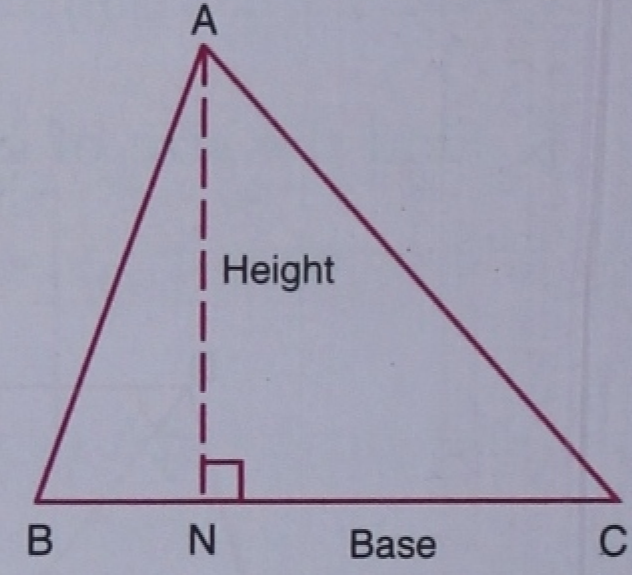
17

THEOREMS ON AREA

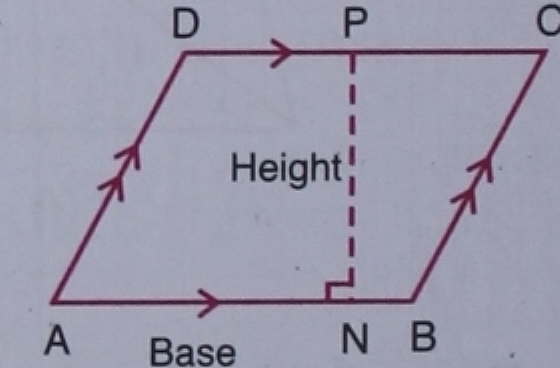
17.1 BASE AND HEIGHT

1. Base and height of a triangle. *Any side of a triangle is called its* ***base,*** *and the length of the perpendicular drawn from the opposite vertex to the base is called the* ***(corresponding) height.***



Let ABC be a triangle and BC be its base. From A, draw AN ⊥ BC, then the length of the line segment AN is the height corresponding to the base BC of Δ ABC.

2. Base and height of a parallelogram. *Any side of a parallelogram is called its* ***base,*** *and the length of perpendicular drawn from any point on the parallel side to the base is called the* ***(corresponding) height.***

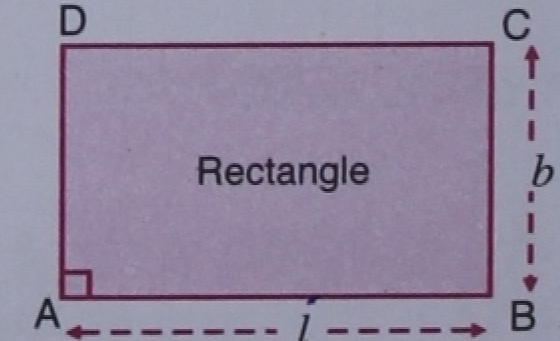


Let ABCD be a parallelogram and AB be its base. Let P be *any point* on CD, draw PN ⊥ AB, then the length of the line segment PN is the height corresponding to the base AB.

17.2 AXIOMS OF AREA

1. Rectangle area axiom.

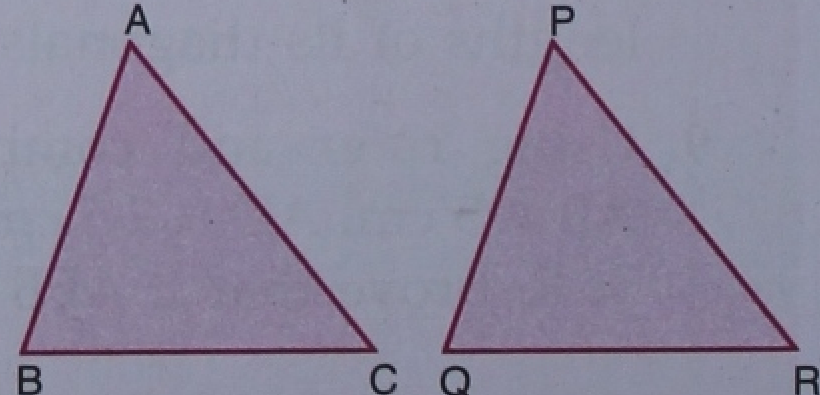
Area of a rectangular region = length × breadth.



In rectangle ABCD, let AB = l (length) and BC = b (breadth), then area of rectangle ABCD = AB × BC = $l \times b$.

2. Congruence area axiom.

If Δs ABC and PQR are congruent, then area of Δ ABC = area of Δ PQR.

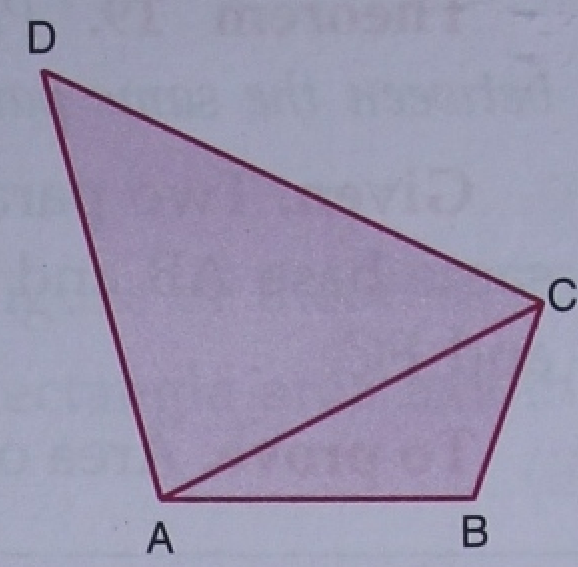


In general, if two polygons ABCDE … and PQRST … are congruent (*i.e.* duplicate of each other), then area of polygon region ABCDE … = area of polygon region PQRST ….

3. Addition area axiom.

If R_1, R_2 are two polygon regions which have no region in common (of course, these regions may have a finite number of points or line segments in common) and region $R = R_1 \cup R_2$, then area of region R

= area of region R_1 + area of region R_2.



Look at the adjoining figure. By the addition area axiom, we get

area of quad. ABCD = area of ΔABC + area of ΔACD.

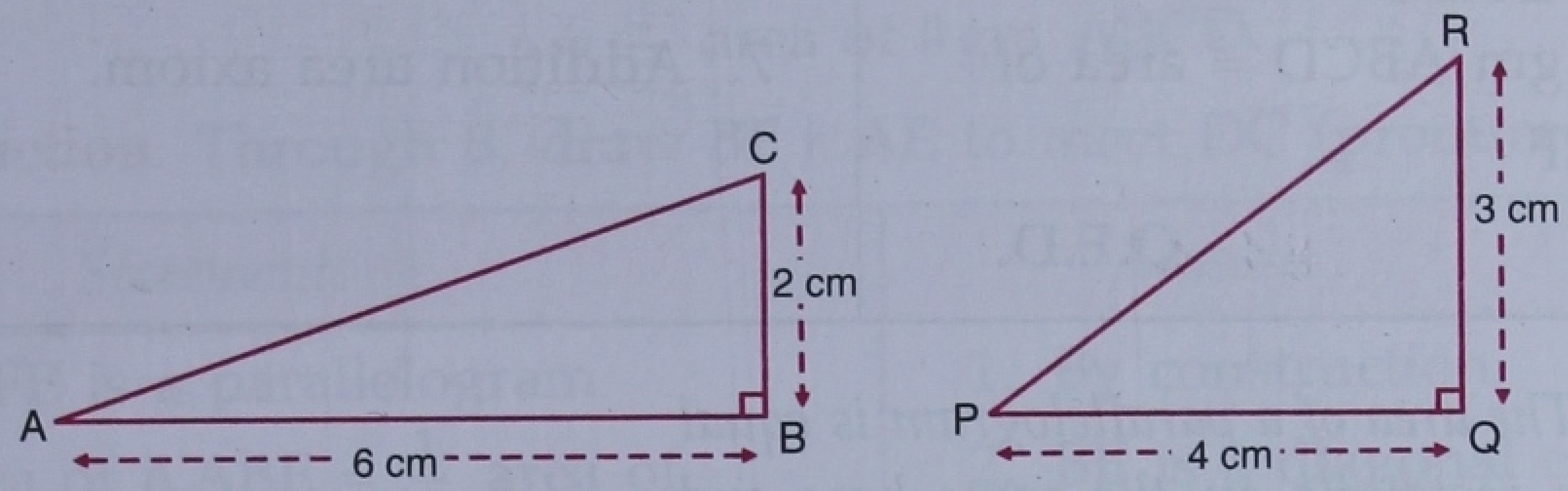
17.3 EQUAL FIGURES

Definition. *Two (plane) figures are called* ***equal*** *if and only if they have equal areas.*

Difference between congruent and equal figures.

Since congruent figures have equal areas (congruence area axiom), *congruent figures are equal figures.*

However, two equal figures may not be congruent. For example, consider the two right-angled triangles ABC and PQR given below :



Area of ΔABC $= \left(\frac{1}{2} \times 6 \times 2\right)$ cm^2 = 6 cm^2.

Area of ΔPQR $= \left(\frac{1}{2} \times 4 \times 3\right)$ cm^2 = 6 cm^2.

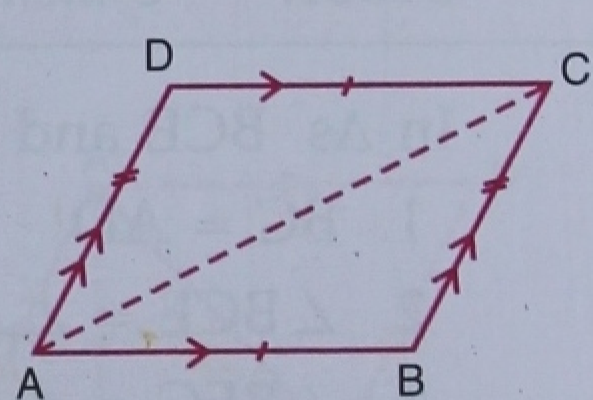
$\therefore$ Area of ΔABC = area of ΔPQR

$\Rightarrow$ Δs ABC and PQR are equal figures.

Clearly, Δs ABC and PQR are not congruent.

17.4 THEOREMS ON AREA

Theorem 18. *A diagonal of a parallelogram divides it into two triangles of equal areas.*

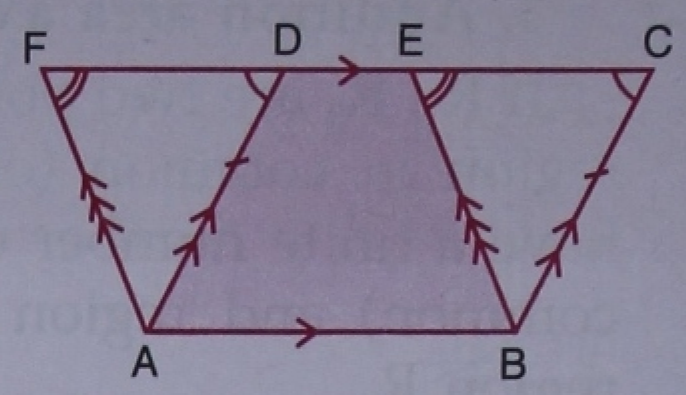


Given. ABCD is a parallelogram and AC is its one diagonal.

To prove. Area of ΔABC = area of ΔACD.

Proof. *Statements*	*Reasons*
In Δs ABC and CDA	
1. AB = DC	1. Opp. sides of ∥gm ABCD.
2. BC = AD	2. Opp. sides of ∥gm ABCD.
3. AC = AC	3. Common.
4. $\Delta ABC \cong \Delta CDA$	4. S.S.S. Axiom of congruency.
5. Area of ΔABC = area of ΔACD	5. Congruence area axiom.
Q.E.D.	

Theorem 19. *Parallelograms on the same base and between the same parallels are equal in area.*

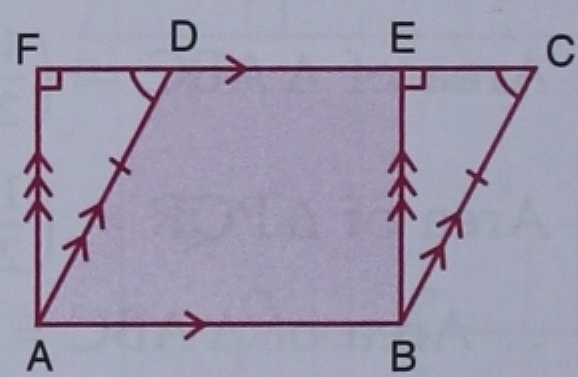


Given. Two parallelograms ABCD and ABEF on the same base AB and between the same parallel lines AB and FC.

To prove. Area of ∥gm ABCD = area of ∥gm ABEF.

Proof. *Statements*	*Reasons*
In Δs BCE and ADF	
1. BC = AD	1. Opp. sides of ∥gm ABCD.
2. ∠BCE = ∠ADF	2. BC ∥ AD, corres. angles.
3. ∠BEC = ∠AFD	3. BE ∥ AF, corres. angles.
4. ΔBCE ≅ Δ ADF	4. A.A.S. Axiom of congruency.
5. Area of ΔBCE = area of ΔADF	5. Congruence area axiom.
6. Area of quad. ABED + area of ΔBCE = area of quad. ABED + area of ΔADF	6. Adding same area on both sides.
7. Area of ∥gm ABCD = area of ∥gm ABEF	7. Addition area axiom.
Q.E.D.	

Theorem 20. *The area of a parallelogram is equal to the area of the rectangle on the same base and between the same parallels.*



Given. A parallelogram ABCD and the rectangle ABEF on the same base AB and between the same parallel lines AB and FC.

To prove. Area of ∥gm ABCD = area of rect. ABEF.

Proof. *Statements*	*Reasons*
In Δs BCE and ADF	
1. BC = AD	1. Opp. sides of ∥gm ABCD.
2. ∠BCE = ∠ADF	2. BC ∥ AD, corres. angles.
3. ∠BEC = ∠AFD	3. BE ∥ AF, corres. angles.
4. ΔBCE ≅ ΔADF	4. A.A.S. Axiom of congruency.
5. Area of ΔBCE = area of ΔADF	5. Congruence area axiom.
6. Area of quad. ABED + area of ΔBCE = area of quad. ABED + area of ΔADF	6. Adding same area on both sides.
7. Area of ∥gm ABCD = area of rect. ABEF	7. Addition area axiom.
Q.E.D.	

Corollary 1. *Area of a parallelogram = base × height.*

Proof. By the above theorem,

area of ‖gm ABCD = area of rectangle ABEF ...(*i*)

(See figure of theorem 20)

Also area of rectangle ABEF = length × breadth (Rectangle area axiom)

= AB × BE ...(*ii*)

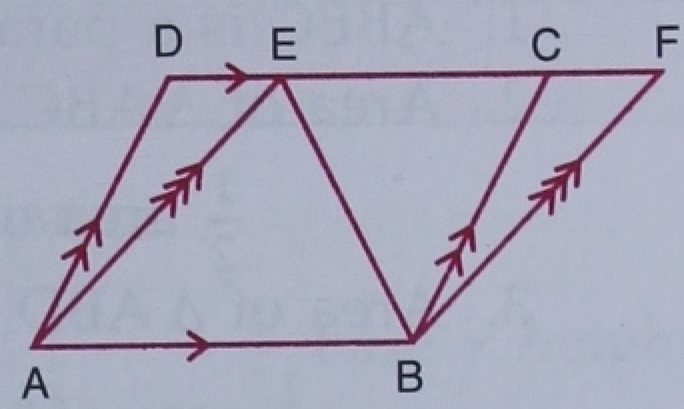
From (*i*) and (*ii*), we get

area of ‖gm ABCD = AB × BE = base × height.

Corollary 2. *Parallelograms with equal bases and between the same parallels are equal in area.*

Theorem 21. *Area of a triangle is half that of a parallelogram on the same base and between the same parallels.*

Given. A triangle ABE and a parallelogram ABCD on the same base AB and between the same parallel lines AB and DC.



To prove. Area of Δ ABE

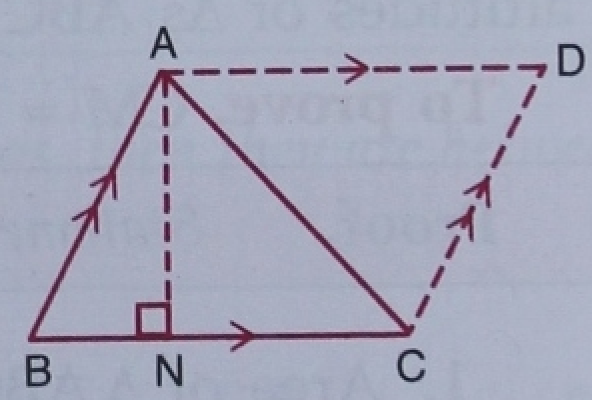
$= \frac{1}{2}$ area of ‖gm ABCD.

Construction. Through B, draw BF ‖ AE to meet DC (produced) at F.

Proof. *Statements*	*Reasons*
1. ABFE is a parallelogram	1. By construction.
2. Area of Δ ABE = $\frac{1}{2}$ area of ‖gm ABFE	2. BE is a diagonal of ‖gm ABFE, and a diagonal divides it into two triangles of equal areas.
3. Area of ‖gm ABCD = area of ‖gm ABFE	3. Parallelograms on the same base and between the same parallels are equal in area.
4. Area of Δ ABE = $\frac{1}{2}$ area of ‖gm ABCD **Q.E.D.**	4. From 2 and 3.

Corollary 1. *Area of a triangle* = $\frac{1}{2}$ *base × height.*

Given. A triangle ABC with BC as its base. AN ⊥ BC, so that height of Δ ABC = AN.



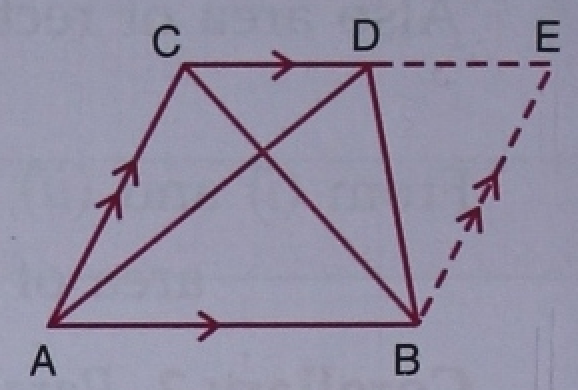
To prove. Area of Δ ABC = $\frac{1}{2}$ BC × AN.

Construction. Through A and C, draw straight lines parallel to BC and BA respectively to meet each other at D.

Proof. *Statements*	*Reasons*
1. ABCD is a parallelogram	1. By construction.
2. Area of Δ ABC = $\frac{1}{2}$ area of ‖gm ABCD	2. Area of a triangle is half that of a parallelogram on the same base and between the same parallels.

3. Area of ∥gm ABCD = BC × AN	3. Area of a ∥gm = base × height.
4. Area of Δ ABC = $\frac{1}{2}$ BC × AN	4. From 2 and 3.
Q.E.D.	

Corollary 2. *Triangles on the same base and between the same parallels are equal in area.*



Given. Two triangles ABC and ABD on the same base AB, and between the same parallel lines AB and CD.

To prove. Area of Δ ABC = area of Δ ABD.

Construction. Through B, draw a st. line parallel to AC to meet CD (produced) at point E.

Proof. *Statements*	*Reasons*
1. ABEC is a parallelogram	1. By construction.
2. Area of Δ ABC = $\frac{1}{2}$ area of ∥gm ABEC	2. Diagonal divides a ∥gm into two Δs of equal area.
3. Area of Δ ABD = $\frac{1}{2}$ area of ∥gm ABEC	3. Area of a Δ is half that of a ∥gm on the same base and between same parallels.
4. Area of Δ ABC = area of Δ ABD	4. From 2 and 3.
Q.E.D.	

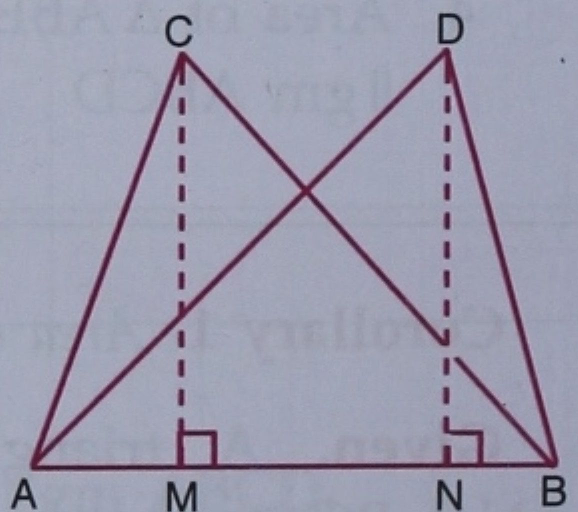
Corollary 3. *If a triangle and a parallelogram have equal bases and are between the same parallels, then the area of triangle is half the area of parallelogram.*

Corollary 4. *Triangles with equal bases and between the same parallels are equal in area.*

Corollary 5. *If two triangles lie between the same parallels (i.e. have equal altitudes), the ratio of their areas equals the ratio of their bases.*

Corollary 6. *If two triangles have equal bases, the ratio of their areas equals the ratio of their altitudes.*

Theorem 22. *Triangles with equal areas on the same base have equal corresponding altitudes.*



Given. Two triangles ABC and ABD on the same AB, and area of Δ ABC = area of Δ ABD. CM and DN are altitudes of Δs ABC and ABD respectively.

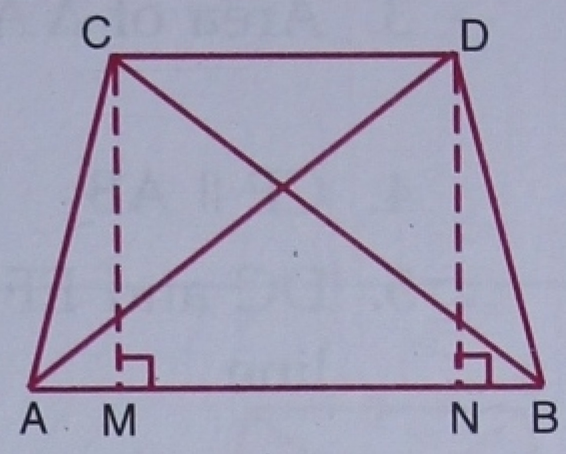
To prove. CM = DN.

Proof. *Statements*	*Reasons*
1. Area of Δ ABC = $\frac{1}{2}$ AB × CM	1. Area of a triangle = $\frac{1}{2}$ base × height
2. Area of Δ ABD = $\frac{1}{2}$ AB × DN	2. Same as above.
3. $\frac{1}{2}$ AB × CM = $\frac{1}{2}$ AB × DN	3. Area of Δ ABC = area of Δ ABD (given)
4. CM = DN	4. From 3, cancelling $\frac{1}{2}$ AB.
Q.E.D.	

Corollary 1. *Triangles with equal areas and on equal bases have equal corresponding altitudes.*

Corollary 2. *Parallelograms with equal areas and on equal bases have equal corresponding altitudes.*

Corollary 3. *If two triangles with same base have equal areas, then the line joining their vertices is parallel to their common base.*



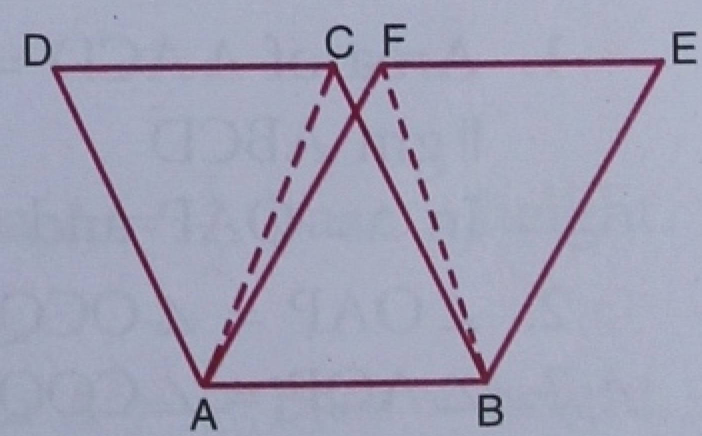
Given. Two triangles ABC and ABD on the same base AB, and area of Δ ABC = area of Δ ABD.

To prove. CD ∥ AB.

Construction. From C and D, draw perpendiculars CM and DN on AB respectively.

Proof. *Statements*	*Reasons*
1. Area of a Δ ABC = $\frac{1}{2}$ AB × CM	1. Area of a triangle = $\frac{1}{2}$ base × height.
2. Area of Δ ABD = $\frac{1}{2}$ AB × DN	2. Same as above.
3. $\frac{1}{2}$ AB × CM = $\frac{1}{2}$ AB × DN	3. Area of Δ ABC = area of Δ ABD (given)
4. CM = DN	4. From 3, cancelling $\frac{1}{2}$ AB.
5. CM ∥ DN	5. CM and DN are both perpendiculars to the same line AB.
6. CMND is a parallelogram	6. Two sides CM and DN of quad. CMND are equal and parallel. (Theorem 17)
7. CD ∥ MN *i.e.* CD ∥ AB **Q.E.D.**	7. By definition of a ∥gm.

Corollary 4. *If two parallelograms with same base have equal areas, then they are between same parallels.*



Given. Two parallelograms ABCD and ABEF, on the same base AB and area of ∥gm ABCD = area of ∥gm ABEF.

To prove. DC and FE lie along same straight line.

Construction. Join AC and BF.

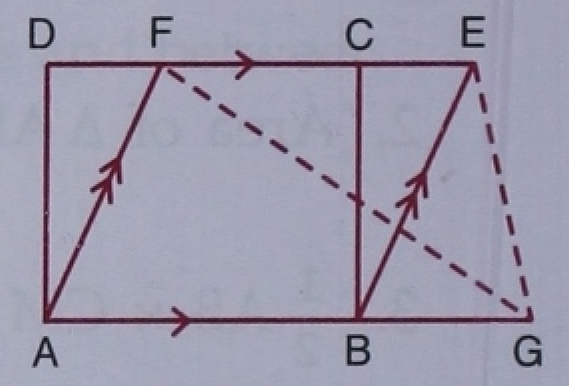
Proof. *Statements*	*Reasons*
1. Area of $\Delta ABC = \frac{1}{2}$ area of $\parallel$gm ABCD	1. Theorem 21.
2. Area of $\Delta ABF = \frac{1}{2}$ area of $\parallel$gm ABEF	2. Same as above.
3. Area of ΔABC = area of ΔABF	3. area of $\parallel$gm ABCD = area of $\parallel$gm ABEF
4. CF $\parallel$ AB	4. By corollary 3.
5. DC and EF lie along same line	5. DC and EF have common points C and F. Therefore, DC, CF and FE are coincident.
Q.E.D	

ILLUSTRATIVE EXAMPLES

Example 1. *In the adjoining diagram, ABCD is a rectangle with sides AB = 8 cm and AD = 5 cm. Compute*

(i) area of parallelogram ABEF

(ii) area of ΔEFG.



Solution. (*i*) Area of $\parallel$gm ABEF

= area of rectangle ABCD

(on the same base AB and between the same parallels AB and DE)

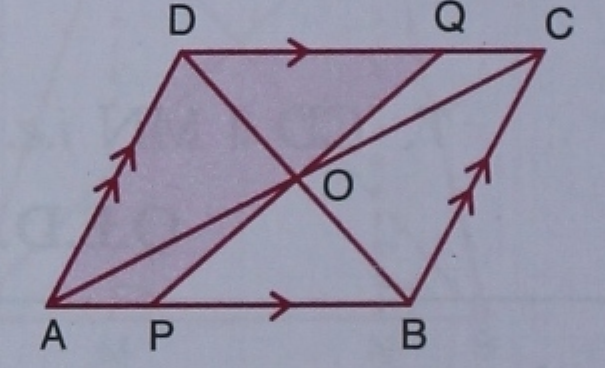
$= (8 \times 5)\ \text{cm}^2 = 40\ \text{cm}^2$.

(*ii*) Area of $\Delta EFG = \frac{1}{2}$ area of $\parallel$gm ABEF

(on the same base FE and between the same parallels FE and AG)

$= \left(\frac{1}{2} \times 40\right)\ \text{cm}^2 = 20\ \text{cm}^2$.

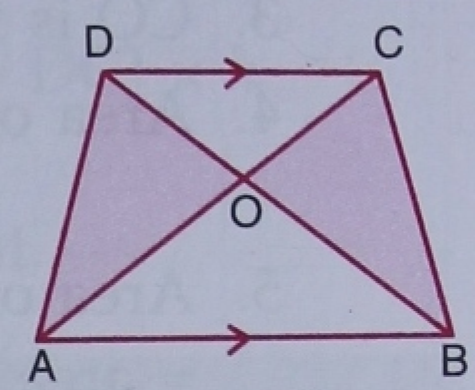
Example 2. *The diagonals of a parallelogram ABCD intersect at O. A st. line through O meets AB at P and the opposite side CD at Q. Prove that area of quad. APQD = $\frac{1}{2}$ area of $\parallel$gm ABCD.*



Proof. *Statements*	*Reasons*
1. Area of $\Delta ACD = \frac{1}{2}$ area of $\parallel$gm ABCD In Δs OAP and OCQ	1. Diagonal divides a $\parallel$gm into two Δs of equal area.
2. $\angle OAP = \angle OCQ$	2. Alt. ∠s.
3. $\angle AOP = \angle COQ$	3. Vert. opp. ∠s.
4. AO = OC	4. Diagonals bisect each other.
5. $\Delta OAP \cong \Delta OCQ$	5. A.S.A. Axiom of congruency.

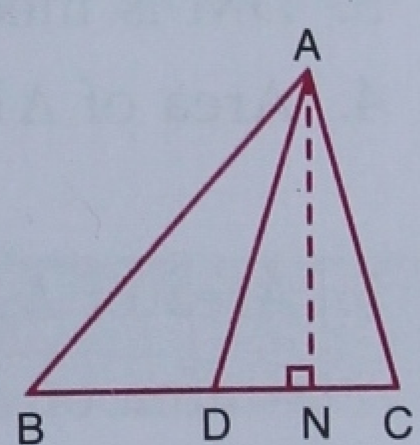
6. Area of ΔOAP = area of ΔOCQ	6. Congruence area axiom.
7. Area of ΔOAP + area of quad. AOQD = area of ΔOCQ + area of quad. AOQD	7. Adding same area on both sides.
8. Area of quad. APQD = area of ΔACD	8. Addition area axiom.
9. Area of quad. APQD = $\frac{1}{2}$ area of ∥gm ABCD **Q.E.D.**	9. From 8 and 1.

Example 3. *ABCD is a trapezium with AB ∥ DC, and diagonals AC and BD meet at O. Prove that area of ΔDAO = area of ΔOBC.*



Proof. *Statements*	*Reasons*
1. AB ∥ DC	1. Given.
2. Area of ΔABD = area of ΔABC	2. Δs on the same base and between the same parallels are equal in area.
3. Area of ΔDAO + area of ΔOAB = area of ΔOBC + area of ΔOAB	3. Addition area axiom.
4. Area of ΔDAO = area of ΔOBC **Q.E.D.**	4. Subtracting same area from both sides.

Example 4. *Prove that a median divides a triangle into two triangles of equal area.*



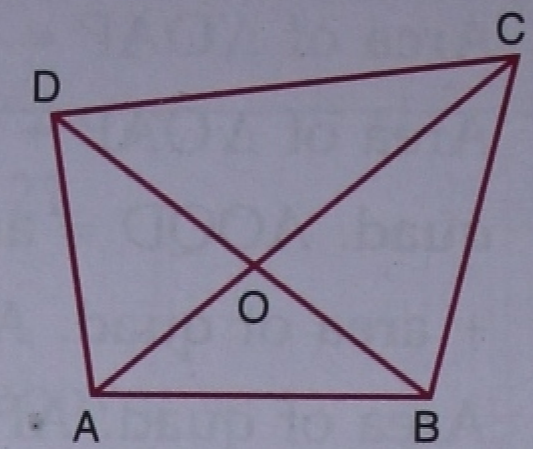
Given. A triangle ABC and AD is a median *i.e.* D is mid-point of BC.

To prove. Area of ΔABD = area of ΔADC.

Construction. From A, draw AN ⊥ BC.

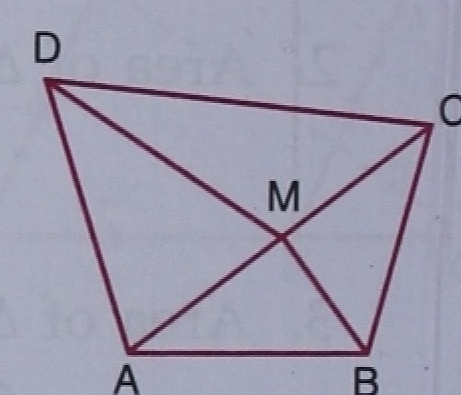
Proof. *Statements*	*Reasons*
1. BD = DC	1. Given.
2. Area of ΔABD = $\frac{1}{2}$ BD × AN	2. Area of a Δ = $\frac{1}{2}$ base × height.
3. Area of ΔADC = $\frac{1}{2}$ DC × AN	3. Area of a Δ = $\frac{1}{2}$ base × height.
4. Area of ΔABD = area of ΔADC **Q.E.D.**	4. From 1, 2 and 3.

Example 5. *The diagonals AC and BD of a quadrilateral ABCD intersect at O. If OB = OD, prove that the triangles ABC and ACD are equal in area.*



Proof. *Statements*	*Reasons*
1.. AO is median of Δ ABD	1. OB = OD (given).
2. Area of Δ OAB = area of Δ OAD	2. Median divides a Δ into two Δ s of equal area.
3. CO is median of Δ CBD	3. OB = OD (given)
4. Area of Δ OBC = area of Δ OCD	4. Median divides a Δ into two Δ s of equal area.
5. Area of Δ OAB + area of Δ OBC = area of Δ OAD + area of Δ OCD	5. Adding 2 and 4
6. Area of Δ ABC = area of Δ ACD **Q.E.D.**	6. Addition area axiom.

Example 6. *In quadrilateral ABCD, M is mid-point of the diagonal AC. Prove that area of quad. ABMD = area of quad. DMBC.*



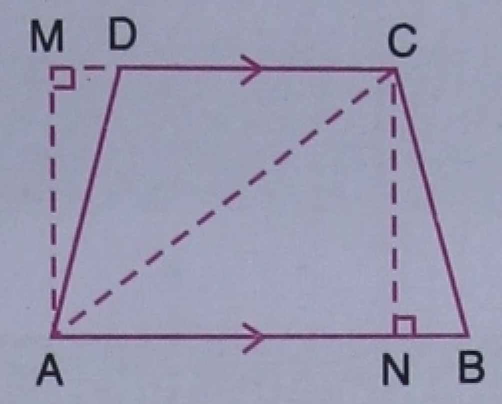
Proof. *Statements*	*Reasons*
1. BM is median of Δ BCA	1. M is mid-point of AC (given).
2. Area of Δ ABM = area of Δ MBC	2. Median divides a Δ into two Δs of equal area.
3. DM is median of Δ DAC	3. M is mid-point of AC (given).
4. Area of Δ DAM = area of Δ DMC	4. Median divides a triangle into two Δs of equal area.
5. Area of Δ ABM + area of Δ DAM = area of Δ MBC + area of Δ DMC	5. Adding 2 and 4.
6. Area of quad. ABMD = area of quad. DMBC **Q.E.D.**	6. Addition area axiom.

Example 7. *Prove that area of a trapezium =* $\frac{1}{2}$ *(sum of parallel sides) × height.*

Solution. Let ABCD be a trapezium with AB ∥ DC.

Join AC. From C, draw CN ⊥ AB and from A, draw AM ⊥ CD (produced if necessary). Then CN = AM = height of trapezium = h (say).

Since area of a triangle = $\frac{1}{2}$ base × height,



$\therefore$ area of $\Delta ABC = \frac{1}{2}\ AB \times CN$ ...(*i*)

and area of $\Delta ACD = \frac{1}{2}\ DC \times AM$ ...(*ii*)

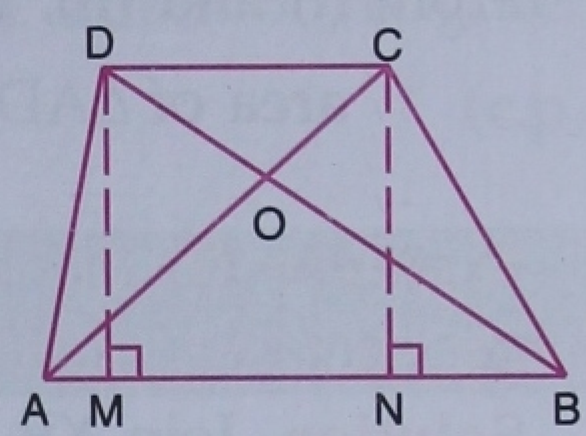
On adding (*i*) and (*ii*), we get

area of ΔABC + area of $\Delta ACD = \frac{1}{2}\ AB \times CN + \frac{1}{2}\ DC \times AM$

$\Rightarrow$ area of trapezium $ABCD = \frac{1}{2}\ (AB \times h + DC \times h) = \frac{1}{2}\ (AB + DC) \times h$

$= \frac{1}{2}$ (sum of parallel sides) × height.

Example 8. *The diagonals AC and BD of a quadrilateral ABCD intersect at O in such a way that area of ΔOAD = area of ΔOBC. Prove that ABCD is a trapezium.*



Solution. Draw $DM \perp AB$ and $CN \perp AB$.

As DM and CN are both perpendiculars to AB, therefore, DM ∥ CN.

Given area of ΔOAD = area of ΔOBC

$\Rightarrow$ area of ΔOAD + area of ΔOAB

= area of ΔOBC + area of ΔOAB (adding same area on both sides)

$\Rightarrow$ area of ΔABD = area of ΔABC

$\Rightarrow \frac{1}{2} AB \times DM = \frac{1}{2} AB \times CN$

$\Rightarrow$ DM = CN.

Thus DM ∥ CN and DM = CN, therefore, DMNC is a parallelogram (Theorem 17)

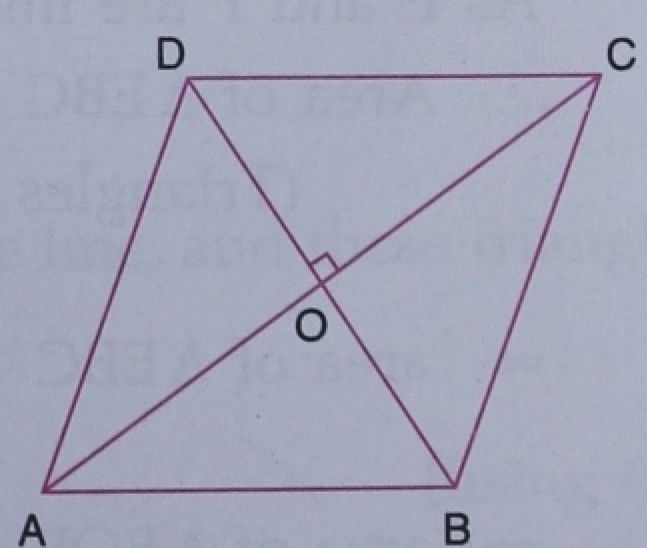
$\Rightarrow$ DC ∥ MN *i.e.* DC ∥ AB.

Hence ABCD is a trapezium.

Example 9. *Prove that area of a rhombus = $\frac{1}{2}$ × product of diagonals.*

Solution. Let ABCD be a rhombus, and let its diagonals intersect at O.

Since the diagonals of a rhombus cut at right angles, $OB \perp AC$ and $OD \perp AC$.



As area of a triangle = $\frac{1}{2}$ base × height,

$\therefore$ area of $\Delta ABC = \frac{1}{2}\ AC \times OB$ ...(*i*)

and area of $\Delta ACD = \frac{1}{2}\ AC \times OD$ ...(*ii*)

On adding (*i*) and (*ii*), we get

area of ΔABC + area of $\Delta ACD = \frac{1}{2}\ AC \times OB + \frac{1}{2}\ AC \times OD$

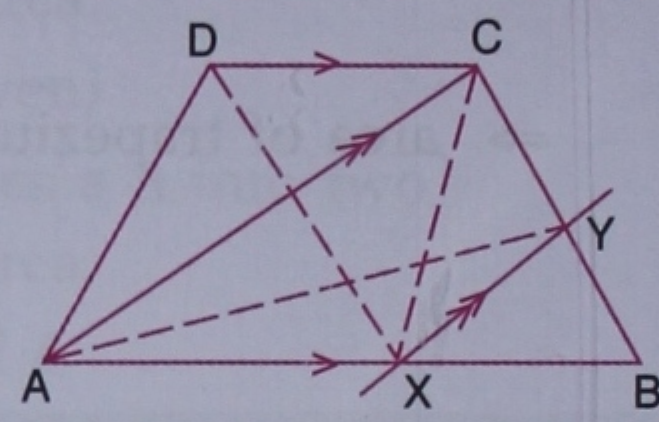
$\Rightarrow$ area of rhombus ABCD $= \frac{1}{2}$ AC $\times$ (OB + OD)

$= \frac{1}{2}$ AC $\times$ BD

$= \frac{1}{2} \times$ product of diagonals.

Example 10. *ABCD is a trapezium with AB ∥ DC. A line parallel to AC intersects AB at X and BC at Y. Prove that: area of ΔADX = area of ΔACY.*

Solution. Join CX.



As triangles ADX and ACX have same base AX and are between the same parallels (AB ∥ DC given, so, AX ∥ DC),

$\therefore$ area of ΔADX = area of ΔACX ...(*i*)

As triangles ACY and ACX have same base AC and are between the same parallels (XY ∥ AC given),

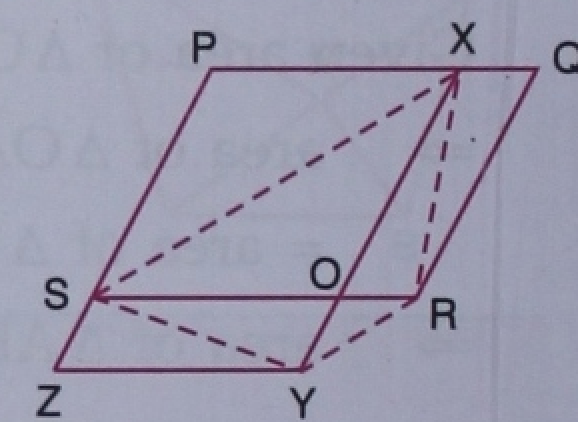
$\therefore$ area of ΔACY = area of ΔACX ...(*ii*)

From (*i*) and (*ii*), we get

area of ΔADX = area of ΔACY.

Example 11. *In the figure, PQRS and PXYZ are two parallelograms of equal area. Prove that SX is parallel to YR.*

Solution. Join XR, SY.



Given area of ∥gm PQSR = area of ∥gm PXYZ.

Subtract area of ∥gm PSOX from both sides.

$\therefore$ Area of ∥gm XORQ = area of ∥gm SZYO

$\Rightarrow$ area of Δ XOR = area of Δ SYO

(because diagonal divides a ∥gm into two equal areas)

Adding area of Δ OYR to both sides, we get

area of Δ XYR = area of Δ SYR.

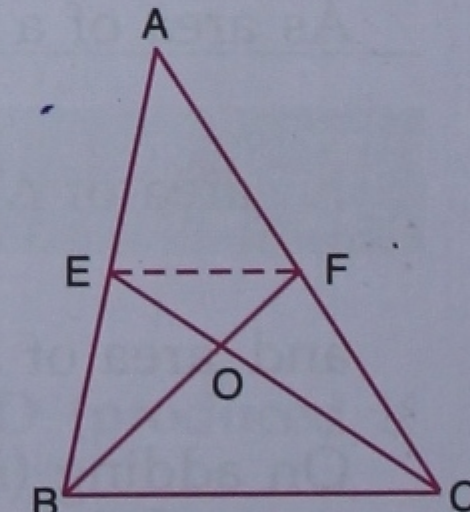
Also the Δs XYR and SYR have the same base YR, therefore, these lie between the same parallels (Cor 3 to theorem 22)

$\Rightarrow$ SX is parallel to YR.

Example 12. *E and F are mid-points of the sides AB and AC respectively of a triangle ABC. If BF and CE meet at O, prove that area of Δ OBC = area of quad. AEOF.*

Solution. Join EF.

As E and F are mid-points of AB and AC respectively, EF ∥ BC.



$\therefore$ Area of Δ EBC = area of Δ FBC.

(Triangles on the same base BC and between same parallels)

$\Rightarrow$ area of Δ EBC – area of Δ OBC

= area of Δ FBC – area of Δ OBC

$\Rightarrow$ area of Δ BOE = area of Δ COF ...(*i*)

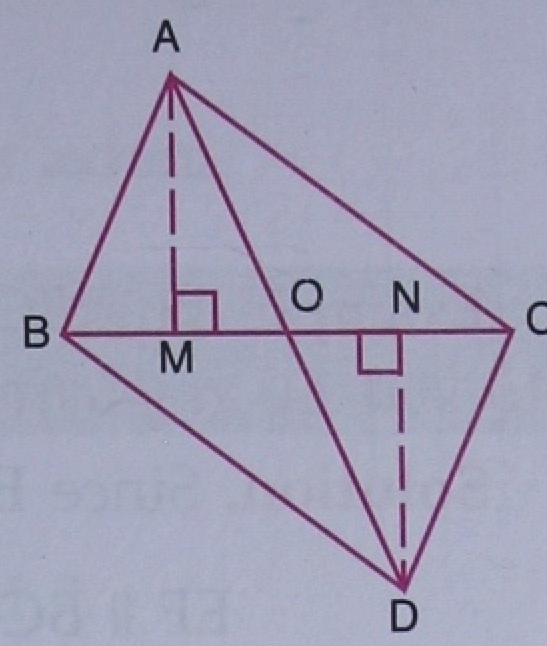
As F is mid-point of AC, area of Δ FBC = area of Δ ABF

($\because$ A median divides a triangle into two triangles of equal area).

$\Rightarrow$ area of Δ FBC – area of Δ COF = area of Δ ABF – area of Δ BOE [using (*i*)]

$\Rightarrow$ area of Δ OBC = area of quad. AEOF (from figure)

Example 13. *Triangles ABC and DBC are on the same base BC with A, D on opposite sides of BC. If area of Δ ABC = area of Δ DBC, prove that BC bisects AD.*



Solution. Let BC and AD intersect at O.

Draw AM $\perp$ BC and DN $\perp$ BC.

Given area of Δ ABC = area of Δ DBC

$\Rightarrow$ $\frac{1}{2}BC \times AM = \frac{1}{2}BC \times DN$

$\Rightarrow$ AM = DN.

In Δs AMO and DNO,

$\angle$ AOM = $\angle$ DON (vert. opp. $\angle$ s)

$\angle$ AMO = $\angle$ DNO (each angle = 90°)

AM = DN (proved above)

$\therefore$ Δ AMO $\cong$ Δ DNO (A.A.S. axiom of congruency)

$\therefore$ AO = DO (c.p.c.t.)

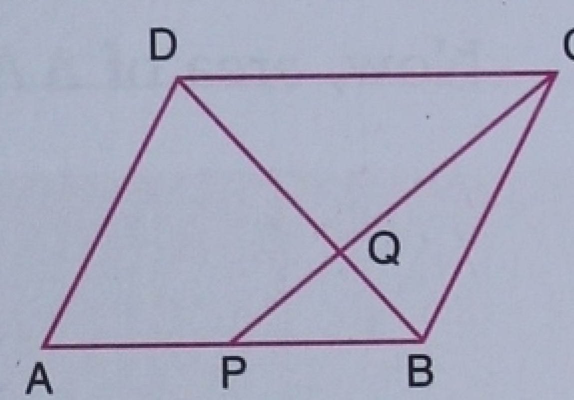
Hence BC bisects AD.

Example 14. *In the adjoining figure, ABCD is a parallelogram. P is mid-point of AB and CP meets the diagonal BD at Q. If area of Δ PBQ = 10 cm², calculate*

(i) PQ : QC

(ii) area of Δ PBC

(iii) area of parallelogram ABCD.



Solution. (*i*) Since P is mid-point of AB, PB = $\frac{1}{2}$ AB.

But AB = DC ($\because$ ABCD is a $\parallel$gm) $\Rightarrow$ PB = $\frac{1}{2}$ DC ...(1)

In Δs PBQ, CDQ

$\angle$ PQB = $\angle$ DQC (vert. opp. $\angle$ s) and $\angle$ PBQ = $\angle$ QDC (alt. $\angle$ s)

$\Rightarrow$ Δ PBQ ~ Δ CDQ

$\therefore$ $\frac{PQ}{QC} = \frac{PB}{DC} = \frac{1}{2}$ (using (1))

$\Rightarrow$ PQ : QC = 1 : 2.

(*ii*) PQ : QC = 1 : 2 $\Rightarrow$ PQ : PC = 1 : 3 ...(2)

Since the bases CP, QP of Δs PBC, PBQ lie along the same line, and these triangles have equal heights, therefore,

$$\frac{\text{area of } \Delta \text{ PBC}}{\text{area of } \Delta \text{ PBQ}} = \frac{PC}{PQ} = \frac{3}{1}$$ (using (2))

$\Rightarrow$ area of Δ PBC = 3 × area of Δ PBQ = (3 × 10) cm² = 30 cm².

(*iii*) Area of ΔABC = 2 × area of ΔPBC

($\because$ median of a triangle divides it into two triangles of equal areas)

$= (2 \times 30)\ \text{cm}^2 = 60\ \text{cm}^2.$

Area of ||gm ABCD = 2 × area of ΔABC.

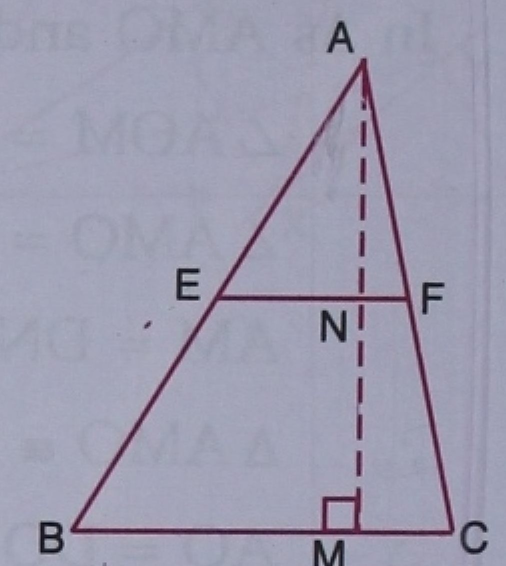
($\because$ diagonal divides a ||gm into two triangles of equal areas)

$= (2 \times 60)\ \text{cm}^2 = 120\ \text{cm}^2.$

Example 15. *ABC is a triangle whose area is 50 cm². E and F are mid-points of the sides AB and AC respectively. Prove that EBCF is a trapezium. Also find its area.*

Solution. Since E and F are mid-points of the sides AB and AC respectively,

EF || BC and $EF = \frac{1}{2} BC$.



As EF || BC, EBCF is a trapezium.

From A, draw AM ⊥ BC.

Let AM meet EF at N.

Since EF || BC, ∠ENA = ∠BMN.

But ∠BMN = 90° ($\because$ AM ⊥ BC),

so ∠ENA = 90° *i.e.* AN ⊥ EF.

Also, as E is mid-point of AB and EN || BM, N is mid-point of AM.

Now, area of $\Delta AEF = \frac{1}{2} EF \times AN = \frac{1}{2}\left(\frac{1}{2} BC \times \frac{1}{2} AM\right)$

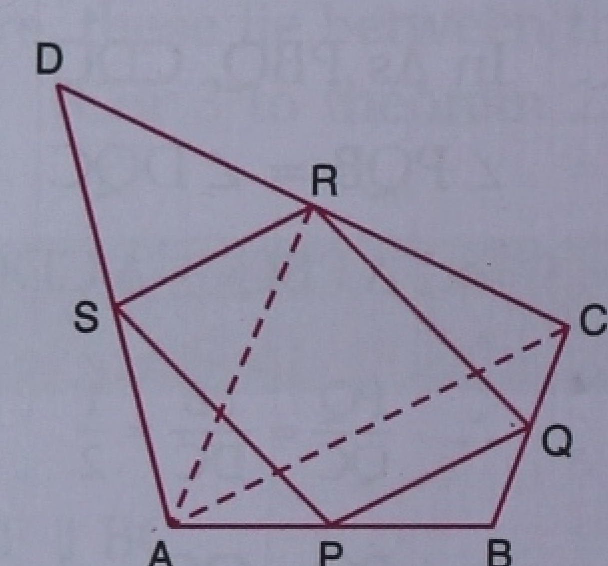
$= \frac{1}{4}\left(\frac{1}{2} BC \times AM\right) = \frac{1}{4}(\text{area of } \Delta ABC)$

$= \frac{1}{4}(50\ \text{cm}^2) = 12{\cdot}5\ \text{cm}^2.$

$\therefore$ Area of trapezium EBCF = area of ΔABC − area of ΔAEF

$= 50\ \text{cm}^2 - 12{\cdot}5\ \text{cm}^2 = 37{\cdot}5\ \text{cm}^2.$

Example 16. *Prove that the area of the quadrilateral formed by joining the mid-points of the adjacent sides of a quadrilateral is half the area of the given quadrilateral.*



Given. A quadrilateral ABCD, and PQRS is the quadrilateral formed by joining mid-points of the sides AB, BC, CD and DA respectively.

To prove. Area of quad. PQRS = $\frac{1}{2}$ area of quad. ABCD.

Construction. Join AC and AR.

Proof. *Statements*	*Reasons*
1. Area of ΔARD = $\frac{1}{2}$ area of ΔACD	1. Median divides a triangle into two triangles of equal area.
2. Area of ΔSRD = $\frac{1}{2}$ area of ΔARD	2. Same as in 1.

3. Area of Δ SRD = $\frac{1}{4}$ area of Δ ACD	3. From 1 and 2.
4. Area of Δ PBQ = $\frac{1}{4}$ area of Δ ABC	4. As in 3.
5. Area of Δ SRD + area of Δ PBQ = $\frac{1}{4}$ (area of Δ ACD + area of Δ ABC)	5. Adding 3 and 4.
6. Area of Δ SRD + area of Δ PBQ = $\frac{1}{4}$ area of quad. ABCD	6. Addition area axiom.
7. Area of Δ APS + area of Δ QCR = $\frac{1}{4}$ area of quad. ABCD	7. Same as in 6.
8. Area of Δ APS + area of Δ PBQ + area of Δ QCR + area of Δ SRD = $\frac{1}{2}$ area of quad. ABCD	8. Adding 6 and 7.
9. Area of Δ APS + area of Δ PBQ + area of Δ QCR + area of Δ SRD + area of quad. PQRS = area of quad. ABCD	9. Addition area axiom.
10. Area of quad. PQRS = $\frac{1}{2}$ area of quad. ABCD **Q.E.D.**	10. Subtracting 8 from 9.

Exercise 17

1. Prove that the line segment joining the mid-points of a pair of opposite sides of a parallelogram divides it into two equal parallelograms.

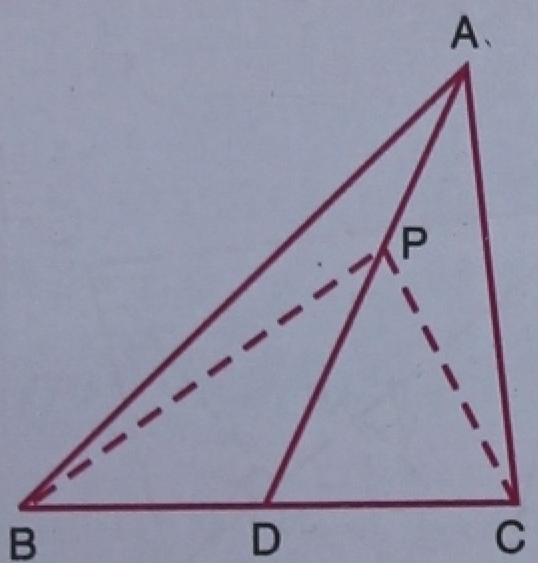
2. Prove that the diagonals of a parallelogram divide it into four triangles of equal area.

3. (*a*) In the figure (1) given below, AD is median of Δ ABC and P is any point on AD. Prove that

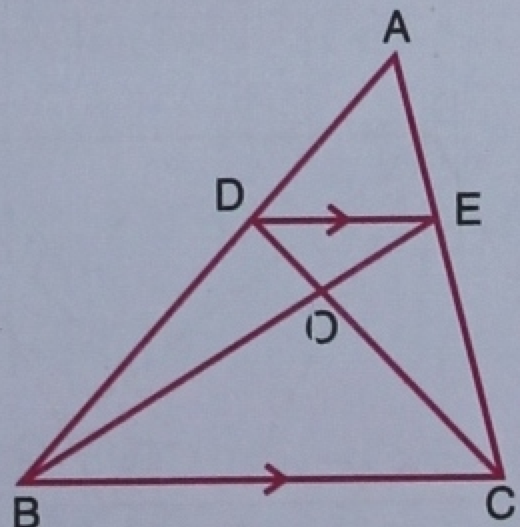
(*i*) area of Δ PBD = area of Δ PDC (*ii*) area of Δ ABP = area of Δ ACP.

(*b*) In the figure (2) given below, DE ∥ BC. Prove that

(*i*) area of Δ ACD = area of Δ ABE (*ii*) area of Δ OBD = area of Δ OCE.



(1)



(2)

Hint

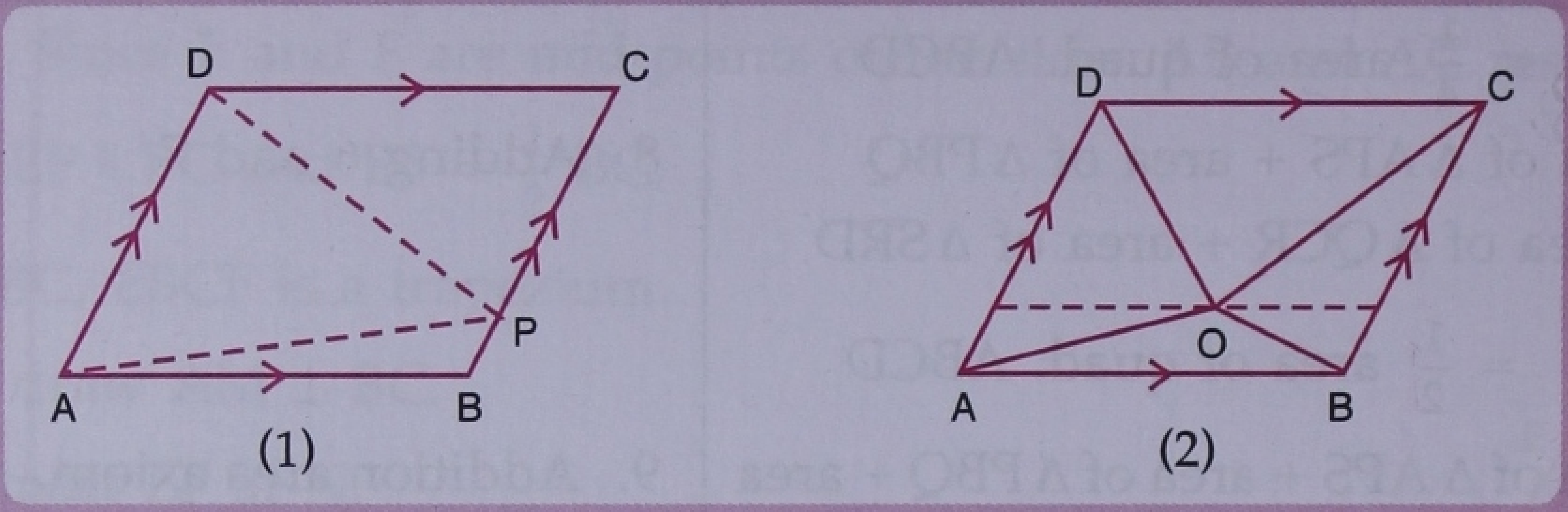
(*b*) (*i*) Area of Δ DEC = area of Δ DEB, add area of Δ ADE to both sides.

4. (*a*) In the figure (1) given below, ABCD is a parallelogram and P is any point in BC. Prove that, area of ΔABP + area of ΔDPC = area of ΔAPD.

(*b*) In the figure (2) given below, O is any point inside a parallelogram ABCD. Prove that

(*i*) area of ΔOAB + area of ΔOCD = $\frac{1}{2}$ area of || gm ABCD.

(*ii*) area of ΔOBC + area of ΔOAD = $\frac{1}{2}$ area of || gm ABCD.



Hint

(*b*) (*i*) Through O, draw a straight line parallel to AB.

5. If E, F, G and H are mid-points of the sides AB, BC, CD and DA respectively of a parallelogram ABCD, prove that area of quad. EFGH = $\frac{1}{2}$ area of || gm ABCD.

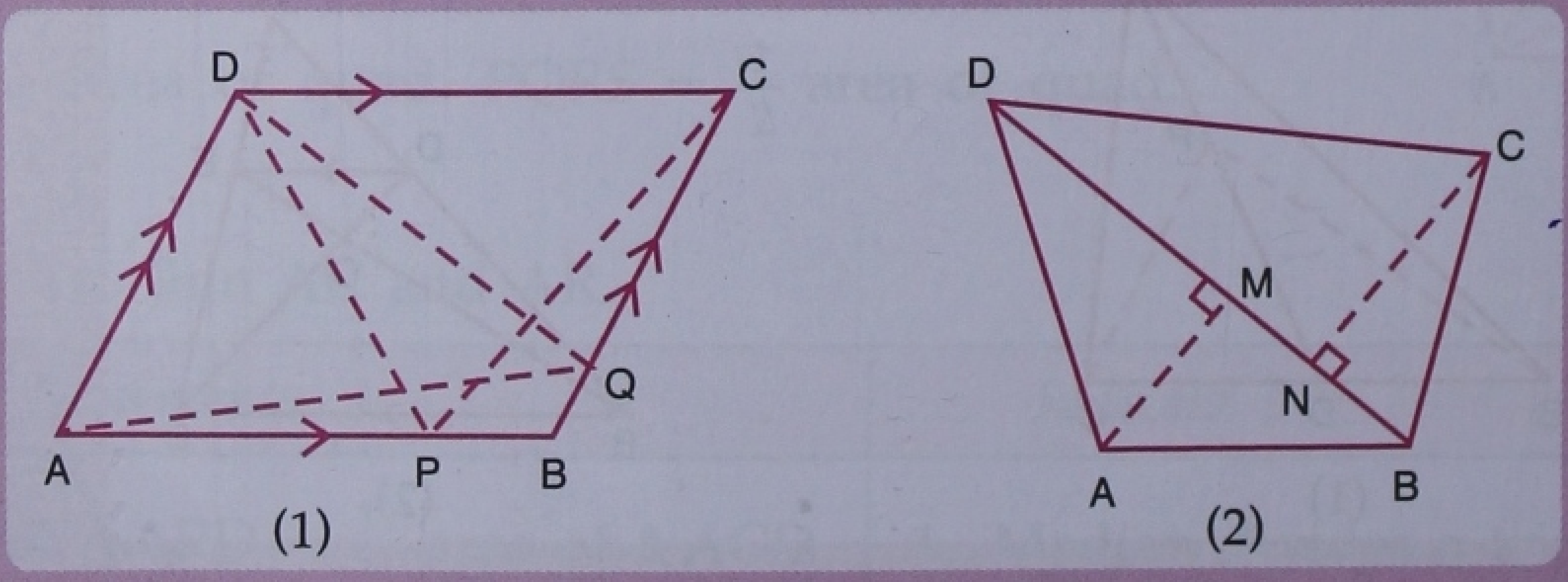
Hint

Join HF. AH = $\frac{1}{2}$AD and BF = $\frac{1}{2}$BC ⇒ AH = BF and AH || BF, so ABFH is a || gm.

∴ Area of ΔEFH = $\frac{1}{2}$area of || gm ABFH.

6. (*a*) In the figure (1) given below, ABCD is a parallelogram. P, Q are any two points on the sides AB and BC respectively. Prove that area of ΔCPD = area of ΔAQD.

(*b*) In the figure (2) given below, BD is a diagonal of the quad. ABCD. AM and CN are perpendiculars from A and C respectively on BD. Prove that area of quad. ABCD = $\frac{1}{2}$ BD × (AM + CN).



7. D, E and F are mid-points of the sides BC, CA and AB respectively of a Δ ABC. Prove that

(*i*) FDCE is a parallelogram (*ii*) area of Δ DEF = $\frac{1}{4}$ area of Δ ABC

(*iii*) area of ∥gm FDCE = $\frac{1}{2}$ area of Δ ABC.

8. If the medians of a triangle ABC intersect at G, prove that area of Δ BCG = $\frac{1}{3}$ area of Δ ABC.

9. Prove that two triangles having equal areas and having one side of one of the triangles equal to one side of the other, have their corresponding altitudes equal.

10. (*a*) In the figure (1) given below, the point D divides the side BC of Δ ABC in the ratio $m : n$. Prove that

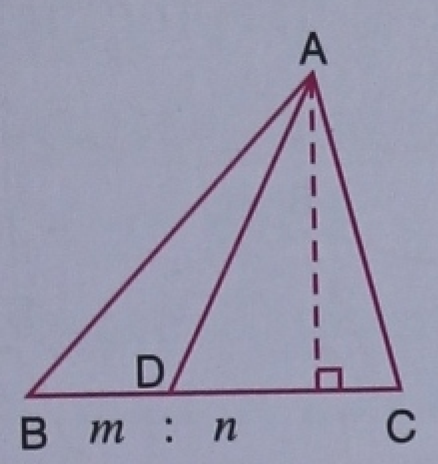
area of Δ ABD : area of Δ ADC = $m : n$.

(*b*) In the figure (2) given below, P is a point on the side BC of Δ ABC such that PC = 2BP, and Q is a point on AP such that QA = 5 PQ, find

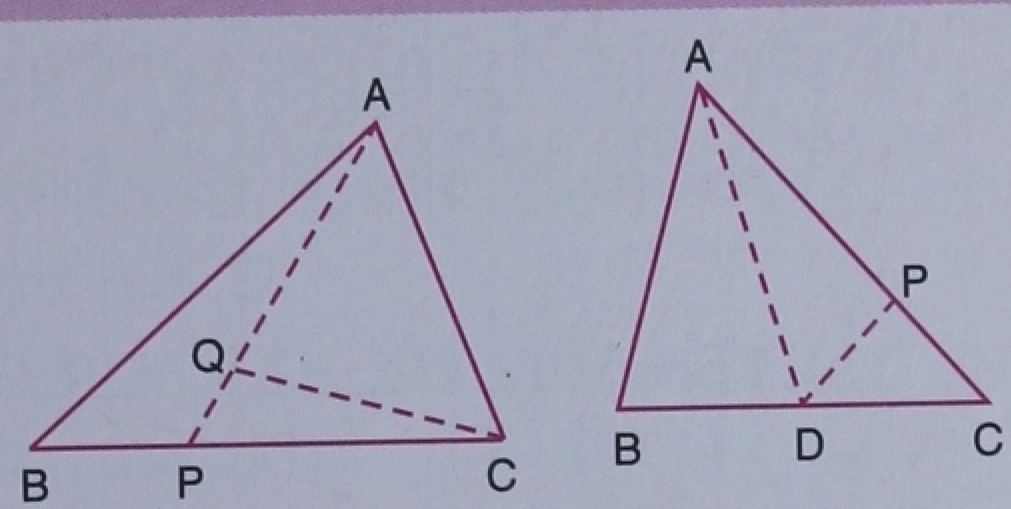
area of Δ AQC : area of Δ ABC.

(*c*) AD is a median of Δ ABC and P is a point in AC such that area of Δ ADP : area of Δ ABD = 2 : 3. Find

(*i*) AP : PC (*ii*) area of Δ PDC : area of Δ ABC



(1)



(2)

Hint

(*b*) PC = 2BP ⇒ PC = $\frac{2}{3}$ BC ⇒ area of Δ APC = $\frac{2}{3}$ area of Δ ABC.

QA= 5PQ ⇒ AQ = $\frac{5}{6}$ AP ⇒ area of Δ AQC = $\frac{5}{6}$ area of ΔAPC

= $\frac{5}{6} \cdot \frac{2}{3}$ area of Δ ABC = $\frac{5}{9}$ area of ΔABC.

(*c*) AD is median of Δ ABC ⇒ area of ABD = area of Δ ADC.

Given area of Δ ADP : area of Δ ABD = 2 : 3

⇒ area of Δ ADP : area of Δ ADC = 2 : 3 ⇒ AP : AC = 2 : 3

⇒ AP : PC = 2 : 1.

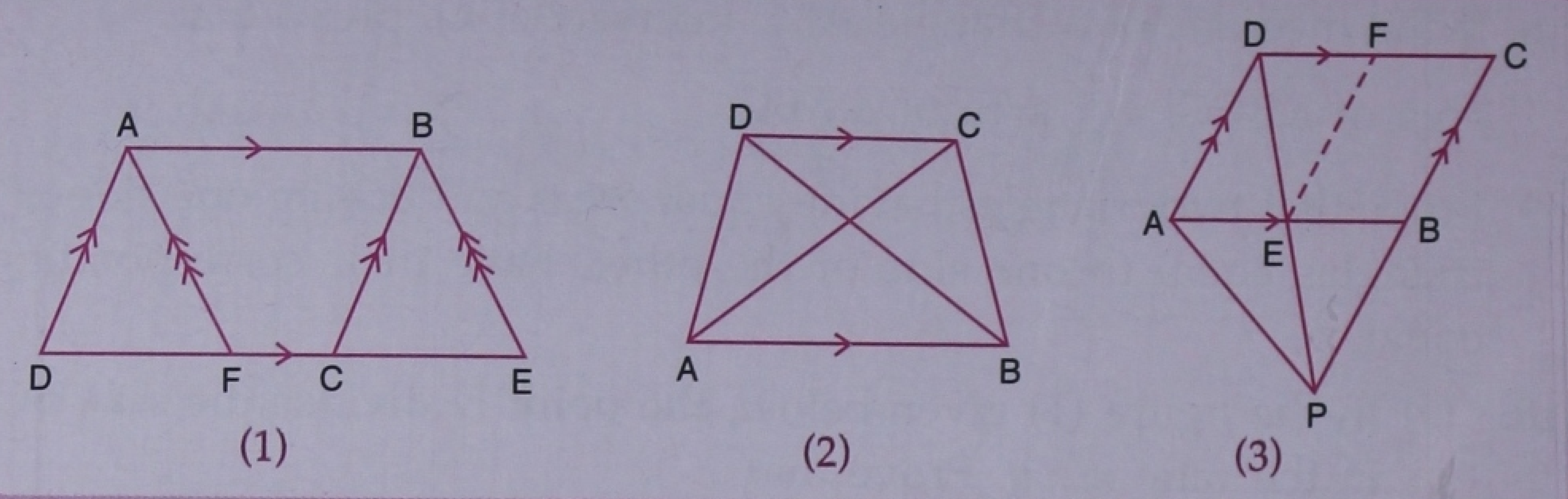
11. (*a*) In the figure (1) given below, area of parallelogram ABCD is 29 cm^2. Calculate the height of parallelogram ABEF if AB = 5·8 cm.

(*b*) In the figure (2) given below, area of Δ ABD is 24 sq. units. If AB = 8 units, find the height of Δ ABC.

(*c*) In the figure (3) given below, E and F are mid-points of sides AB and CD respectively of parallelogram ABCD. If the area of parallelogram ABCD is 36 cm^2,

(*i*) state the area of Δ APD.

(*ii*) name the parallelogram whose area is equal to the area of Δ APD.

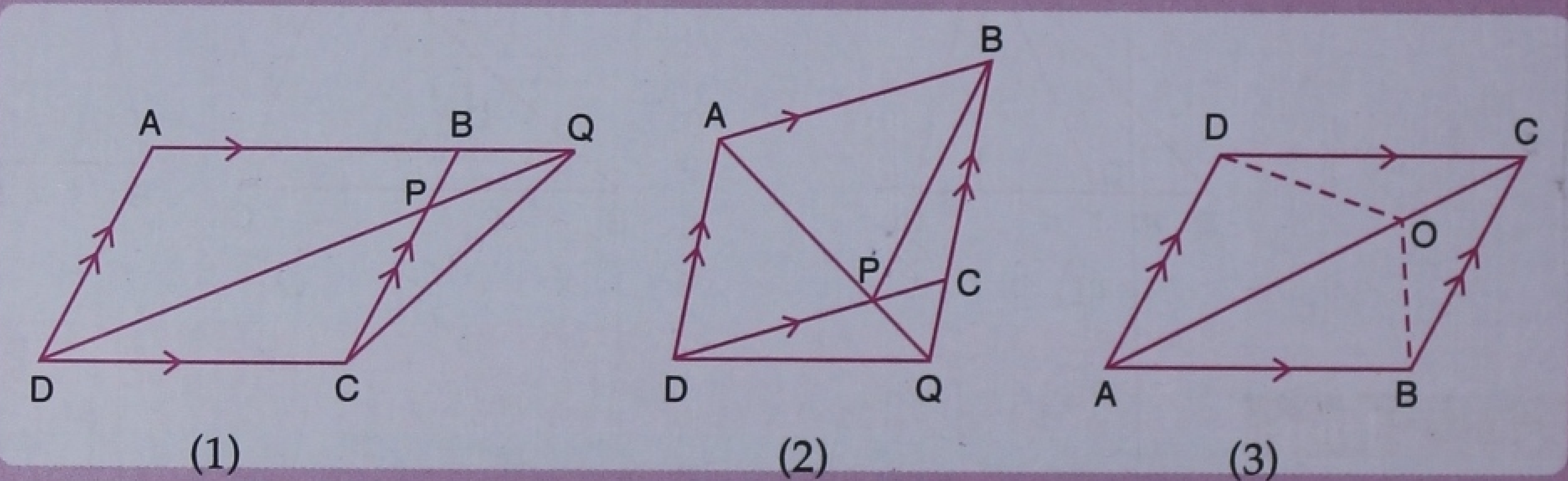


12. (*a*) In the figure (1) given below, ABCD is a parallelogram. P is a point on BC such that BP : PC = 1 : 2. DP produced meets AB produced at Q. Given area of triangle CPQ = 20 cm^2, calculate :

(*i*) area of triangle CDP (*ii*) area of parallelogram ABCD.

(*b*) In the figure (2) given below, ABCD is a parallelogram. Any line through A cuts DC at a point P and BC produced at Q. Prove that Δ BPC is equal in area to Δ DPQ.

(*c*) In the figure (3) given below, ABCD is a parallelogram. O is any point on the diagonal AC of the parallelogram. Show that the area of Δ AOB is equal to the area of Δ AOD.



Hint

(*b*) Join AC. Area of ΔBPC = area of ΔAPC ...(*i*)

Area of ΔACQ = area of ΔDCQ

⇒ area of ΔACQ – area of ΔPCQ = area of ΔDCQ – area of ΔPCQ

⇒ area of ΔAPC = area of ΔDPQ ...(*ii*)

From (*i*) and (*ii*), we get, area of ΔBPC = area of ΔDPQ.

(*c*) Join BD. Let diagonals AC and BD of ||gm ABCD meet at P.

Then AP is median of ΔABD ⇒ area of ΔABP = area of ΔADP.

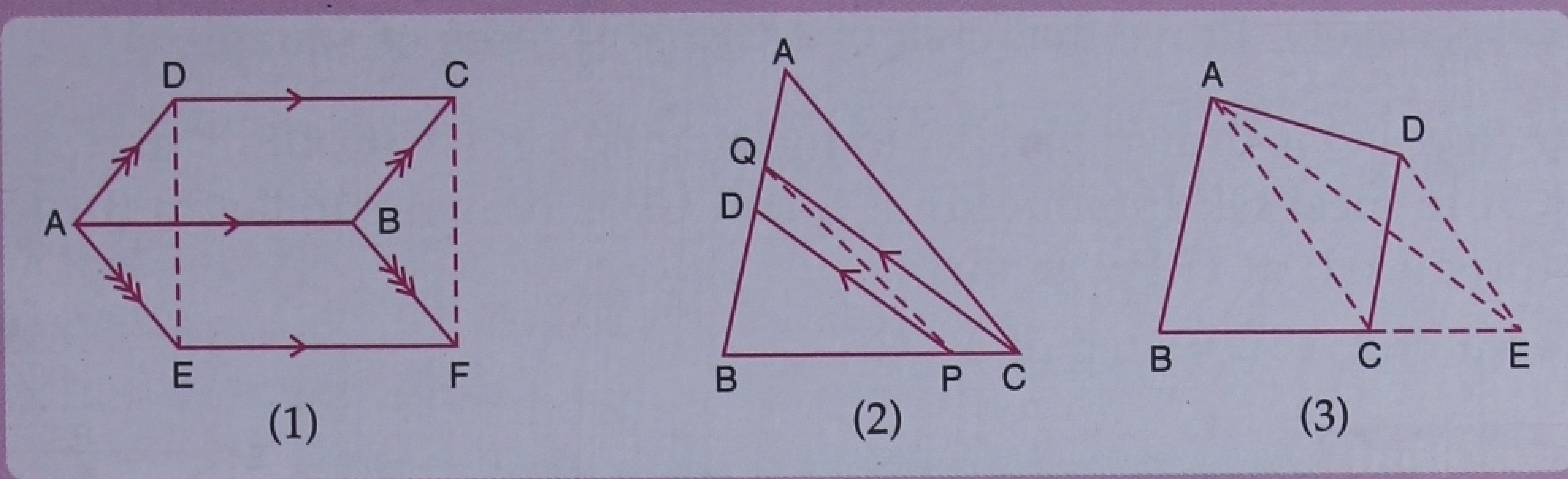
Similarly, area of ΔPBO = area of ΔPDO.

13. (*a*) In the figure (1) given below, two parallelograms ABCD and AEFB are drawn on opposite sides of AB, prove that

area of ||gm ABCD + area of ||gm AEFB = area of ||gm EFCD.

(*b*) In the figure (2) given below, D is mid-point of the side AB of Δ ABC. P is any point on BC, CQ is drawn parallel to PD to meet AB in Q. Show that area of Δ BPQ = $\frac{1}{2}$ area of Δ ABC.

(*c*) In the figure (3) given below, DE is drawn parallel to the diagonal AC of the quadrilateral ABCD to meet BC produced at the point E. Prove that area of quad. ABCD = area of Δ ABE.



Hint

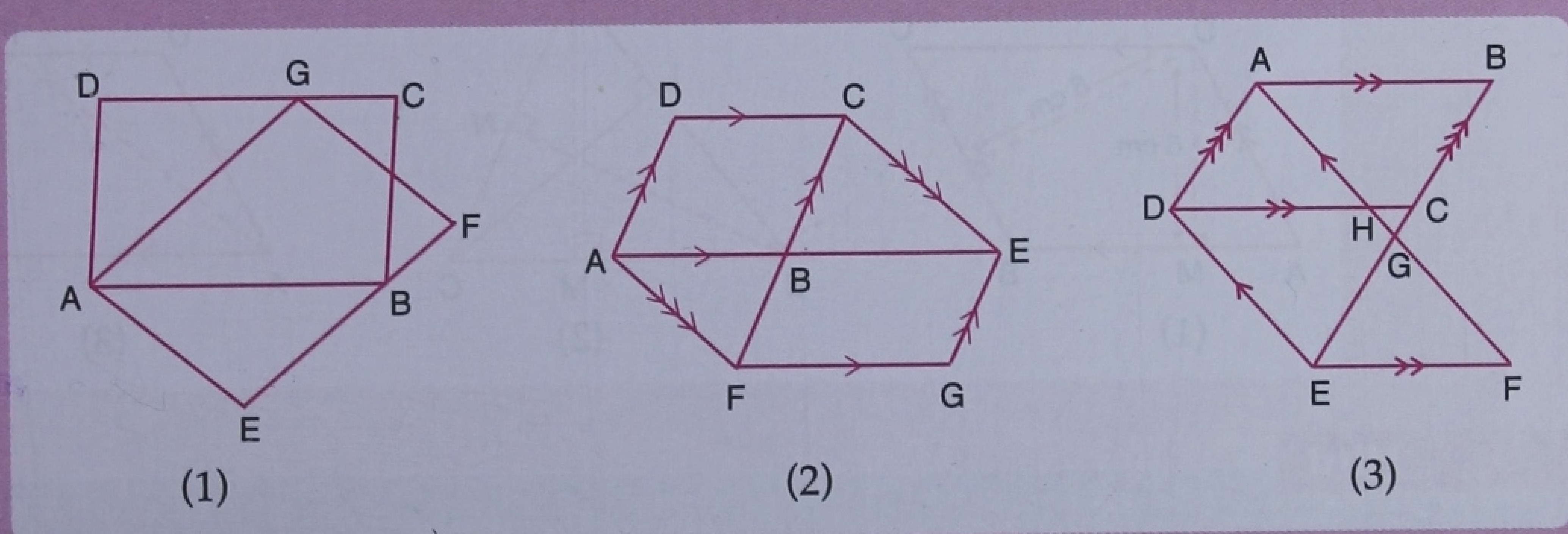
(*b*) Join CD, area of Δ QDP = area of Δ CPD. Add area of Δ DBP to both sides.

(*c*) Area of Δ ACE = area of Δ ACD.

14. (*a*) In the figure (1) given below, ABCD and AEFG are two parallelograms. Prove that area of ∥gm ABCD = area of ∥gm AEFG.

(*b*) In the figure (2) given below, the side AB of the parallelogram ABCD is produced to E. A st. line through A is drawn parallel to CE to meet CB produced at F, and parallelogram BFGE is completed. Prove that area of ∥gm BFGE = area of ∥gm ABCD.

(*c*) In the figure (3) given below, AB ∥ DC ∥ EF, AD ∥ BE and DE ∥ AF. Prove that the area of DEFH is equal to the area of ABCD.



Hint

(*a*) Join BG. Area of Δ ABG = $\frac{1}{2}$(area of ∥gm ABCD);

area of Δ ABG = $\frac{1}{2}$(area of ∥gm AEFG).

(*b*) Join AC and EF. Area of Δ CAF = area of Δ EAF.

15. Any point D is taken on the side BC of a Δ ABC and AD is produced to E such that AD = DE, prove that area of Δ BCE = area of Δ ABC.

16. ABCD is a rectangle and P is mid-point of AB. DP is produced to meet CB at Q. Prove that area of rectangle ABCD = area of Δ DQC.

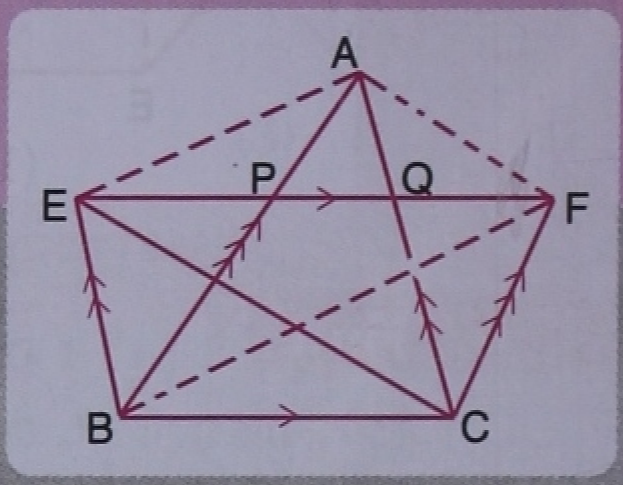
Hint

$\Delta APD \cong \Delta PQB$.

17. ABCD is a square. E and F are mid-points of the sides AB and AD respectively. Prove that area of Δ CEF = $\frac{3}{8}$ (area of square ABCD).

18. A line PQ is drawn parallel to the side BC of Δ ABC. BE is drawn parallel to CA to meet QP (produced) at E and CF is drawn parallel to BA to meet PQ (produced) at F. Prove that

area of Δ ABE = area of Δ ACF.



Hint

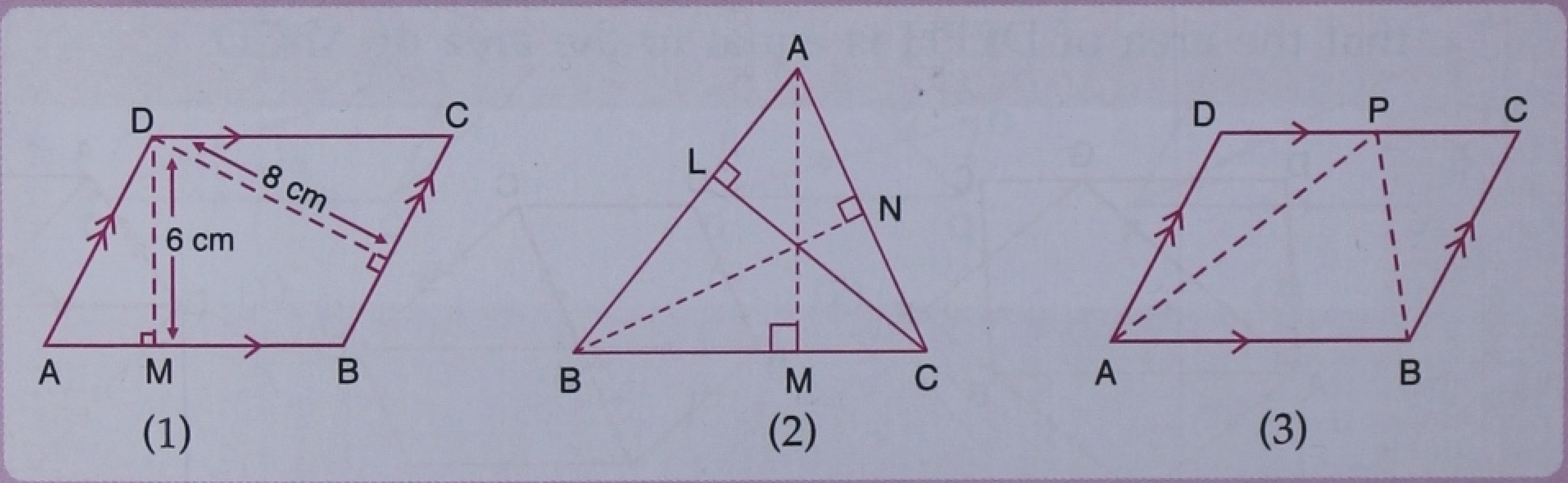
Area of Δ ABE = area of Δ EBC,
area of Δ ACF = area of Δ FBC.
But area of Δ EBC = area of Δ FBC.

19. (*a*) In the figure (1) given below, the perimeter of parallelogram is 42 cm. Calculate the lengths of the sides of the parallelogram.

(*b*) In the figure (2) given below, the perimeter of Δ ABC is 37 cm. If the lengths of the altitudes AM, BN and CL are $5x$, $6x$ and $4x$ respectively, calculate the lengths of the sides of Δ ABC.

(*c*) In the figure (3) given below, ABCD is a parallelogram. P is a point on DC such that area of Δ DAP = 25 cm^2 and area of ΔBCP = 15 cm^2. Find

(*i*) area of ∥gm ABCD (*ii*) DP : PC.



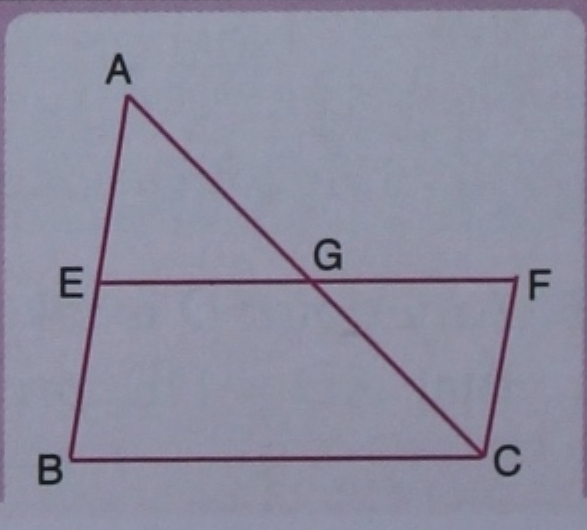
Hint

(*a*) Let AB = p, then BC = $\frac{1}{2} \cdot 42 - p$. Also $6 \cdot p = 8 \cdot (21 - p)$.

(*b*) Let BC = p, CA = q, then AB = $37 - p - q$.

Also $\frac{1}{2} p \cdot 5x = \frac{1}{2} q \cdot 6x = \frac{1}{2} (37 - p - q) \cdot 4x$.

20. In the adjoining figure, E is mid-point of the side AB of a triangle ABC and EBCF is a parallelogram. If the area of Δ ABC is 25 sq. units, find the area of ∥gm EBCF.



Hint

Let EF meet AC at G. As E is mid-point of AB and EF ∥ BC, G is mid-point of AC *i.e.* AG = GC.

Also ∠EAG = ∠GCF and ∠EGA = ∠CGF

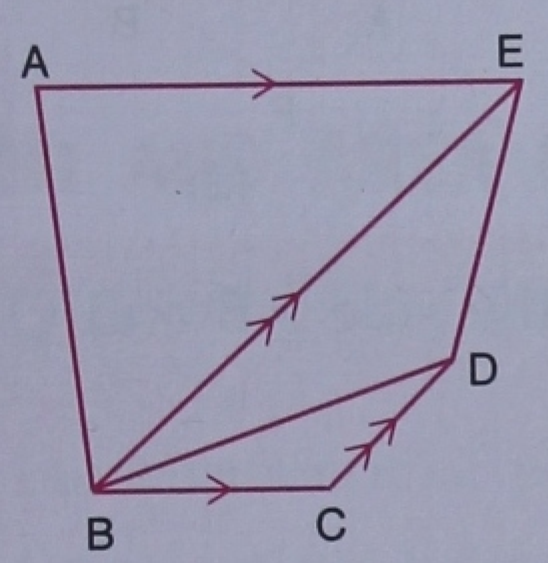
⇒ Δ AEG ≅ Δ CFG.

21. (*a*) In the figure (1) given below, BC ∥ AE and CD ∥ BE. Prove that area of Δ ABC = area of Δ EBD.

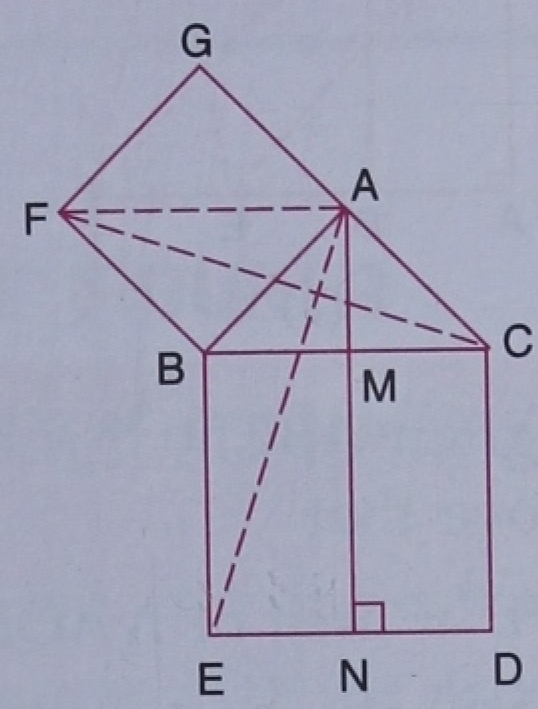
(*b*) In the figure (2) given below, ABC is right angled triangle at A. AGFB is a square on the side AB and BCDE is a square on the hypotenuse BC. If AN ⊥ ED, prove that

(*i*) Δ BCF ≅ Δ ABE.

(*ii*) area of square ABFG = area of rectangle BENM.



(1)



(2)

Hint

(*a*) Join CE. Area of Δ ABC = area of Δ EBC. Area of Δ BCD = area of Δ ECD.

(*b*) (*ii*) Area of Δ BCF = $\frac{1}{2}$ area of square ABFG.

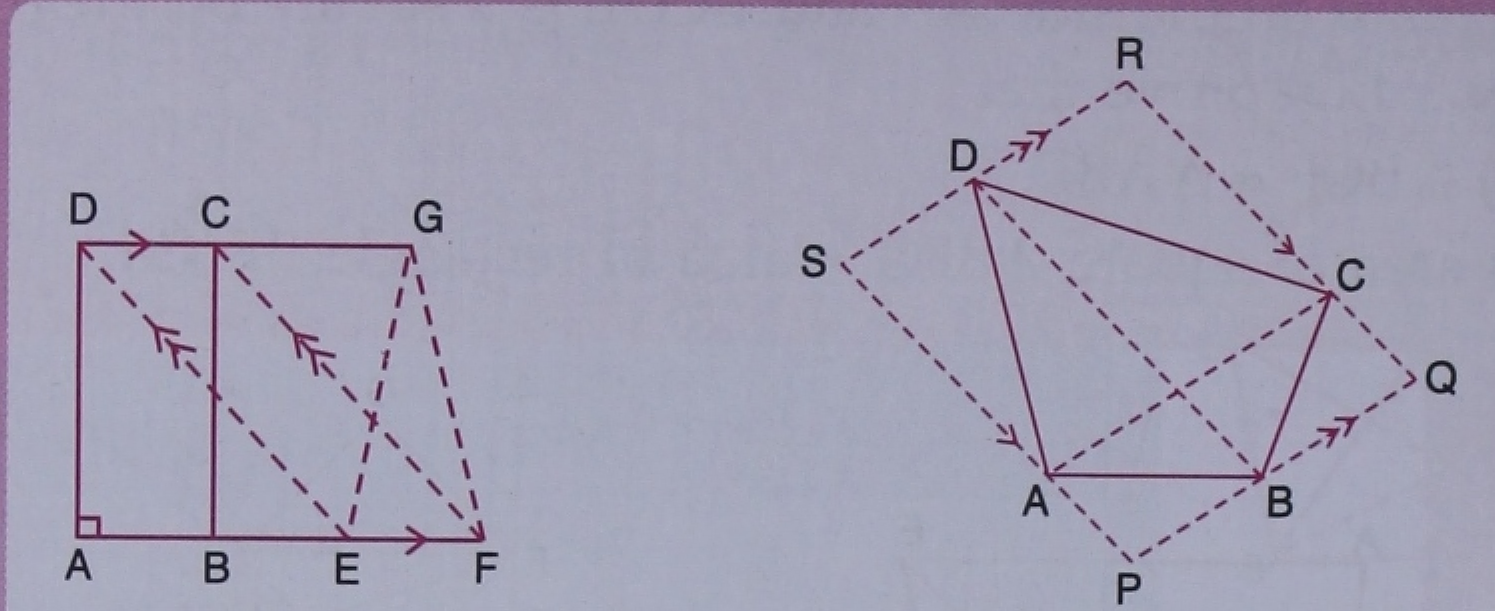
CHAPTER TEST

1. (*a*) In the figure (1) given below, ABCD is a rectangle (not drawn to scale) with side AB = 4 cm and AD = 6 cm. Find

   (*i*) the area of parallelogram DEFC (*ii*) area of ΔEFG.

   [Ans. (*i*) 24 cm^2 (*ii*) 12 cm^2.]

   (*b*) In the figure (2) given below, PQRS is a parallelogram formed by drawing lines parallel to the diagonals of a quadrilateral ABCD through its corners. Prove that area of ∥gm PQRS = 2 × area of quad. ABCD.



(1) (2)

2. In the parallelogram ABCD, P is a point on the side AB and Q is a point on the side BC. Prove that

   (*i*) area of Δ CPD = area of Δ AQD

   (*ii*) area of Δ ADQ = area of Δ APD + area of Δ CPB.

3. If D is a point on the base BC of a triangle ABC such that 2BD = DC, prove that area of Δ ABD = $\frac{1}{3}$ area of Δ ABC.

4. Perpendiculars are drawn from a point within an equilateral triangle to the three sides. Prove that the sum of the three perpendiculars is equal to the altitude of the triangle.

5. If each diagonal of a quadrilateral divides it into two triangles of equal areas, then prove that the quadrilateral is a parallelogram.

Hint

Let ABCD be a quadrilateral such that each diagonal divide it into triangles of equal areas, then area of Δ ABC = $\frac{1}{2}$ (area of quad. ABCD) and area of Δ ABD = $\frac{1}{2}$ (area of quad. ABCD). So, area of Δ ABC = area of Δ ABD. By corollary 3 to Theorem 22, DC ∥ AB.

6. In the adjoining figure, ABCDE is a pentagon. BP drawn parallel to AC meets DC produced at P and EQ drawn parallel to AD meets CD produced at Q. Prove that the area of Δ APQ is equal to the area of pentagon ABCDE.