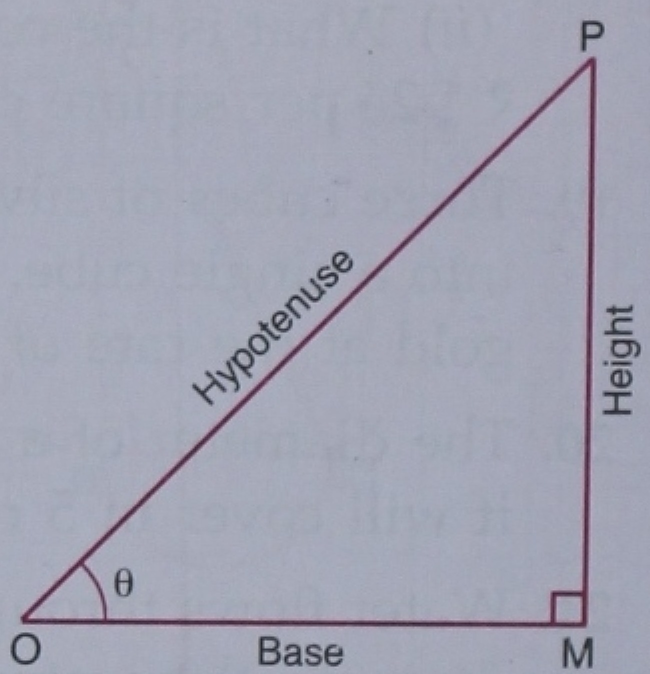
19

TRIGONOMETRICAL RATIOS

19.1 TRIGONOMETRICAL RATIOS

Trigonometry is that branch of Mathematics which deals with the measurement of the angles and the sides of a triangle, and of the various relationships existing between them.

Let OMP be a right-angled triangle at M and ∠MOP = θ (theta), then the *trigonometrical ratios* (abbreviated *t-ratios*) are defined as



(1) $\frac{MP}{OP}$ is called **sine** of θ and is written as sin θ.

Thus $\sin\theta = \frac{MP}{OP}$.

(2) $\frac{OM}{OP}$ is called **cosine** of θ and is written as cos θ.

Thus $\cos\theta = \frac{OM}{OP}$.

(3) $\frac{MP}{OM}$ is called **tangent** of θ and is written as tan θ.

Thus $\tan\theta = \frac{MP}{OM}$.

(4) $\frac{OM}{MP}$ is called **cotangent** of θ and is written as cot θ. Thus

$$\cot\theta = \frac{OM}{MP}.$$

(5) $\frac{OP}{OM}$ is called **secant** of θ and is written as sec θ. Thus

$$\sec\theta = \frac{OP}{OM}.$$

(6) $\frac{OP}{MP}$ is called **cosecant** of θ and is written as cosec θ. Thus

$$\text{cosec}\,\theta = \frac{OP}{MP}.$$

In reference to $\angle MOP$ in Δ OMP, OM is called *base* or *adjoining side*, MP is called *height* or *opposite side* and OP is the *hypotenuse*. The *six trigonometrical ratios* can be defined as :

(1) $\sin\theta = \dfrac{\text{height}}{\text{hypotenuse}}$. (2) $\cos\theta = \dfrac{\text{base}}{\text{hypotenuse}}$.

(3) $\tan\theta = \dfrac{\text{height}}{\text{base}}$. (4) $\cot\theta = \dfrac{\text{base}}{\text{height}}$.

(5) $\sec\theta = \dfrac{\text{hypotenuse}}{\text{base}}$. (6) $\operatorname{cosec}\theta = \dfrac{\text{hypotenuse}}{\text{height}}$.

Notes

1. In right-angled triangle OMP, $\angle MOP$ lies between 0° to 90° *i.e.* $\angle MOP$ is acute angle *i.e.* θ is acute and all the six trigonometrical ratios are positive.
2. $\sin\theta$ is one symbol *i.e.* $\sin\theta \neq \sin \times \theta$. Similar is the case for other *t*-ratios.
3. *Each trigonometrical ratio is a (unitless) real number.*

19.1.1 Reciprocal relations

From the right-angled triangle OMP, we get

(1) $\sin\theta = \dfrac{MP}{OP}$ and $\operatorname{cosec}\theta = \dfrac{OP}{MP}$

$\Rightarrow \sin\theta = \dfrac{1}{\operatorname{cosec}\theta}$ and $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$

$\Rightarrow$ *$\sin\theta$ and $\operatorname{cosec}\theta$ are reciprocals of each other.*

(2) $\cos\theta = \dfrac{OM}{OP}$ and $\sec\theta = \dfrac{OP}{OM}$

$\Rightarrow \cos\theta = \dfrac{1}{\sec\theta}$ and $\sec\theta = \dfrac{1}{\cos\theta}$

$\Rightarrow$ *$\cos\theta$ and $\sec\theta$ are reciprocal of each other.*

(3) $\tan\theta = \dfrac{OM}{OP}$ and $\cot\theta = \dfrac{OM}{MP}$

$\Rightarrow \tan\theta = \dfrac{1}{\cot\theta}$ and $\cot\theta = \dfrac{1}{\tan\theta}$

$\Rightarrow$ *$\tan\theta$ and $\cot\theta$ are reciprocal of each other.*

From above, it follows that :

(*i*) $\sin\theta \times \operatorname{cosec}\theta = 1$

(*ii*) $\cos\theta \times \sec\theta = 1$

(*iii*) $\tan\theta \times \cot\theta = 1$.

19.1.2 Quotient relations

From the right-angled triangle OMP, we get

(1) $\dfrac{\sin\theta}{\cos\theta} = \dfrac{\frac{MP}{OP}}{\frac{OM}{OP}} = \dfrac{MP}{OP} \times \dfrac{OP}{OM} = \dfrac{MP}{OM} = \tan\theta$,

$\therefore$ $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.

(2) $\dfrac{\cos\theta}{\sin\theta} = \dfrac{\frac{OM}{OP}}{\frac{MP}{OP}} = \dfrac{OM}{OP} \times \dfrac{OP}{MP} = \dfrac{OM}{MP} = \cot\theta,$

$\therefore$ **$\cot\theta = \dfrac{\cos\theta}{\sin\theta}$.**

19.1.3 Square Relations

From the right-angled triangle OMP, by Pythagoras theorem, we get

$$MP^2 + OM^2 = OP^2 \quad \ldots(i)$$

(1) Dividing both sides of (*i*) by OP^2, we get

$$\left(\frac{MP}{OP}\right)^2 + \left(\frac{OM}{OP}\right)^2 = 1$$

$\Rightarrow$ **$\sin^2\theta + \cos^2\theta = 1$.**

(2) Dividing both sides of (*i*) by OM^2, we get

$$\left(\frac{MP}{OM}\right)^2 + 1 = \left(\frac{OP}{OM}\right)^2$$

$\Rightarrow \tan^2\theta + 1 = \sec^2\theta$ *i.e.* **$1 + \tan^2\theta = \sec^2\theta$.**

(3) Dividing both sides of (*i*) by MP^2, we get

$$1 + \left(\frac{OM}{MP}\right)^2 = \left(\frac{OP}{MP}\right)^2$$

$\Rightarrow$ **$1 + \cot^2\theta = \mathrm{cosec}^2\theta$.**

From above, it follows that :

(*i*) $1 - \sin^2\theta = \cos^2\theta$ (*ii*) $1 - \cos^2\theta = \sin^2\theta$

(*iii*) $\sec^2\theta - 1 = \tan^2\theta$ (*iv*) $\sec^2\theta - \tan^2\theta = 1$

(*v*) $\mathrm{cosec}^2\theta - 1 = \cot^2\theta$ (*vi*) $\mathrm{cosec}^2\theta - \cot^2\theta = 1$.

Note

$\sin^2\theta$ means $(\sin\theta)^2$ and $\sin^2\theta$ is read as *sine squared* θ. Similarly $\cos^2\theta$ means $(\cos\theta)^2$ etc.

19.2 THE VALUE OF ANY T-RATIO IN TERMS OF A GIVEN T-RATIO

We shall explain the method of finding the value of any *t*-ratio in terms of a given *t*-ratio with the help of the following examples :

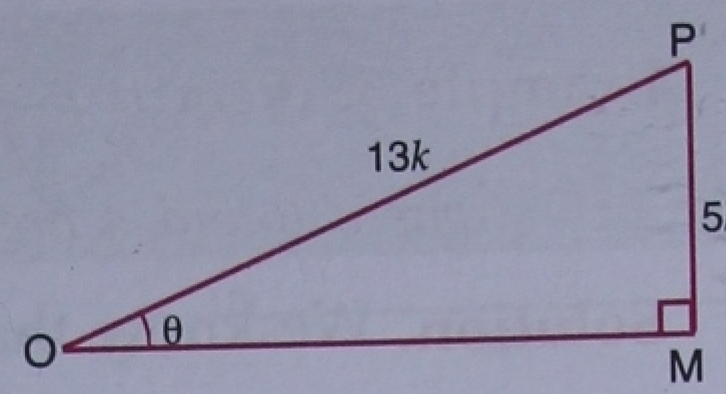
ILLUSTRATIVE EXAMPLES

Example 1. *If $\sin\theta = \frac{5}{13}$ and θ is acute angle, find the value of $\tan\theta + \frac{1}{\cos\theta}$.*

Solution. We know that $\sin\theta = \dfrac{MP}{OP}$ and $\sin\theta = \dfrac{5}{13}$ (given)

$\Rightarrow \quad \dfrac{MP}{OP} = \dfrac{5}{13}$.

Draw a triangle OMP right-angled at M (shown in the adjoining figure) such that MP = $5k$ and OP = $13k$, where k is a positive real number.



From right-angled ΔOMP, by Pythagoras theorem, we get

$$OP^2 = OM^2 + MP^2$$

$$\Rightarrow \quad OM^2 = OP^2 - MP^2$$

$$\Rightarrow \quad OM^2 = (13k)^2 - (5k)^2 = 169k^2 - 25k^2 = 144k^2$$

$$\Rightarrow \quad OM = 12k.$$

$$\therefore \quad \tan\theta = \frac{MP}{OM} = \frac{5k}{12k} = \frac{5}{12} \text{ and } \cos\theta = \frac{OM}{OP} = \frac{12k}{13k} = \frac{12}{13}.$$

$$\therefore \quad \tan\theta + \frac{1}{\cos\theta} = \frac{5}{12} + \frac{1}{\frac{12}{13}} = \frac{5}{12} + \frac{13}{12} = \frac{5+13}{12} = \frac{18}{12} = \frac{3}{2}.$$

Alternatively

We know that $\sin^2\theta + \cos^2\theta = 1$

$$\Rightarrow \quad \cos^2\theta = 1 - \sin^2\theta = 1 - \left(\frac{5}{13}\right)^2 \qquad \left(\because \sin\theta = \frac{5}{13} \text{ given}\right)$$

$$\Rightarrow \quad \cos^2\theta = 1 - \frac{25}{169} = \frac{169-25}{169} = \frac{144}{169}$$

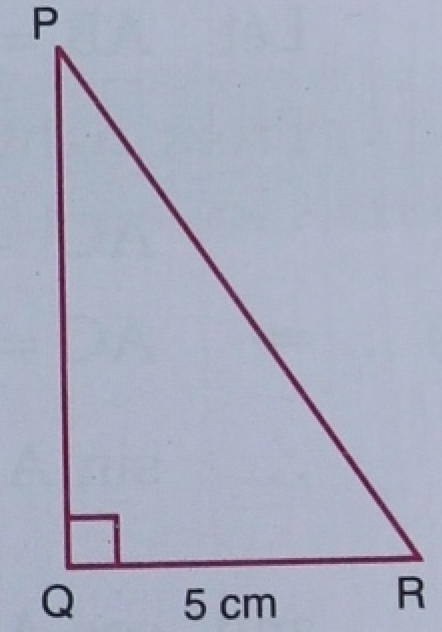
$$\Rightarrow \quad \cos\theta = \frac{12}{13}$$

(As θ is acute, $\cos\theta$ is +ve, so we take +ve value of the square root)

$$\therefore \quad \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{5}{13} \div \frac{12}{13} = \frac{5}{13} \times \frac{13}{12} = \frac{5}{12}.$$

$$\therefore \quad \tan\theta + \frac{1}{\cos\theta} = \frac{5}{12} + \frac{1}{\frac{12}{13}} = \frac{5}{12} + \frac{13}{12} = \frac{5+13}{12} = \frac{18}{12} = \frac{3}{2}.$$

Example 2. *In the adjoining figure, PQR is a right-angled triangle, right-angled at Q. If QR = 5 cm and PR – PQ = 1 cm, then find the values of sin P and sec P.*



Solution. Given, QR = 5 cm and PR – PQ = 1 cm

$\Rightarrow$ PR = (1 + PQ) cm.

From right-angled ΔPQR, by Pythagoras theorem, we get

$$PR^2 = PQ^2 + QR^2$$

$$\Rightarrow \quad (1 + PQ)^2 = PQ^2 + 5^2$$

$$\Rightarrow \quad 1 + PQ^2 + 2PQ = PQ^2 + 25$$

$$\Rightarrow \quad 2PQ = 25 - 1 \Rightarrow 2PQ = 24 \Rightarrow PQ = 12 \text{ cm}.$$

$$\therefore \quad PR = (1 + 12) \text{ cm} = 13 \text{ cm}$$

$$\therefore \quad \sin P = \frac{QR}{PR} = \frac{5}{13} \text{ and } \sec P = \frac{PR}{PQ} = \frac{13}{12}.$$

Example 3. *If* $\sin\theta = \frac{\sqrt{3}}{2}$ *and* $\cos\phi = \frac{1}{\sqrt{2}}$, *find the value of* $\frac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi}$ *(where θ and φ are acute).*

Solution. We know that $\sin^2\theta + \cos^2\theta = 1$

$$\Rightarrow \quad \cos^2\theta = 1 - \sin^2\theta = 1 - \left(\frac{\sqrt{3}}{2}\right)^2 = 1 - \frac{3}{4} = \frac{1}{4}$$

$$\Rightarrow \quad \cos\theta = \frac{1}{2} \qquad (\because \theta \text{ is acute, so } \cos\theta \text{ is +ve})$$

$$\therefore \quad \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{3}}{2} \times \frac{2}{1} = \sqrt{3}.$$

Also we know that $\sin^2\phi + \cos^2\phi = 1$

$$\Rightarrow \quad \sin^2\phi = 1 - \cos^2\phi = 1 - \left(\frac{1}{\sqrt{2}}\right)^2 = 1 - \frac{1}{2} = \frac{1}{2}$$

$$\Rightarrow \quad \sin\phi = \frac{1}{\sqrt{2}}. \qquad (\because \phi \text{ is acute, so } \sin\phi \text{ is +ve})$$

$$\therefore \quad \tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{1} = 1.$$

$$\therefore \quad \frac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi} = \frac{\sqrt{3} - 1}{1 + \sqrt{3} \times 1} = \frac{\sqrt{3} - 1}{\sqrt{3} + 1} = \frac{\sqrt{3} - 1}{\sqrt{3} + 1} \times \frac{\sqrt{3} - 1}{\sqrt{3} - 1}$$

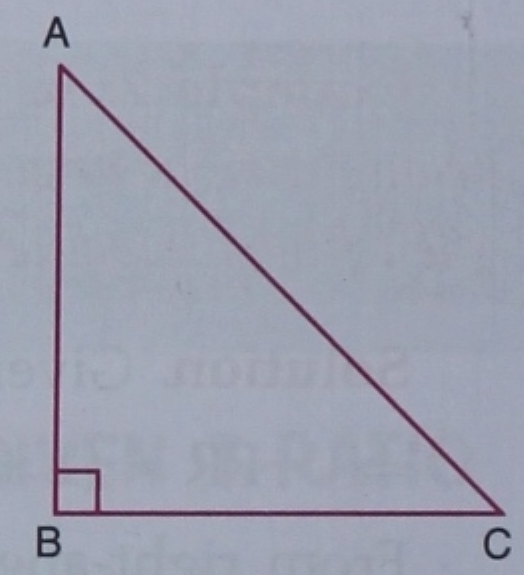
$$= \frac{(\sqrt{3} - 1)^2}{(\sqrt{3})^2 - 1^2} = \frac{3 + 1 - 2\sqrt{3}}{3 - 1} = \frac{4 - 2\sqrt{3}}{2} = 2 - \sqrt{3}.$$

Example 4. *In a right-angled triangle ABC, right-angled at B, if cot A = 1, prove that 2 sin A cos A = 1.*

Solution. Given, a triangle ABC in which $\angle B = 90°$

and $\cot A = 1 \Rightarrow \frac{AB}{BC} = 1$

$\Rightarrow \quad AB = BC.$



Let $AB = BC = k$ units, where k is positive real number.

From right-angled ΔABC, by Pythagoras theorem, we get

$$AC^2 = AB^2 + BC^2 = k^2 + k^2 = 2k^2$$

$$\Rightarrow \quad AC = \sqrt{2}k \text{ units.}$$

$$\therefore \quad \sin A = \frac{BC}{AC} = \frac{k}{\sqrt{2}k} = \frac{1}{\sqrt{2}}$$

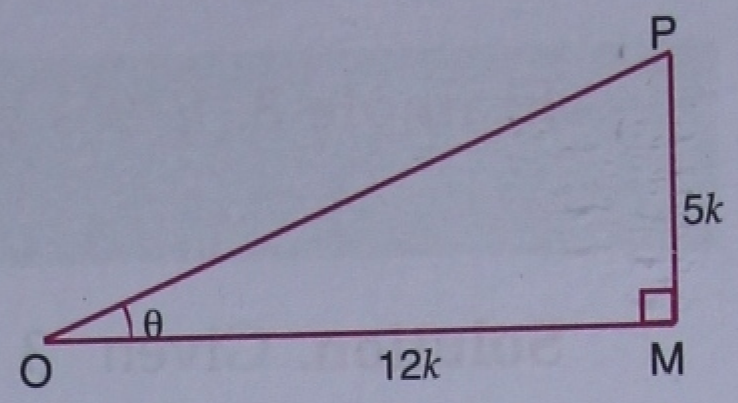
$$\text{and} \quad \cos A = \frac{AB}{AC} = \frac{k}{\sqrt{2}k} = \frac{1}{\sqrt{2}}.$$

$$\therefore \quad 2\sin A\cos A = 2 \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = \frac{2}{2} = 1.$$

Example 5. *If* $\tan\theta = \frac{5}{12}$, *find sec θ and sec θ + cosec θ, where θ is acute.*

Solution. Given $\tan\theta = \frac{5}{12}$ but $\tan\theta = \frac{MP}{OM}$

$$\Rightarrow \quad \frac{MP}{OM} = \frac{5}{12}.$$



Draw a triangle OMP right angled at M (shown in the adjoining figure) such that MP = $5k$ and OM = $12k$, where k is a positive real number.

From right angled ΔOMP, by Pythagoras theorem, we get

$$OP^2 = OM^2 + MP^2 = (12k)^2 + (5k)^2 = 144k^2 + 25k^2 = 169k^2$$

$\Rightarrow \quad OP = 13k.$

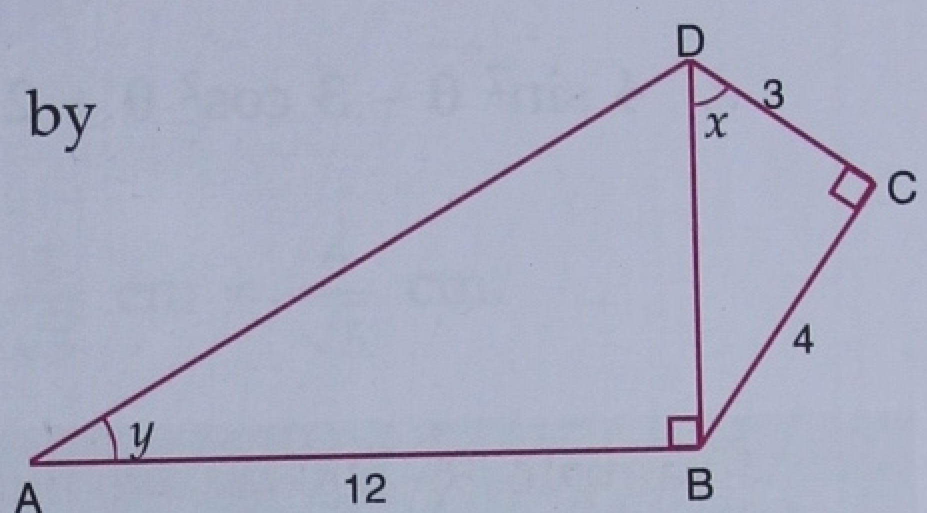
$\therefore \quad \sec\theta = \frac{OP}{OM} = \frac{13k}{12k} = \frac{13}{12}$ and $\operatorname{cosec}\theta = \frac{OP}{MP} = \frac{13k}{5k} = \frac{13}{5}.$

$$\therefore \sec\theta + \operatorname{cosec}\theta = \frac{13}{12} + \frac{13}{5} = 13\left(\frac{1}{12} + \frac{1}{5}\right)$$

$$= 13 . \frac{5+12}{60}$$

$$= \frac{13 \times 17}{60} = \frac{221}{60} = 3\frac{41}{60}.$$

Example 6. *From the adjoining figure, find*
(i) sin x (ii) cos y.



Solution. (*i*) From right-angled Δ BCD, by Pythagoras theorem, we get

$$BD^2 = BC^2 + CD^2$$

$\Rightarrow \quad BD^2 = 4^2 + 3^2 = 16 + 9 = 25$

$\Rightarrow \quad BD = 5.$

$\therefore \quad \sin x = \frac{\text{height}}{\text{hypotenuse}} = \frac{4}{5}.$

(*ii*) From right-angled Δ ABD, by Pythagoras theorem, we get

$$AD^2 = AB^2 + BD^2$$

$\Rightarrow \quad AD^2 = (12)^2 + 5^2 = 144 + 25 = 169 \Rightarrow AD = 13.$

$\therefore \quad \cos y = \frac{\text{base}}{\text{hypotenuse}} = \frac{12}{13}.$

Example 7. *If 5 tan θ = 4, find the value of* $\frac{5\sin\theta - 3\cos\theta}{5\sin\theta + 2\cos\theta}$.

Solution. Given $5\tan\theta = 4 \Rightarrow \tan\theta = \frac{4}{5}$ ... (*i*)

Now $\frac{5\sin\theta - 3\cos\theta}{5\sin\theta + 2\cos\theta} = \frac{5\frac{\sin\theta}{\cos\theta} - 3\frac{\cos\theta}{\cos\theta}}{5\frac{\sin\theta}{\cos\theta} + 2\frac{\cos\theta}{\cos\theta}}$

(Dividing the numerator and denominator by cos θ)

$= \frac{5\tan\theta - 3}{5\tan\theta + 2}$ $\left(\because \tan\theta = \frac{\sin\theta}{\cos\theta}\right)$

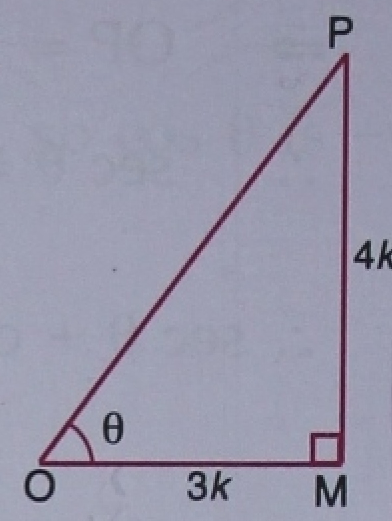
$= \frac{5.\frac{4}{5} - 3}{5.\frac{4}{5} + 2}$ [using (*i*)]

$= \frac{4-3}{4+2} = \frac{1}{6}.$

Example 8. *If θ is acute and 3 sin θ = 4 cos θ, find the value of $4\sin^2\theta - 3\cos^2\theta + 2$.*

Solution. Given $3\sin\theta = 4\cos\theta \Rightarrow \frac{\sin\theta}{\cos\theta} = \frac{4}{3} \Rightarrow \tan\theta = \frac{4}{3}$.

But $\tan\theta = \frac{MP}{OM} \Rightarrow \frac{MP}{OM} = \frac{4}{3}$.



Draw a triangle OMP right angled at M (shown in the adjoining figure) such that MP = $4k$ and OM = $3k$, where k is a positive real number.

From right-angled Δ OMP, by Pythagoras theorem, we get

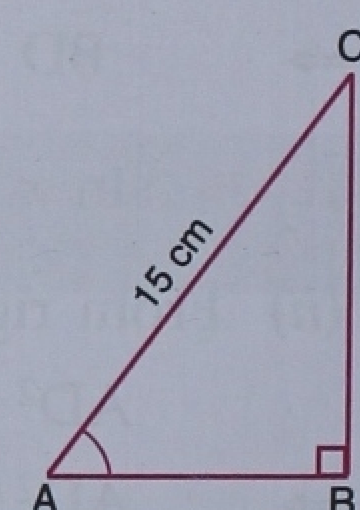
$$OP^2 = OM^2 + MP^2$$
$$\Rightarrow OP^2 = (3k)^2 + (4k)^2 = 9k^2 + 16k^2 = 25k^2$$
$$\Rightarrow OP = 5k.$$

$$\therefore \sin\theta = \frac{MP}{OP} = \frac{4k}{5k} = \frac{4}{5} \text{ and } \cos\theta = \frac{OM}{OP} = \frac{3k}{5k} = \frac{3}{5}.$$

$$\therefore 4\sin^2\theta - 3\cos^2\theta + 2 = 4.\left(\frac{4}{5}\right)^2 - 3.\left(\frac{3}{5}\right)^2 + 2 = 4.\frac{16}{25} - 3.\frac{9}{25} + 2$$

$$= \frac{64 - 27 + 50}{25} = \frac{87}{25} = 3\frac{12}{25}.$$

Example 9. *In the adjoining figure, Δ ABC is right-angled at B and tan A = $\frac{4}{3}$. If AC = 15 cm, find the lengths of AB and BC.*



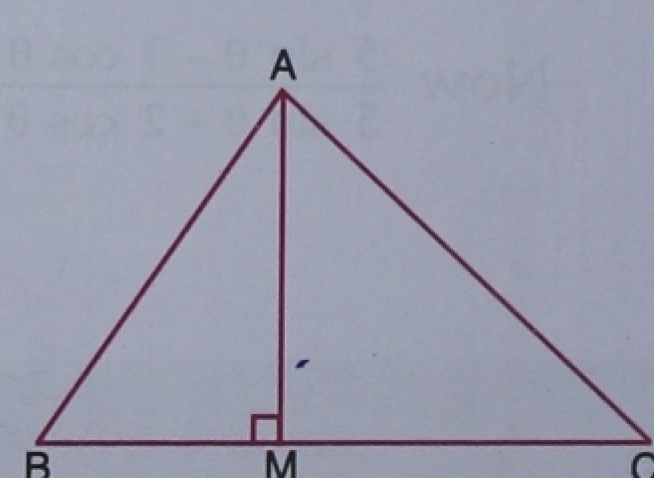
Solution. Given $\tan A = \frac{4}{3} \Rightarrow \frac{BC}{AB} = \frac{4}{3}$.

Let BC = $4x$ cm , then AB = $3x$ cm.

From right-angled ΔABC, by Pythagoras theorem, we get

$$AB^2 + BC^2 = AC^2 \Rightarrow (3x)^2 + (4x)^2 = 15^2$$
$$\Rightarrow 9x^2 + 16x^2 = 225 \Rightarrow 25x^2 = 225$$
$$\Rightarrow x^2 = 9 \Rightarrow x = 3.$$
$$\therefore AB = 3x \text{ cm} \Rightarrow (3 \times 3) \text{ cm} = 9 \text{ cm and } BC = 4x \text{ cm} \Rightarrow (4 \times 3) \text{ cm} = 12 \text{ cm}.$$

Example 10. *In the adjoining figure, AM is perpendicular to BC. If tan B = $\frac{3}{4}$, tan C = $\frac{5}{12}$ and BC = 56 cm, calculate the length of AM.*



Solution. Given $\tan B = \frac{3}{4} \Rightarrow \frac{AM}{BM} = \frac{3}{4}$.

Let AM = $3x$ cm, then BM = $4x$ cm.

$\therefore$ MC = BC – BM = $(56 - 4x)$ cm.

Also $\tan C = \frac{5}{12}$ (given) $\Rightarrow \frac{AM}{MC} = \frac{5}{12}$

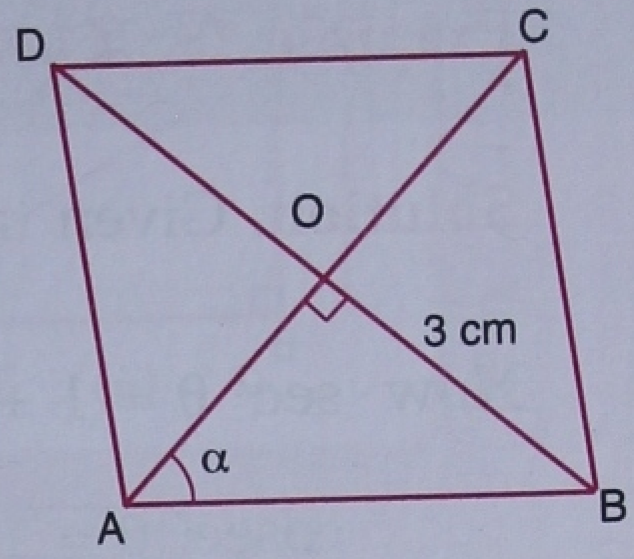
$\Rightarrow \quad \dfrac{3x}{56-4x} = \dfrac{5}{12} \Rightarrow 36\,x = 280 - 20\,x$

$\Rightarrow \quad 56\,x = 280 \Rightarrow x = 5.$

$\therefore \quad AM = 3x \text{ cm} = (3 \times 5) \text{ cm} = 15 \text{ cm}.$

Example 11. *ABCD is a rhombus whose diagonal AC makes an angle α with AB. If $\cos\alpha = \frac{2}{3}$ and OB = 3 cm, then find the side and the diagonals of the rhombus.*

Solution. We know that the diagonals of a rhombus bisects each other at right angles.



From right-angled triangle OAB,

$\cos\alpha = \dfrac{OA}{AB} = \dfrac{2}{3}$ (given).

Let OA = $2x$ cm , then AB = $3x$ cm.

By Pythagoras theorem, $AB^2 = OA^2 + OB^2$

$\Rightarrow \quad (3x)^2 = (2x)^2 + 3^2 \Rightarrow 9x^2 = 4x^2 + 9$

$\Rightarrow \quad 5x^2 = 9 \Rightarrow x = \dfrac{3}{\sqrt{5}}.$

$\therefore \quad AB = 3 \times \dfrac{3}{\sqrt{5}} \text{ cm} = \dfrac{9}{\sqrt{5}} \text{ cm and } OA = 2 \times \dfrac{3}{\sqrt{5}} \text{ cm} = \dfrac{6}{\sqrt{5}} \text{ cm}.$

$\therefore \quad BD = 2 \times OB = (2 \times 3) \text{ cm} = 6 \text{ cm and } AC = 2 \times OA = \left(2 \times \dfrac{6}{\sqrt{5}}\right) \text{ cm} = \dfrac{12}{\sqrt{5}} \text{ cm}.$

$\therefore \quad$ Each side = $\dfrac{9}{\sqrt{5}}$ cm, diagonal BD = 6 cm and diagonal AC = $\dfrac{12}{\sqrt{5}}$ cm.

Example 12. *If $\tan x + \cot x = 2$, find the value of $\tan^2 x + \cot^2 x$.*

Solution. Given $\tan x + \cot x = 2$, on squaring both sides, we get

$(\tan x + \cot x)^2 = 2^2$

$\Rightarrow \quad \tan^2 x + \cot^2 x + 2 \tan x \times \cot x = 4$

$\Rightarrow \quad \tan^2 x + \cot^2 x + 2 \times \tan x \times \dfrac{1}{\tan x} = 4 \qquad \left(\because \cot x = \dfrac{1}{\tan x}\right)$

$\Rightarrow \quad \tan^2 x + \cot^2 x + 2 = 4$

$\Rightarrow \quad \tan^2 x + \cot^2 x = 4 - 2$

$\Rightarrow \quad \tan^2 x + \cot^2 x = 2.$

Example 13. *Prove that $\tan^2\theta - \dfrac{1}{\cos^2\theta} + 1 = 0.$*

Solution. L.H.S. $= \tan^2\theta - \dfrac{1}{\cos^2\theta} + 1$

$= \tan^2\theta - \sec^2\theta + 1 \qquad \left(\because \dfrac{1}{\cos\theta} = \sec\theta\right)$

$= \tan^2\theta - (1 + \tan^2\theta) + 1 \qquad (\because \sec^2\theta = 1 + \tan^2\theta)$

$= \tan^2\theta - 1 - \tan^2\theta + 1 = 0 =$ R.H.S.

Example 14. *If 8 cot θ = 15, find the value of* $\frac{(2+2\sin\theta)(1-\sin\theta)}{(1+\cos\theta)(2-2\cos\theta)}$.

Solution. Given $8\cot\theta = 15 \Rightarrow \cot\theta = \frac{15}{8}$ ...(*i*)

Now $\frac{(2+2\sin\theta)(1-\sin\theta)}{(1+\cos\theta)(2-2\cos\theta)} = \frac{2(1+\sin\theta)(1-\sin\theta)}{2(1+\cos\theta)(1-\cos\theta)}$

$$= \frac{1-\sin^2\theta}{1-\cos^2\theta} = \frac{\cos^2\theta}{\sin^2\theta} = \left(\frac{\cos\theta}{\sin\theta}\right)^2$$

$$= (\cot^2\theta) = \left(\frac{15}{8}\right)^2 = \frac{225}{64} = 3\frac{33}{64}.$$

Example 15. *If* $\tan\theta = \frac{1}{\sqrt{5}}$, *find the value of* $\frac{\text{cosec}^2\,\theta - \sec^2\theta}{\text{cosec}^2\,\theta + \sec^2\theta}$.

Solution. Given $\tan\theta = \frac{1}{\sqrt{5}} \Rightarrow \cot\theta = \sqrt{5}$.

Now $\sec^2\theta = 1 + \tan^2\theta = 1 + \left(\frac{1}{\sqrt{5}}\right)^2 = 1 + \frac{1}{5} = \frac{6}{5}$ and

$\text{cosec}^2\,\theta = 1 + \cot^2\theta = 1 + (\sqrt{5})^2 = 1 + 5 = 6.$

$$\therefore \quad \frac{\text{cosec}^2\,\theta - \sec^2\theta}{\text{cosec}^2\,\theta + \sin^2\theta} = \frac{6-\frac{6}{5}}{6+\frac{6}{5}} = \frac{30-6}{30+6} = \frac{24}{36} = \frac{2}{3}.$$

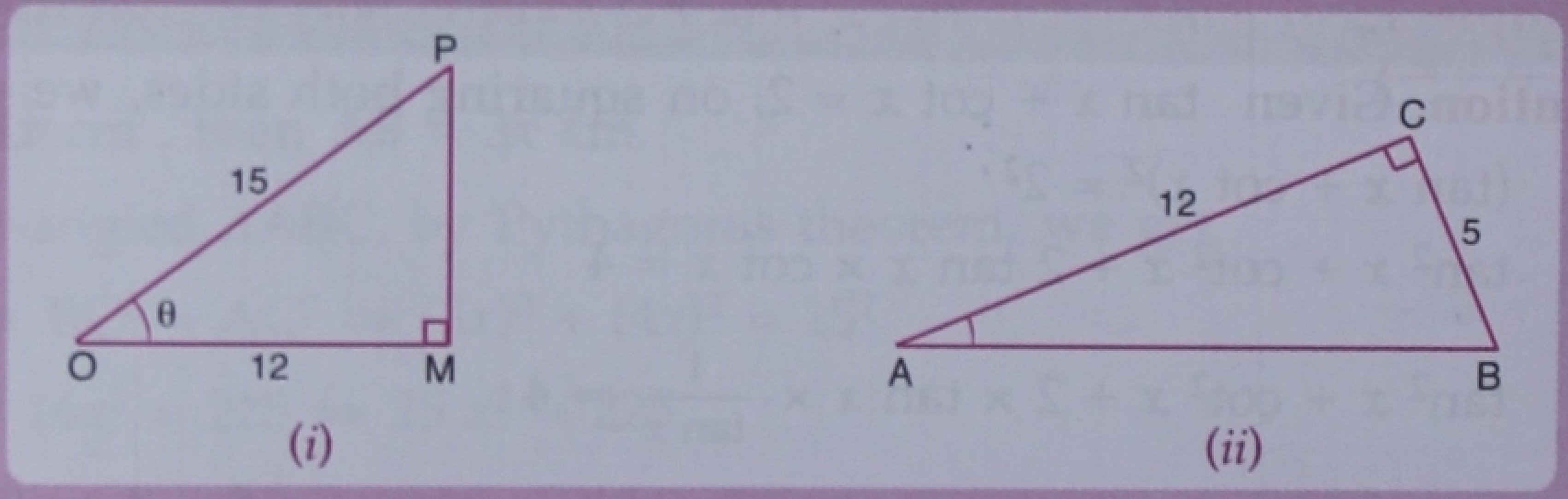
Exercise 19

1. (*a*) From the figure (*i*) given below, find the values of :

(*i*) sin θ (*ii*) cos θ (*iii*) tan θ (*iv*) cot θ (*v*) sec θ (*vi*) cosec θ.

(*b*) From the figure (*ii*) given below, find the values of :

(*i*) sin A (*ii*) cos A (*iii*) $\sin^2 A + \cos^2 A$ (*iv*) $\sec^2 A - \tan^2 A$.



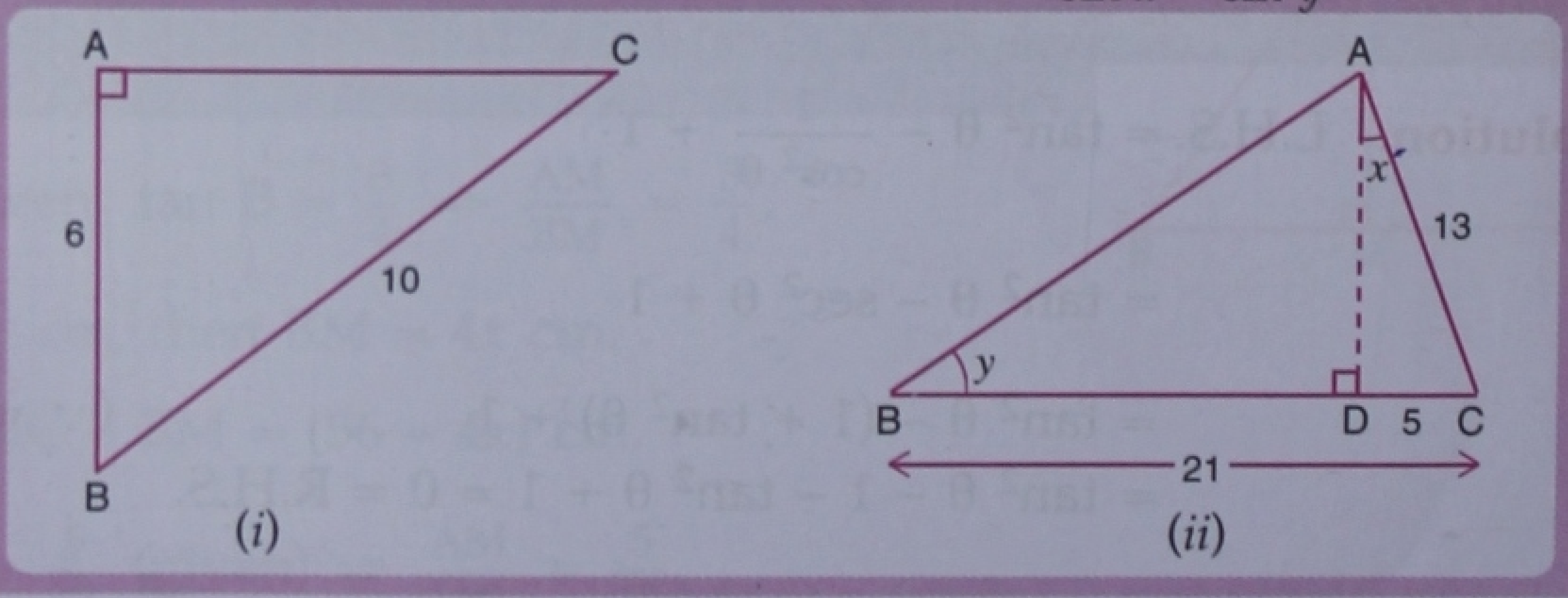
(*i*) (*ii*)

2. (*a*) From the figure (*i*) given below, find the values of :

(*i*) sin B (*ii*) cos C (*iii*) sin B + sin C (*iv*) sin B cos C + sin C cos B.

(*b*) From the figure (*ii*) given below, find the values of :

(*i*) tan *x* (*ii*) cos *y* (*iii*) $\text{cosec}^2\, y - \cot^2 y$ (*iv*) $\frac{5}{\sin x} + \frac{3}{\sin y} - 3\cot y$.



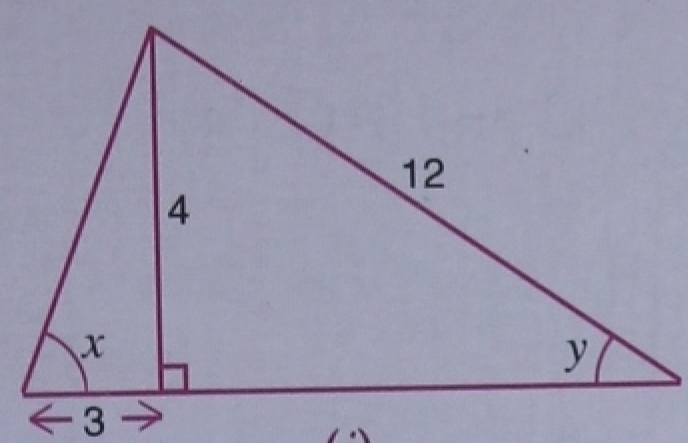
(*i*) (*ii*)

3. (a) From the figure (i) given below, find the values of :

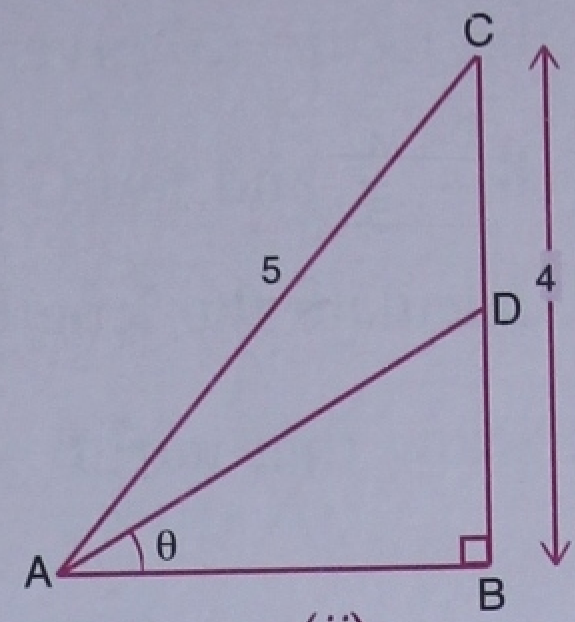
(i) $\sin x$ (ii) $\cot x$ (iii) $\cot^2 x - \operatorname{cosec}^2 x$ (iv) $\sec y$ (v) $\tan^2 y - \frac{1}{\cos^2 y}$.

(b) In the figure (ii) given below, ΔABC is right-angled at B, D is mid-point of BC, AC = 5, BC = 4 and $\angle BAD = \theta$, find the values of :

(i) $\tan \theta$ (ii) $\sin \theta$ (iii) $\sin^2 \theta + \cos^2 \theta$.



(i)



(ii)

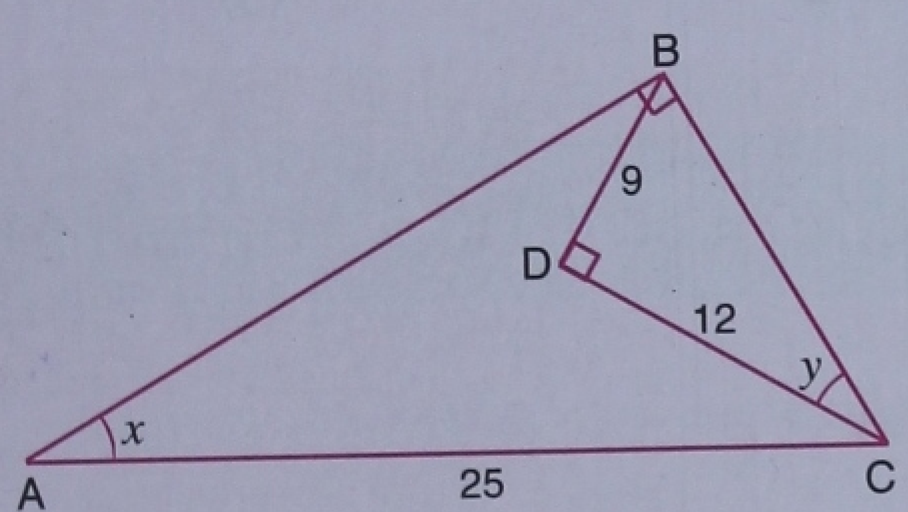
4. (a) From the figure (i) given below, find the values of :

(i) $2 \sin y - \cos y$ (ii) $2 \sin x - \cos x$

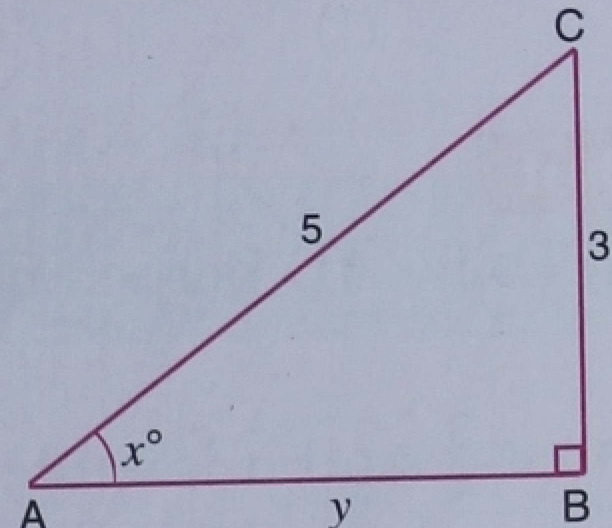
(iii) $1 - \sin x + \cos y$ (iv) $2 \cos x - 3 \sin y + 4 \tan x$.

(b) In the figure (ii) given below, ΔABC is right-angled at B. If AB = y units, BC = 3 units and CA = 5 units, find

(i) $\sin x°$ (ii) y.



(i)



(ii)

5. In a right-angled triangle, it is given that angle A is an acute angle and that $\tan A = \frac{5}{12}$. Find the values of :

(i) $\cos A$ (ii) $\operatorname{cosec} A - \cot A$.

6. (a) In ΔABC, $\angle A = 90°$. If AB = 7 cm and BC – AC = 1 cm, find :

(i) $\sin C$ (ii) $\tan B$

(b) In ΔPQR, $\angle Q = 90°$. If PQ = 40 cm and PR + QR = 50 cm, find :

(i) $\sin P$ (ii) $\cos P$ (iii) $\tan R$.

7. In Δ ABC, AB = AC = 15 cm, BC = 18 cm. Find (i) $\cos \angle ABC$ (ii) $\sin \angle ACB$.

Hint

Draw AD perpendicular to BC, then D is mid-point of BC, so BD = 9 cm. By Pythagoras theorem, AD = 12 cm.

8. (*a*) In the figure (1) given below, Δ ABC is isosceles with AB = AC = 5 cm and BC = 6 cm. Find

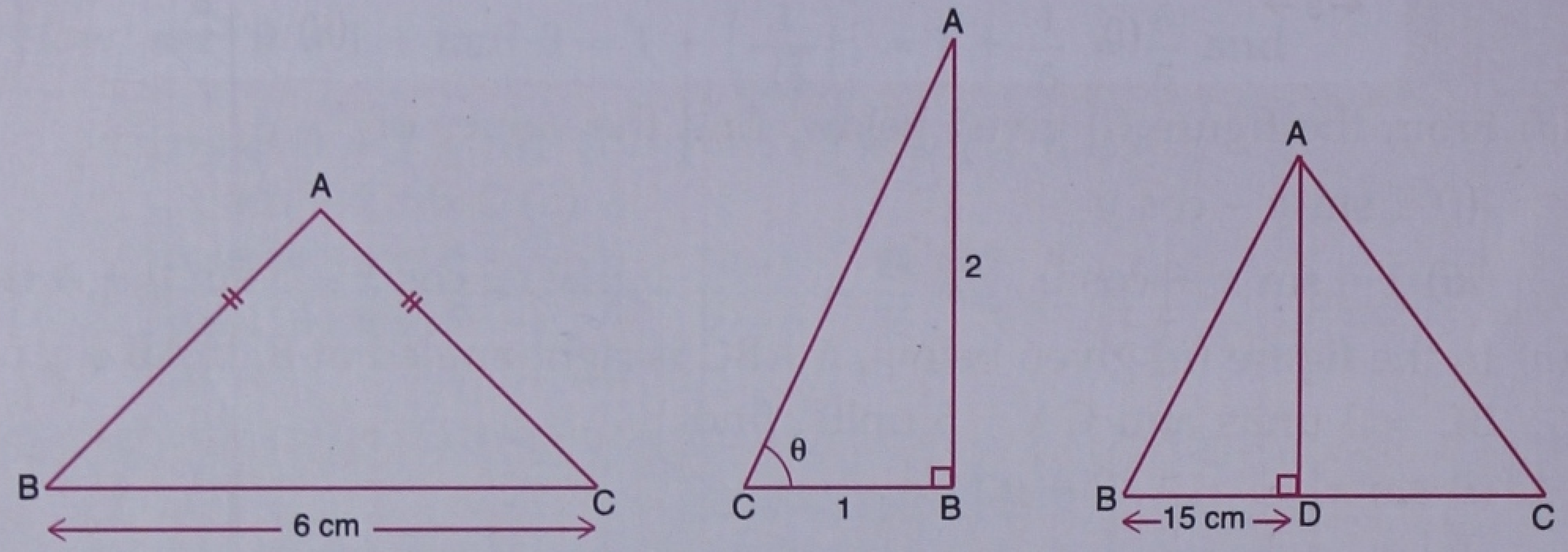
(*i*) sin C (*ii*) tan B (*iii*) tan C – cot B.

(*b*) In the figure (2) given below, Δ ABC is right-angled at B. Given that $\angle ACB = \theta$, side AB = 2 units and side BC = 1 unit, find the value of $\sin^2\theta + \tan^2\theta$.

(*c*) In the figure (3) given below, AD is perpendicular to BC, BD = 15 cm, $\sin B = \frac{4}{5}$ and tan C = 1.

(*i*) Calculate the lengths of AD, AB, DC and AC.

(*ii*) Show that $\tan^2 B - \frac{1}{\cos^2 B} = -1$.



(1) (2) (3)

Hint

(*a*) Draw AD perpendicular to BC, then BD = DC = 3 cm and AD = 4 cm.

9. If $\sin\theta = \frac{3}{5}$ and θ is acute angle, find

(*i*) $\cos\theta$ (*ii*) $\tan\theta$.

10. Given that $\tan\theta = \frac{5}{12}$ and θ is an acute angle, find $\sin\theta$ and $\cos\theta$.

11. If $\sin\theta = \frac{6}{10}$, find the value of $\cos\theta + \tan\theta$.

12. If $\tan\theta = \frac{4}{3}$, find the value of $\sin\theta + \cos\theta$ (both $\sin\theta$ and $\cos\theta$ are positive).

13. If $\text{cosec}\,\theta = \sqrt{5}$ and θ is less than 90°, find the value of $\cot\theta - \cos\theta$.

14. Given $\sin\theta = \frac{p}{q}$, find $\cos\theta + \sin\theta$ in terms of p and q.

15. If θ is an acute angle and $\tan\theta = \frac{8}{15}$, find the value of $\sec\theta + \text{cosec}\,\theta$.

16. Given A is an acute angle and 13 sin A = 5, evaluate :

$\frac{5\sin A - 2\cos A}{\tan A}$.

17. Given A is an acute angle and $\text{cosec}\,A = \sqrt{2}$, find the value of

$\frac{2\sin^2 A + 3\cot^2 A}{\tan^2 A - \cos^2 A}$.

18. The diagonals AC and BD of a rhombus ABCD meet at O. If AC = 8 cm and BD = 6 cm, find sin ∠OCD.

19. If $\tan\theta = \frac{5}{12}$, find the value of $\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}$.

20. Given 5 cos A – 12 sin A = 0, find the value of $\frac{\sin A + \cos A}{2\cos A - \sin A}$.

21. If $\tan\theta = \frac{p}{q}$, find the value of $\frac{p\sin\theta - q\cos\theta}{p\sin\theta + q\cos\theta}$.

22. If 3 cot θ = 4, find the value of $\frac{5\sin\theta - 3\cos\theta}{5\sin\theta + 3\cos\theta}$.

23. (*i*) If 5 cos θ – 12 sin θ = 0, find the value of $\frac{\sin\theta + \cos\theta}{2\cos\theta - \sin\theta}$.

(*ii*) If $\operatorname{cosec}\theta = \frac{13}{12}$, find the value of $\frac{2\sin\theta - 3\cos\theta}{4\sin\theta - 9\cos\theta}$.

Hint

(*i*) $5\cos\theta - 12\sin\theta = 0 \Rightarrow 5\cos\theta = 12\sin\theta \Rightarrow \cot\theta = \frac{12}{5}$.

(*ii*) $\cot^2\theta = \operatorname{cosec}^2\theta - 1 = \left(\frac{13}{12}\right)^2 - 1 = \frac{169}{144} - 1 = \frac{25}{144} \Rightarrow \cot\theta = \frac{5}{12}$.

24. If 5 sin θ = 3, find the value of $\frac{\sec\theta - \tan\theta}{\sec\theta + \tan\theta}$.

25. If θ is an acute angle and sin θ = cos θ, find the value of

$2\tan^2\theta + \sin^2\theta - 1$.

26. Prove the following :

(*i*) cos θ tan θ = sin θ (*ii*) sin θ cot θ = cos θ (*iii*) $\frac{\sin^2\theta}{\cos\theta} + \cos\theta = \frac{1}{\cos\theta}$.

27. If in Δ ABC, ∠C = 90° and $\tan A = \frac{3}{4}$, prove that

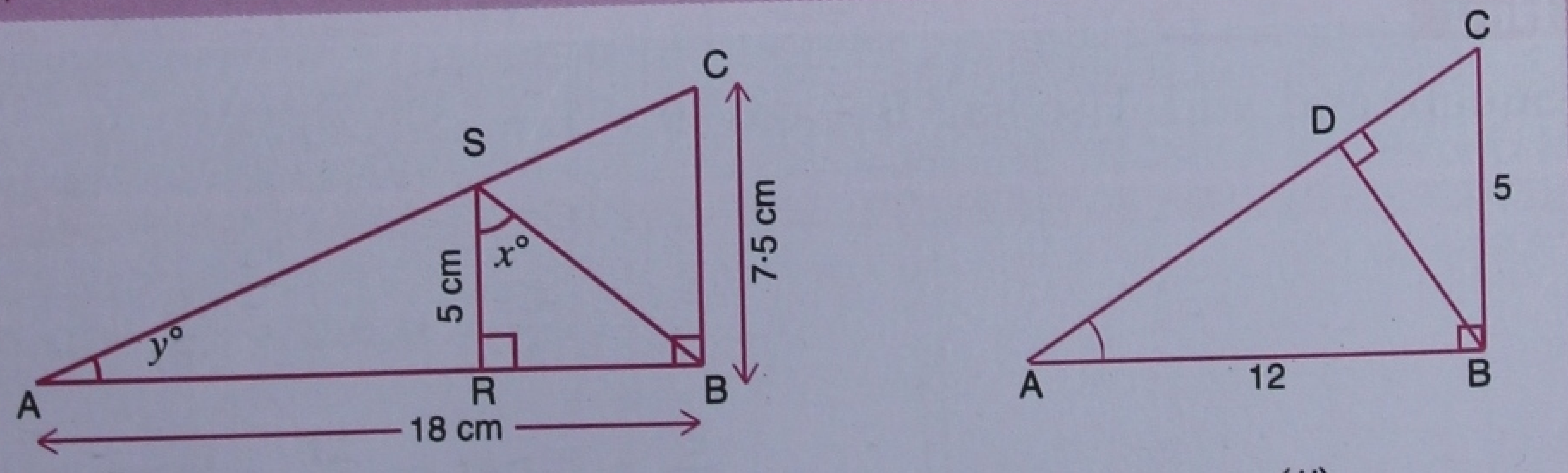
sin A cos B + cos A sin B = 1.

28. (*a*) In the figure (*i*) given below, Δ ABC is right-angled at B and Δ BRS is right-angled at R. If AB = 18 cm, BC = 7·5 cm, RS = 5 cm, ∠BSR = $x°$ and ∠SAB = $y°$, then find :

(*i*) tan $x°$ (*ii*) sin $y°$.

(*b*) In the figure (*ii*) given below, Δ ABC is right angled at B and BD is perpendicular to AC. Find

(*i*) cos ∠CBD (*ii*) cot ∠ABD.



(*i*) (*ii*)

Hint

(*a*) Δs ARS and ABC are similar,

$\therefore \frac{AR}{AB} = \frac{RS}{BC} \Rightarrow \frac{AR}{18} = \frac{5}{7 \cdot 5} \quad \Rightarrow AR = 12$ cm.

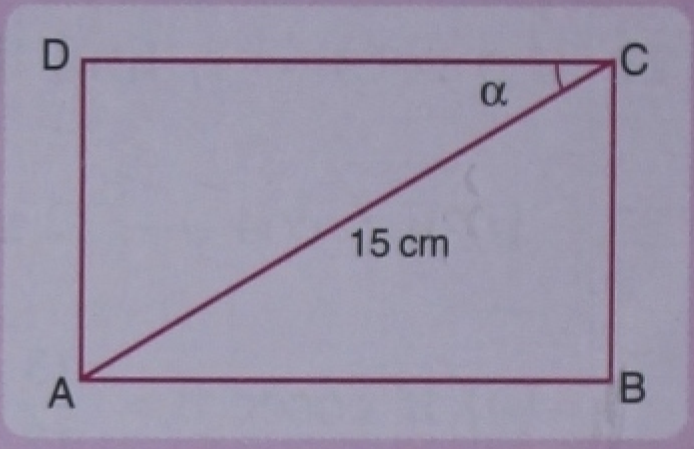
$\therefore$ RB = AB – AR = 18 cm – 12 cm = 6 cm.

Also $AC^2 = AB^2 + BC^2 = 18^2 + (7 \cdot 5)^2 = 380 \cdot 25$

$\Rightarrow$ AC = 19·5 cm.

(*b*) $\angle CBD = \angle A$ and $\angle ABD = \angle C$.

29. In the adjoining figure, ABCD is a rectangle. Its diagonal AC = 15 cm and $\angle ACD = \alpha$. If $\cot \alpha = \frac{3}{2}$, find the perimeter and the area of the rectangle.

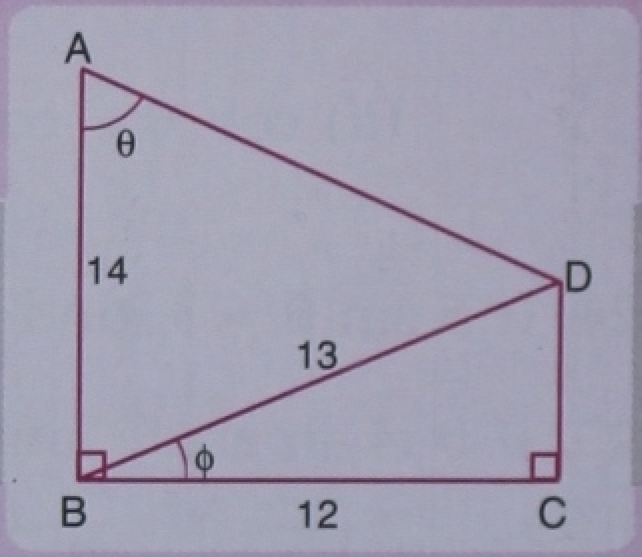


30. Using the measurements given in the figure alongside,

(*a*) find the values of :

(*i*) $\sin \phi$ (*ii*) $\tan \theta$.

(*b*) write an expression for AD in terms of θ.



Hint

(*b*) CD = 5. Draw DE perpendicular to AB, BE = 5, EA = 9.

31. Prove the following :

(*i*) $(\sin A + \cos A)^2 + (\sin A - \cos A)^2 = 2$

(*ii*) $\cot^2 A - \frac{1}{\sin^2 A} + 1 = 0$

(*iii*) $\frac{1}{1 + \tan^2 A} + \frac{1}{1 + \cot^2 A} = 1$

32. Simplify $\sqrt{\frac{1 - \sin^2 \theta}{1 - \cos^2 \theta}}$.

33. If $\sin \theta + \text{cosec}\, \theta = 2$, find the value of $\sin^2 \theta + \text{cosec}^2 \theta$.

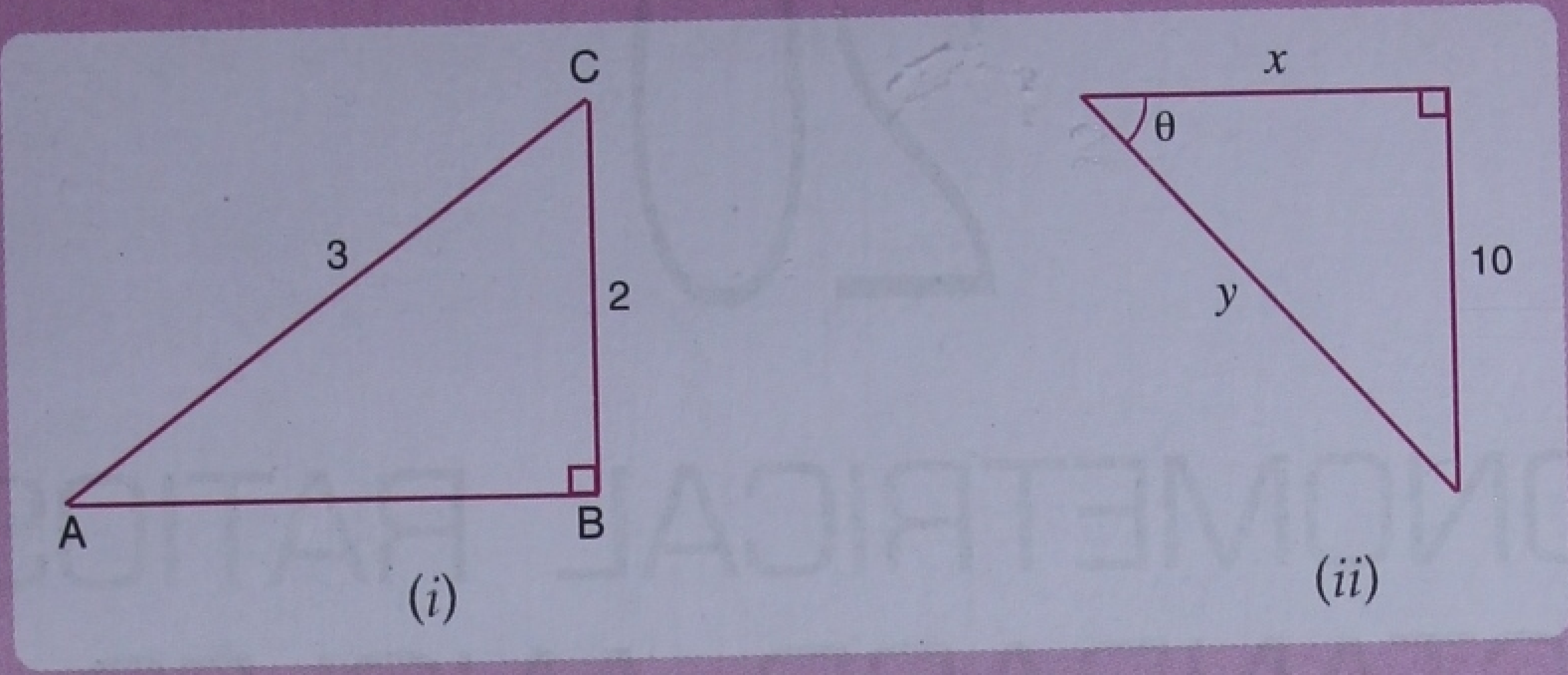
34. If $x = a \cos \theta + b \sin \theta$ and $y = a \sin \theta - b \cos \theta$, prove that $x^2 + y^2 = a^2 + b^2$.

Hint

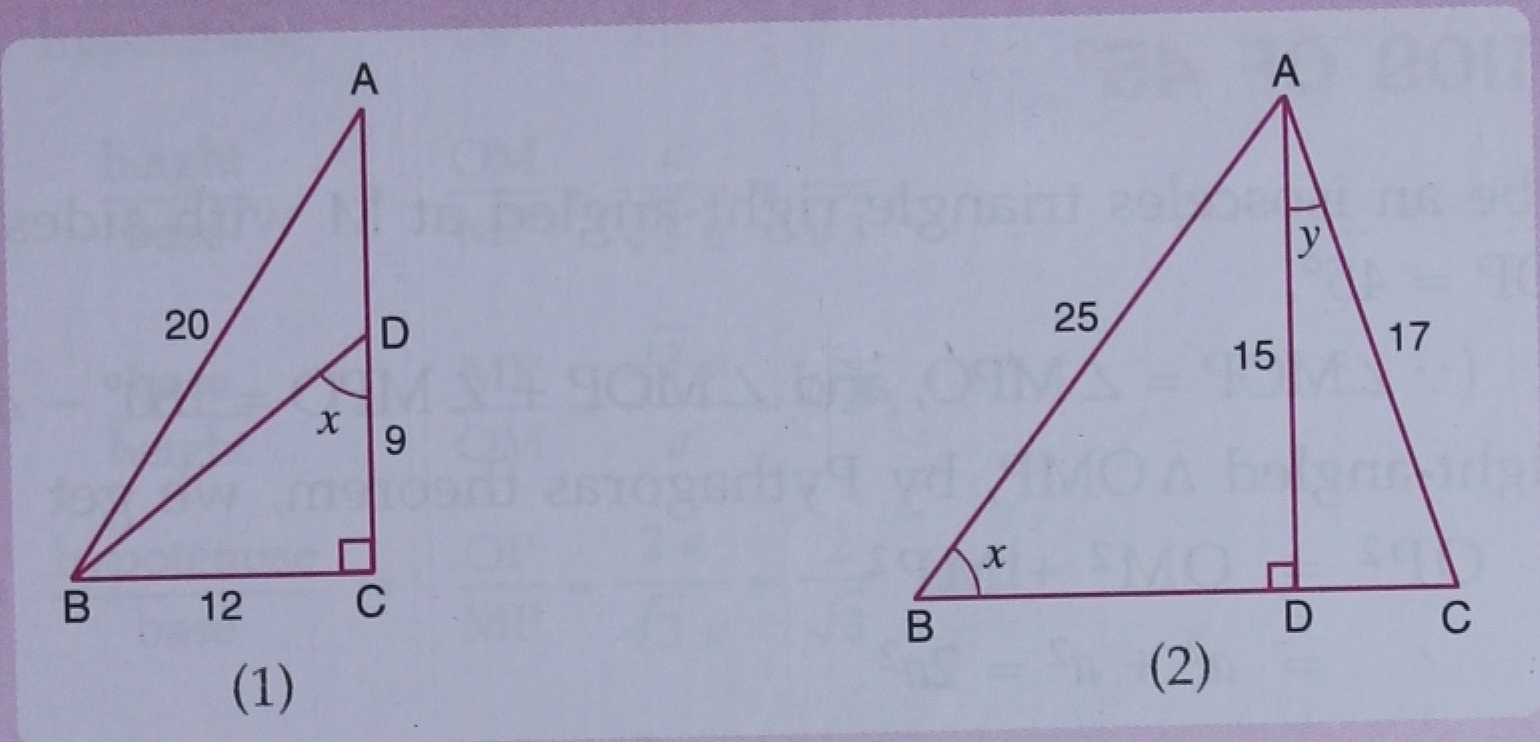
Square and add. Use $\sin^2 \theta + \cos^2 \theta = 1$.

CHAPTER TEST

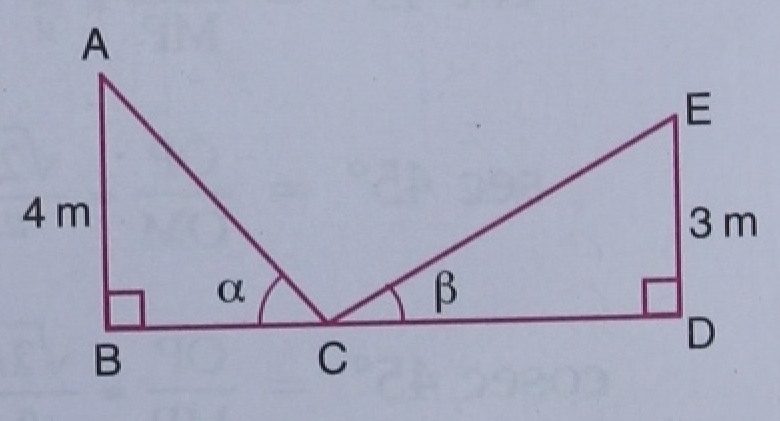
1. (*a*) From the figure (*i*) given below, calculate all the six *t*-ratios for both acute angles.
   (*b*) From the figure (*ii*) given below, find the values of x and y in terms of *t*-ratios of θ.



2. (*a*) From the figure (1) given below, find the values of :
   (*i*) $\sin \angle ABC$ (*ii*) $\tan x - \cos x + 3 \sin x$.
   (*b*) From the figure (2) given below, find the values of :
   (*i*) $5 \sin x$ (*ii*) $7 \tan x$ (*iii*) $5 \cos x - 17 \sin y - \tan x$.



3. If $q \cos \theta = p$, find $\tan \theta - \cot \theta$ in terms of p and q.
4. Given $4 \sin \theta = 3 \cos \theta$, find the values of :
   (*i*) $\sin \theta$ (*ii*) $\cos \theta$ (*iii*) $\cot^2 \theta - \text{cosec}^2 \theta$.
5. If $2 \cos \theta = \sqrt{3}$, prove that $3 \sin \theta - 4 \sin^3 \theta = 1$.
6. If $\frac{\sec \theta - \tan \theta}{\sec \theta + \tan \theta} = \frac{1}{4}$, find $\sin \theta$.
7. If $\sin \theta + \text{cosec}\, \theta = 3\frac{1}{3}$, find the value of $\sin^2 \theta + \text{cosec}^2 \theta$.
8. In the adjoining figure, AB = 4 m and ED = 3 m.
   If $\sin \alpha = \frac{3}{5}$ and $\cos \beta = \frac{12}{13}$, find the length of BD.



Hint

$\sin \alpha = \frac{3}{5} \Rightarrow \tan \alpha = \frac{3}{4}$.

Also $\cos \beta = \frac{12}{13} \Rightarrow \tan \beta = \frac{5}{12}$.