\sin 33°}{\cos 57°} - \frac{3\cos 35°}{\sin 55°} + \tan^2 60°$

(ii) $\frac{\cos 77°}{\sin 13°} + \sin^2 52° + \sin^2 38° - 3\tan^2 45°$.

9. In the adjoining figure, ABC is right-angled triangle at B and ABD is right angled triangle at A. If $BD \perp AC$ and $BC = 2\sqrt{3}$ cm, find the length of AD.



Hint

In ΔABC, $\tan 30° = \frac{BC}{AB} \Rightarrow \frac{1}{\sqrt{3}} = \frac{2\sqrt{3}\text{ cm}}{AB}$

$\Rightarrow AB = 6$ cm.

For ΔABE, $90° = 30° + \angle ABE$ ($\because$ ext $\angle$ = sum of two opp. int. $\angle$s)

$\Rightarrow \angle ABE = 60°$.

In ΔABD, $\tan 60° = \frac{AD}{AB} \Rightarrow \sqrt{3} = \frac{AD}{6\text{ cm}}$.