16

RECTILINEAR FIGURES

16.1 RECTILINEAR FIGURE

A plane figure bounded by line segments is called a ***rectilinear figure****.*

Polygon. A closed plane figure bounded by line segements is called a **polygon**.

The line segments are called its **sides** and the points of intersection of consecutive sides are called its **vertices**. An angle formed by two consecutive sides of a polygon is called an **interior angle** or simply an **angle** of the polygon.

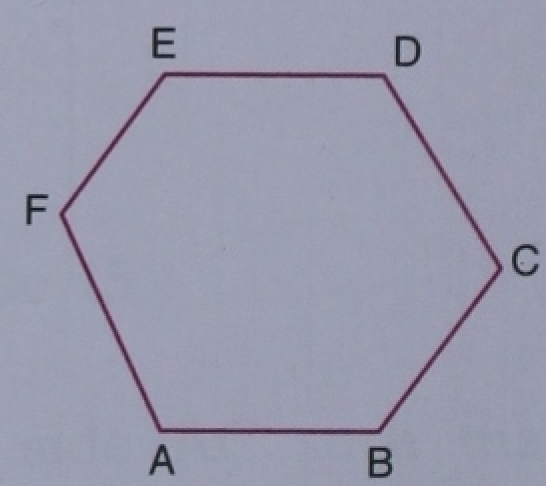
A polygon is named according to the number of sides it has.

No. of sides	3	4	5	6	7	8	10
Name	Triangle	Quadrilateral	Pentagon	Hexagon	Heptagon	Octagon	Decagon

In general, a polygon having n sides is called *n-gon*. Thus, a polygon having 18 sides is called 18-gon.

Diagonal of a polygon. *Line segment joining any two non-consecutive vertices of a polygon is called its* ***diagonal****.*

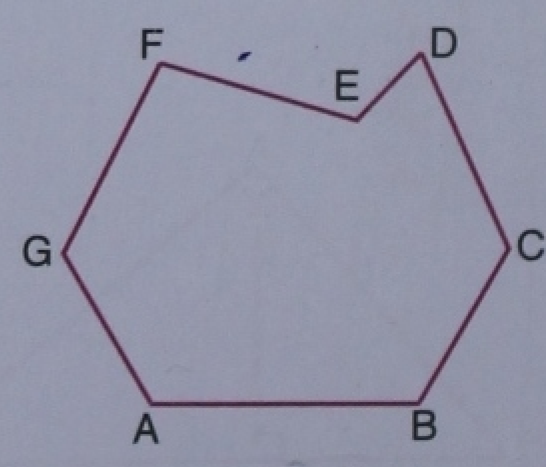
Convex polygon. *If all the (interior) angles of a polygon are less than 180°, it is called a* **convex polygon.**



Convex polygon

In the adjoining figure, ABCDEF is a convex polygon. In fact, it is a convex hexagon.

Concave polygon. *If one or more of the (interior) angles of a polygon is greater than 180° i.e. reflex, it is called* **concave** (or **re-entrant**) polygon.

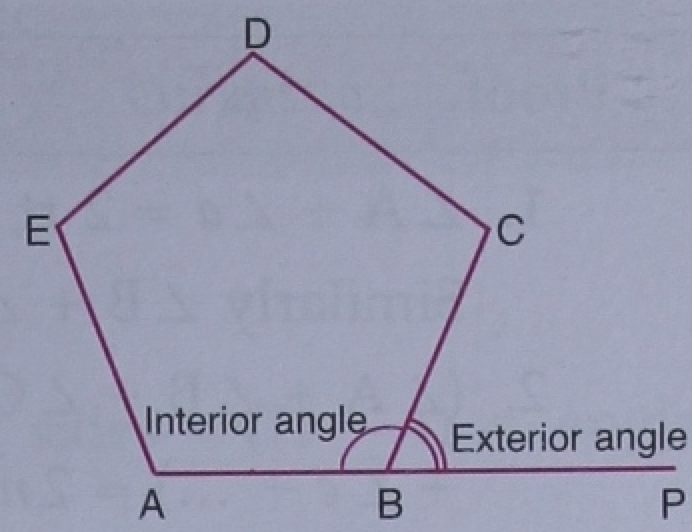


Concave polygon

In the adjoining figure, ABCDEFG is a concave polygon. In fact, it is a concave heptagon.

However, we shall be dealing with convex polygons only.

Exterior angle of convex polygon



If we produce a side of a polygon, the angle it makes with the next side is called an **exterior angle**.

In the adjoining diagram, ABCDE is a pentagon. Its side AB has been produced to P, then $\angle CBP$ is an exterior angle.

Note that corresponding to each interior angle, there is an exterior angle. Also, as an exterior angle and its adjacent interior angle make a straight line, we have :

an exterior angle + adjacent interior angle = 180°

Regular polygon. *A polygon is called* **regular polygon** *if all its sides have equal length and all its angles have equal size.*

Thus, in a regular polygon :

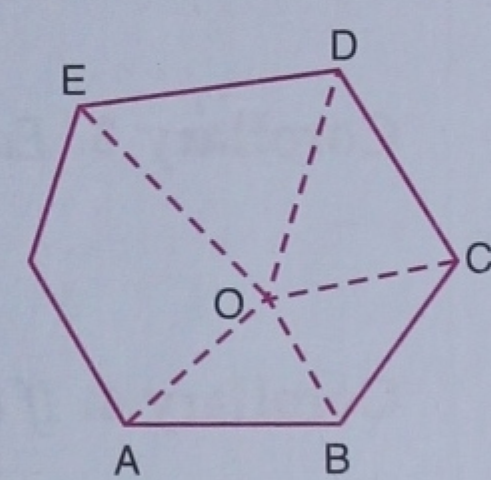
(*i*) *all sides are equal in length*

(*ii*) *all interior angles are equal in size*

(*iii*) *all exterior angles are equal in size.*

All regular polygons are convex.

Theroem 13. *The sum of the interior angles of a convex polygon of n sides is* $(2n - 4)$ *right angles.*



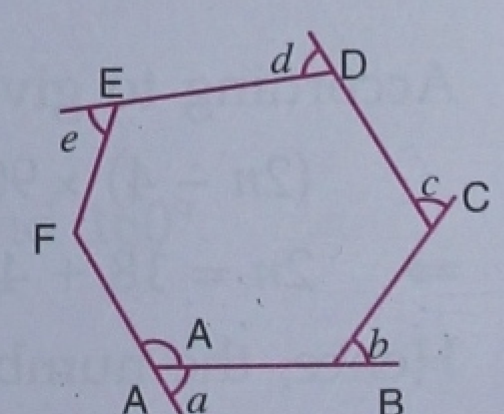
Given. A convex polygon ABCDE … of n sides.

To prove. Sum of interior angles = $(2n - 4)$ right angles.

Construction. Take any point O inside the polygon. Join OA, OB, OC, OD….

Proof. *Statements*	*Reasons*
1. Polygon ABCDE… is divided into n triangles OAB, OBC, OCD, …	1. Polygon has n sides and on each side, one triangle is formed.
2. Sum of all the angles of n triangles = $2n$ right angles	2. Sum of all three angles of a triangle is 2 rt. angles.
3. Sum of interior angles of polygon + sum of angles at O = $2n$ right angles	3. From figure and 2.
4. Sum of angles at O = 4 right angles	4. Sum of angles around any point = 4 right angles.
5. Sum of interior angles of polygon + 4 right angles = $2n$ right angles	5. From 3 and 4.
Hence, sum of interior angles of the polygon = $(2n - 4)$ right angles.	
Q.E.D.	

Corollary 1. *The sum of the exterior angles of a convex polygon is 4 right angles.*



Given. A convex polygon ABCDE… of n sides.

To prove. Sum of exterior angles = 4 right angles.

Here $\angle a$, $\angle b$, $\angle c$, … are exterior angles.

Proof. *Statements*	*Reasons*
1. $\angle A + \angle a = 2$ rt. angles Similarly $\angle B + \angle b = 2$ rt. angles etc.	1. FA is a st. line. AB is a st. line.
2. $(\angle A + \angle B + \angle C + \ldots) + (\angle a + \angle b + \angle c + \ldots) = 2n$ rt. angles	2. Since polygon has n sides, and from 1.
3. $(2n - 4)$ rt. $\angle$s $+ (\angle a + \angle b + \angle c + \ldots) = 2n$ rt. angles	3. Sum of interior angles of a polygon of n sides = $(2n - 4)$ rt. angles.
4. $\angle a + \angle b + \angle c + \ldots = (2n - (2n - 4))$ rt. angles Hence, $\angle a + \angle b + \angle c + \ldots = 4$ rt. angles *i.e.* sum of exterior angles = 4 rt. angles. **Q.E.D.**	4. From 3.

Corollary 2. *Each interior angle of a regular polygon of n sides* = $\frac{2n-4}{n}$ *right angles.*

[$\because$ all interior angles of a regular polygon are equal]

Corollary 3. *Each exterior angle of a regular polygon of n sides*

$$= \frac{4}{n} \text{ right angles} = \left(\frac{360}{n}\right)^{\circ}.$$

Corollary 4. *If each exterior angle of a regular polygon is $x°$, then the number of sides in the polygon* = $\frac{360}{x}$.

Corollary 5. *Greater the number of sides in a regular polygon, greater is the value of its each interior angle and smaller is the value of each exterior angle.*

Remark

If a polygon has n sides, then the number of diagonals of the polygon

$$= \frac{n(n-1)}{2} - n.$$

ILLUSTRATIVE EXAMPLES

Example 1. *If the sum of interior angles of a polygon is 1620°, find its number of sides.*

Solution. Let the number of sides of the polygon be n.

Then sum of interior angles = $(2n - 4)$ right angles

$= ((2n - 4) \times 90)°$.

According to given,

$(2n - 4) \times 90 = 1620 \quad \Rightarrow 2n - 4 = 18$

$\Rightarrow \quad 2n = 18 + 4 = 22 \quad \Rightarrow n = 11.$

Hence, the number of sides = 11.

Example 2. *A heptagon has 4 equal angles each of 132° and three equal angles. Find the size of equal angles.*

Solution. A heptagon has 7 sides

Sum of its interior angles = $(2 \times 7 - 4)$ rt. angles.

($\because$ sum of interior angles of a polygon of n sides = $(2n - 4)$ rt. angles)

$= 10 \times 90° = 900°$.

Let the size of each of the three equal angles be $x°$, so we have

$4 \times 132° + 3x° = 900°$

$\Rightarrow \quad 3x = 900 - 528 = 372 \Rightarrow x = 124$.

Hence, the size of each equal angle = 124°.

Example 3. *The exterior angle of a regular polygon is $\frac{1}{3}$rd of its interior angle. Find the number of sides of the polygon.*

Solution. Let an exterior angle of the given regular polygon be $x°$, then its interior angle = $(180 - x)°$.

According to given $x = \frac{1}{3}(180 - x)$

$\Rightarrow \quad 3x = 180 - x \Rightarrow 4x = 180 \Rightarrow x = 45$.

$\therefore \quad$ Each exterior angle = 45°.

$\therefore \quad$ The number of sides of the polygon = $\frac{360}{45} = 8$.

Example 4. *Is it possible to have a regular polygon whose each interior angle is 105°?*

Solution. Since each interior angle is 105°,

$\therefore \quad$ each exterior angle = $(180 - 105)° = 75°$.

Let n be the number of sides of the regular polygon, then

$n = \frac{360}{75} = \frac{24}{5} = 4\frac{4}{5}$, which is not a natural number.

Therefore, there is no regular polygon whose each interior angle is 105°.

Example 5. *One angle of a pentagon is 140°. If the remaining angles are in the ratio 1 : 2 : 3 : 4, calculate the size of the greatest angle.*

Solution. One angle of the pentagon is 140°. Since the remaining angles are in the ratio 1 : 2 : 3 : 4, therefore, let the remaining angles be $x°$, $(2x)°$, $(3x)°$ and $(4x)°$.

But the sum of interior (angles) of a pentagon = $(2.5 - 4) \times 90° = 6 \times 90° = 540°$

($\because$ sum of angles of a polygon of n sides = $(2n - 4)$ right angles, here $n = 5$)

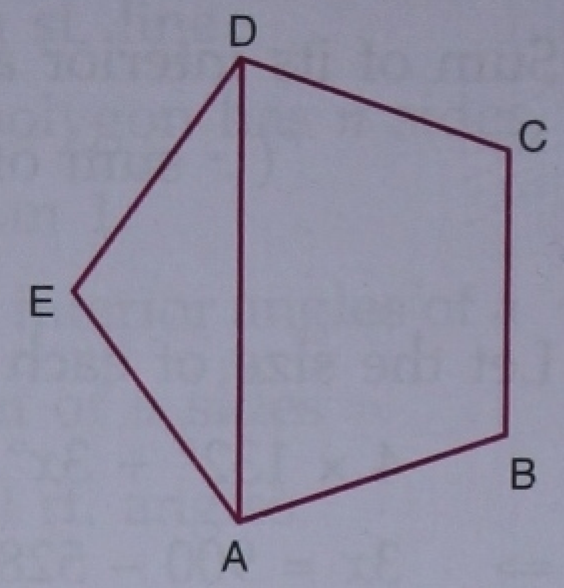
$\therefore \quad 140 + x + 2x + 3x + 4x = 540$

$\Rightarrow \quad 10x = 540 - 140 \Rightarrow 10x = 400 \Rightarrow x = 40$.

$\therefore \quad$ The angles of the pentagon are 140°, 40°, 80°, 120° and 160°.

Hence the size of the greatest angle = 160°.

Example 6. *ABCDE is a regular pentagon. Diagonal AD divides ∠CDE into two parts. Find the ratio* $\frac{\angle ADE}{\angle ADC}$.



Solution. Here number of sides = 5.

∴ Each interior angle = $\frac{2n-4}{n}$ rt. angles

$= \frac{2.5-4}{5} \times 90$ degrees = 108°.

∴ In Δ AED, ∠ AED = 108° and AE = ED

∴ ∠ EDA = ∠ EAD = $\frac{180-108}{2}$ degrees = 36°.

∴ ∠ ADC = ∠ EDC – ∠ EDA = (108 – 36)° = 72°.

∴ $\frac{\angle ADE}{\angle ADC} = \frac{36°}{72°} = \frac{1}{2}$.

Example 7. *Two angles of a convex polygon are right angles and each of the other is 120°. How many sides has the polygon?*

Solution. Let the polygon have n sides. Since two angles are right angles,

∴ the number of angles each of 120° = $n - 2$.

∴ The sum of interior angles = $[2 \times 90 + (n-2) \times 120]$ degrees.

Also sum of interior angles of a polygon of n sides = $(2n - 4) \times 90$ degrees,

∴ $180 + (n-2) \times 120 = (2n-4) \times 90$

⇒ $6 + (n-2) \times 4 = (2n-4) \times 3$ (dividing by 30)

⇒ $4n - 8 + 6 = 6n - 12$

⇒ $2n = 10 \Rightarrow n = 5$.

∴ The polygon has 5 sides.

Example 8. *Each interior angle of a regular polygon is 144°. Find the interior angle of a regular polygon which has double the number of sides as the first polygon.*

Solution. Since each interior angle of the first polygon = 144° (given),

∴ Each exterior angle of the first polygon = 180° – 144° = 36°.

∴ The number of sides of the first polygon = $\frac{360}{36}$ = 10.

∴ The number of sides of the second polygon = 2 × 10 = 20.

∴ Each exterior angle of the second polygon = $\left(\frac{360}{20}\right)^\circ$ = 18°.

∴ Each interior angle of the second polygon = 180° – 18° = 162°.

Example 9. *The difference between an exterior angle of (n – 1) sided regular polygon and an exterior angle of (n + 2) sided regular polygon is 6°. Find the value of n.*

Solution. Each ext. angle of $(n-1)$ sided regular polygon = $\left(\frac{360}{n-1}\right)^\circ$, and each ext. angle of $(n + 2)$ sided regular polygon = $\left(\frac{360}{n+2}\right)^\circ$.

According to given, $\frac{360}{n-1} - \frac{360}{n+2} = 6$

[Since greater is the number of sides, smaller is the value of each ext. angle.]

$\Rightarrow$ $360(n+2) - 360(n-1) = 6(n+2)(n-1)$

$\Rightarrow$ $60(n+2-n+1) = n^2 + 2n - n - 2$

$\Rightarrow$ $180 = n^2 + n - 2$

$\Rightarrow$ $n^2 + n - 182 = 0$

$\Rightarrow$ $(n+14)(n-13) = 0$ [by factors]

$\Rightarrow$ either $n + 14 = 0$ or $n - 13 = 0$

$\Rightarrow$ $n = -14$ or $n = 13$ but n cannot be negative,

$\therefore$ $n = 13$.

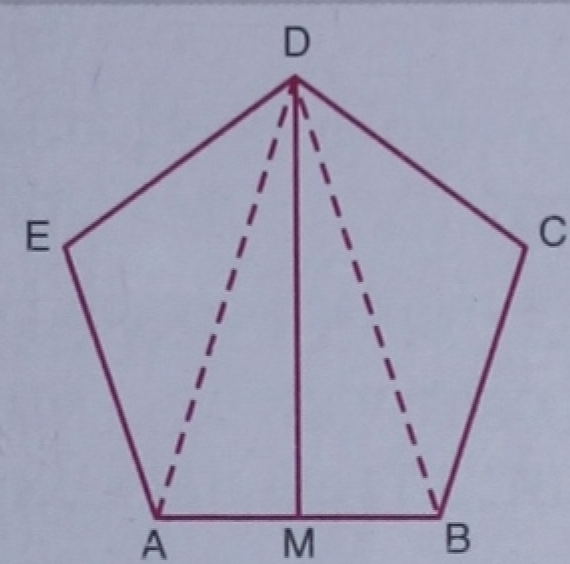
Example 10. *Show that the line joining any vertex of a regular pentagon to the mid-point of the opposite side is at right angles to the side.*

Given. A regular pentagon ABCDE.

M is mid-point of AB.

To prove. DM ⊥ AB.

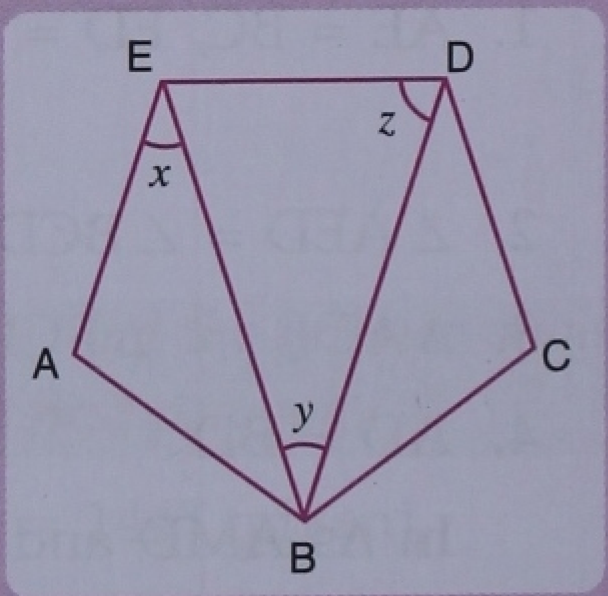
Construction. Join AD and BD.



Proof. *Statements*	*Reasons*
In Δs ADE and BCD	
1. AE = BC, ED = CD	1. ABCDE is a regular pentagon, all sides are equal.
2. ∠AED = ∠BCD	2. All angles are equal.
3. ΔADE ≅ ΔBCD	3. S.A.S. (Axiom of congruency)
4. AD = BD	4. 'c.p.c.t.'
In Δs AMD and BMD	
5. AD = BD	5. From 4.
6. AM = MB	6. M is mid-point of AB.
7. MD = MD	7. Common.
8. ΔAMD ≅ ΔBMD	8. S.S.S. (Axiom of congruency)
9. ∠AMD = ∠DMB	9. 'c.p.c.t.'
10. ∠AMD + ∠DMB = 180°	10. AMB is a st. line.
11. 2∠AMD = 180° ⇒ ∠AMD = 90°. Hence, MD ⊥ AB. **Q.E.D.**	11. From 9 and 10.

Exercise 16.1

1. Find the sum of interior angles (in degrees) of a polygon of
   (*i*) 7 sides (*ii*) 10 sides (*iii*) 12 sides.
2. Find the measure (in degrees) of an angle of a regular
   (*i*) hexagon (*ii*) octagon (*iii*) 15-gon.
3. Find the number of sides of a regular polygon if each of its exterior angle is
   (*i*) 36° (*ii*) 40° (*iii*) 45° (*iv*) $\left(51\frac{3}{7}\right)^\circ$.
4. Find the number of sides of a regular polygon if each of its interior angles is
   (*i*) 108° (*ii*) 160° (*iii*) 135° (*iv*) $\left(147\frac{3}{11}\right)^\circ$.
5. Find the number of sides of a polygon whose sum of interior angles is
   (*i*) 28 right angles (*ii*) 38 right angles (*iii*) 1440°.
6. Can a regular polygon be described whose each exterior angle is
   (*i*) 24° (*ii*) 50° (*iii*) 70° ?
7. Can a regular polygon be described whose each interior angle is
   (*i*) 120° (*ii*) 150° (*iii*) 110°?
8. Can a regular polygon be described whose sum of interior angles is
   (*i*) 9 right angles (*ii*) 520° (*iii*) 1260°?
9. AB, BC, CD are three consecutive sides of a regular polygon. If ∠BAC = 20°, find
   (*i*) exterior angle of the polygon.
   (*ii*) the number of sides of the polygon.
10. ABCDE is a regular pentagon. Find the measures of the angles marked *x*, *y* and *z*.



11. The interior angle of a regular polygon is double the exterior angle. Find the number of sides in the polygon.
12. If the ratio of interior angle to the exterior angle of a regular polygon is 7 : 2, find the number of sides in the polygon.
13. The sum of interior angles of a polygon is 6 times the sum of its exterior angles. Find the number of sides in the polygon.
14. Three angles of a quadrilateral are 75°, 110°, 91°. Find its fourth angle.
15. If the angles of a quadrilateral are in the ratio 2 : 3 : 5 : 2, find the angles. How many right angles are there in it?
16. If the angles of a pentagon are in the ratio 4 : 5 : 6 : 7 : 5, find the angles.
17. The angles of a pentagon are $x°$, $(x - 10)°$, $(x + 20)°$, $(2x - 44)°$ and $(2x - 70)°$. Calculate x.

18. Three of the exterior angles of a hexagon are 40°, 51°, 86°. If each of the remaining exterior angles is $x°$, calculate the value of x.
19. The angles of a hexagon are $(2x + 5)°$, $(3x - 5)°$, $(x + 40)°$, $(2x + 20)°$, $(2x + 25)°$ and $(2x + 35)°$. Find the value of x.
20. One angle of a 7-gon is 162° and each of the other angles is $x°$. Find the value of x.
21. In a polygon, there are 5 right angles and the remaining angles are 195° each. Find the number of sides in the polygon.
22. ABCDE is a pentagon in which AE ∥ BC, ∠C = 153°, ∠D = $x°$ and ∠E = $(2x)°$. Find the value of x.

Hint

Draw a rough sketch. As AE ∥ BC, ∠A + ∠B = 180° (sum of co-int. ∠s).
$180° + 153° + x° + 2x° = (2 \times 5 - 4) \times 90° \Rightarrow x = 69$.

23. ABCDE is a pentagon in which AB ∥ ED, ∠B = 140°. If ∠C : ∠D = 5 : 6, find ∠C and ∠D.
24. ABCDE is a pentagon in which ∠A = 110°, ∠B = 142° and ∠D = ∠E. The sides AB and DC when produced meet at right angles, compute ∠BCD and ∠E.
25. If the difference between an exterior angle of a regular polygon of n sides and an exterior angle of another regular polygon of $n + 1$ sides is 5°, find the value of n.
26. Find the number of diagonals in a
    (*i*) pentagon (*ii*) hexagon (*iii*) octagon.

Hint

No. of diagonals in a polygon of n sides $= \dfrac{n(n-1)}{2} - n$.

27. Find the number of sides of a polygon whose number of diagonals is
    (*i*) 5 (*ii*) 14 (*iii*) 27.
28. The ratio between the number of sides of two regular polygons is 3 : 4 and the ratio between the sum of their interior angles is 2 : 3. Find the number of sides in each polygon.

Hint

Let the number of sides of two regular polygons be $3n$ and $4n$, then $(2 \times 3n - 4) : (2 \times 4n - 4) = 2 : 3$.

29. The number of sides of two regular polygons are in the ratio 1 : 2 and their interior angles are in the ratio 3 : 4. Find the number of sides in each polygon.
30. Prove that each interior angle of a regular pentagon is three times of each exterior angle of a regular decagon.
31. Show that the diagonals of a regular pentagon are equal.
32. If ABCDE is a regular pentagon, show that AB is parallel to EC.

33. ABCDE is a regular pentagon. AC and EB intersect at P. Calculate ∠EAP and ∠BPA. Also prove that EA = EP.

34. In a regular hexagon, show that pairs of opposite sides are parallel.

35. ABCDEF is a regular haxagon. Prove that ACE is an equilateral triangle.

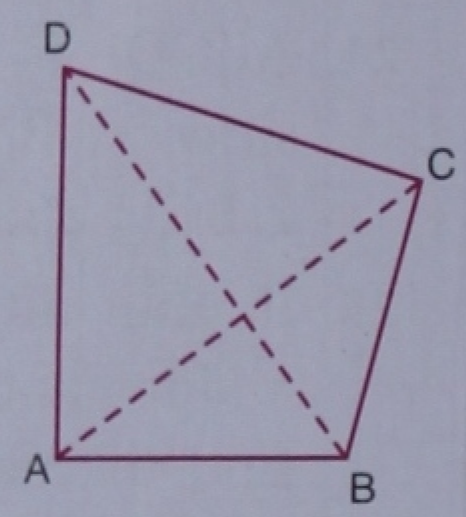
36. ABCDE is a pentagon in which AB = AE, BC = ED and ∠ABC = ∠AED.
   (*a*) Prove that (*i*) AC = AD (*ii*) ∠BCD = ∠EDC.
   (*b*) If BC and ED are produced to meet at X, prove that BX = EX.

Hint

(*a*) Join AC and AD, Δ ABC ≅ Δ AED ⇒ AC = AD.
(*b*) ∠XDC = ∠XCD (Why?) ⇒ DX = CX.

16.2 QUADRILATERALS

Quadrilateral. *A closed plane figure bounded by four line segments is called a **quadrilateral** (abbreviated quad.)*



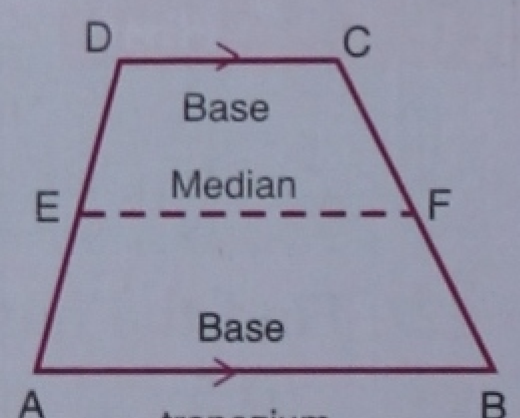
In the adjoining figure, ABCD is a quadrilateral. The points A, B, C and D are its vertices. The line segments AB, BC, CD and DA are its sides, and AC, BD are its diagonals. It has four interior angles—∠A, ∠B ∠C and ∠D.

Sum of (interior) angles of a quadrilateral

$= (2 \times 4 - 4)$ rt. angles = 4 rt. angles = 360°.

16.2.1 Types of quadrilaterals

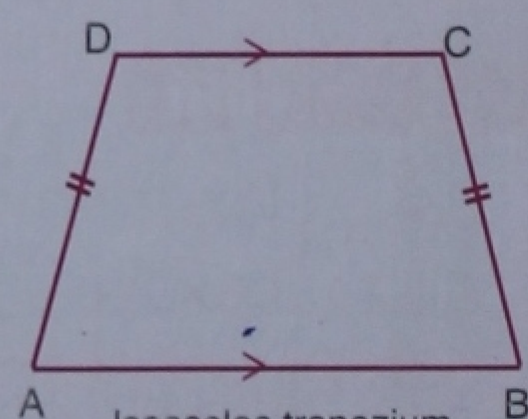
1. Trapezium. *A quadrilateral in which one pair of opposite sides is parallel is called a **trapezium** (abbreviated trap.)*



trapezium

The parallel sides are called **bases** of the trapezium. The line segment joining mid-points of non-parallel sides is called its **median.**

In the adjoining quadrilateral, AB ∥ DC whereas AD and BC are non-parallel, so ABCD is a trapezium, AB and CD are its *bases*, and EF is its *median* where E, F are mid-points of the sides AD, BC respectively.



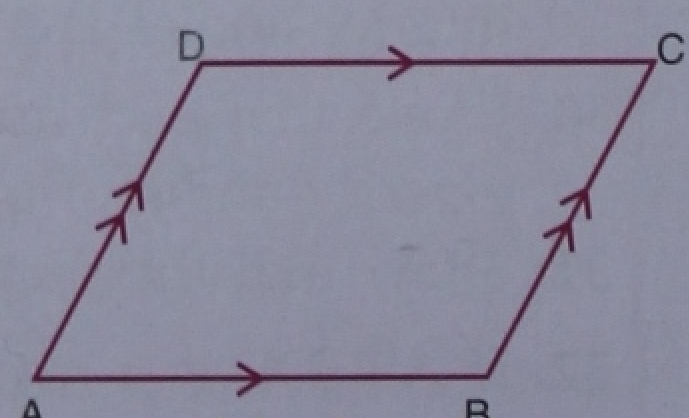
Isosceles trapezium

Isosceles trapezium. *If non-parallel sides of a trapezium are equal, then it is called an **isosceles trapezium.***

Here AB ∥ DC, AD and BC are non-parallel but AD = BC, so ABCD is an isosceles trapezium.

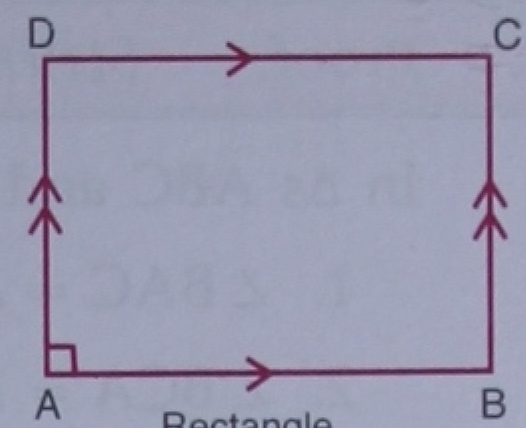
2. Parallelogram. *A quadrilateral in which both pairs of opposite sides are parallel is called a **parallelogram**. It is usually written as '∥gm'.*

In the adjoining quadrilateral, AB ∥ DC and AD ∥ BC, so ABCD is a parallelogram.



Parallelogram

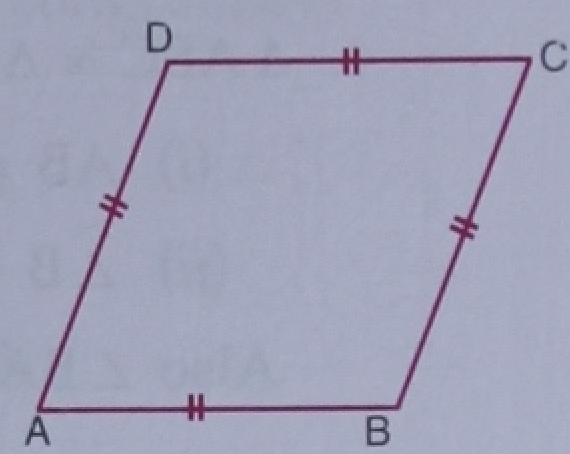
3. Rectangle. *If one of the angles of a parallelogram is a right angle, then it is called a* ***rectangle***.



Rectangle

In the adjoining parallelogram, $\angle A = 90°$, so ABCD is a rectangle. Of course, the remaining angles will also be right angles.

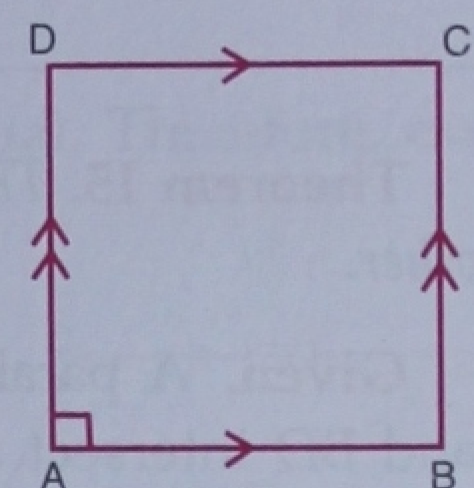
4. Rhombus. *If all the sides of a quadrilateral are equal, then it is called a* ***rhombus.***



In the adjoining quadrilateral AB = BC = CD = DA, so ABCD is a rhombus.

[Every rhombus is a parallelogram, see corollary to theorem 16.]

5. Square. *If two adjacent sides of a rectangle are equal, then it is called a* ***square***. Alternatively, if one angle of a rhombus is a right angle, it is called a ***square***.

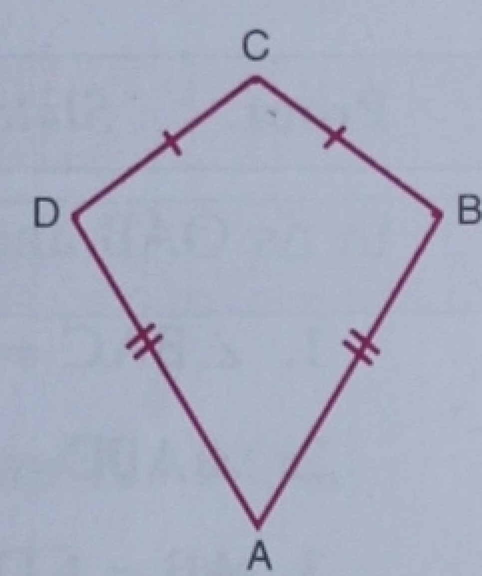


Square

In the adjoining rectangle, AB = AD, so ABCD is a square.

Of course, the remaining sides are also equal.

6. Kite. *A quadrilateral in which two pairs of adjacent sides are equal is called a* ***kite*** *(or* ***diamond****).*



Kite

In the adjoining quadrilateral, AD = AB and DC = BC, so ABCD is a kite.

Remark

From the above definitions it follows that parallelograms include rectangles, squares and rhombi (plural of rhombus), therefore, any result which is true for a parallelogram is certainly true for all these figures.

Theorem 14. (*i*) *The opposite sides of a parallelogram are equal.*

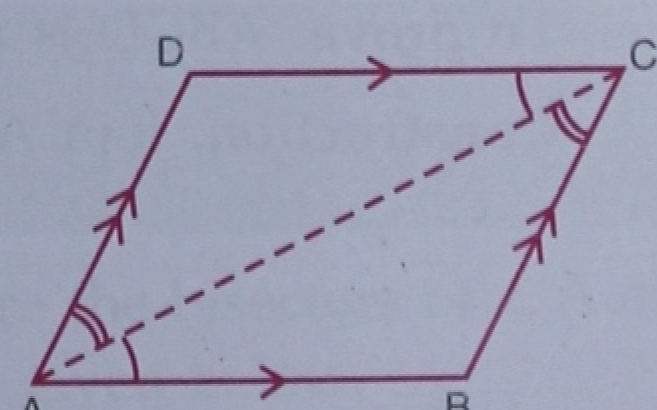
(*ii*) *The opposite angles of a parallelogram are equal.*

Given. A parallelogram ABCD.

To prove. (*i*) AB = DC and BC = AD.

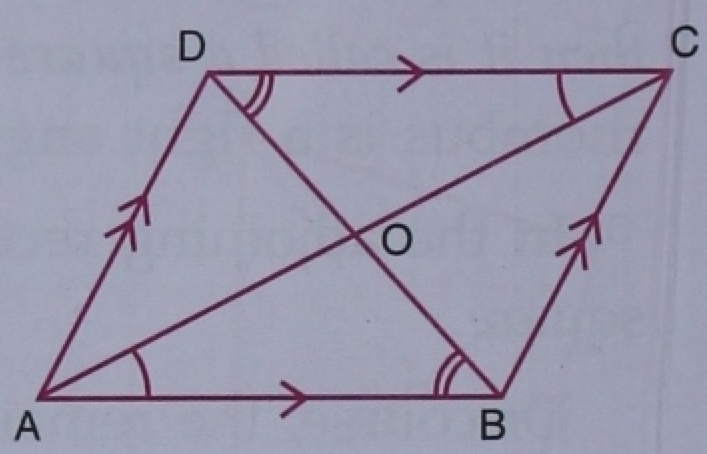
(*ii*) $\angle B = \angle D$ and $\angle A = \angle C$.

Construction. Join AC.



Proof. *Statements*	*Reasons*
In ∆s ABC and CDA	
1. ∠BAC = ∠ACD	1. Alt. ∠s are equal, since AB ∥ DC.
2. ∠BCA = ∠CAD	2. Alt. ∠s are equal, since AD ∥ BC.
3. AC = AC	3. Common.
4. ∆ABC ≅ ∆CDA	4. A.S.A. (Axiom of congruency)
∴ (*i*) AB = DC and BC = AD	'c.p.c.t.'
(*ii*) ∠B = ∠D	'c.p.c.t.'
Also ∠BAC + ∠CAD	
= ∠ACD + ∠BCA	Adding 1 and 2
⇒ ∠A = ∠C.	
Q.E.D.	

Theorem 15. *The diagonals of a parallelogram bisect each other.*



Given. A parallelogram ABCD whose diagonals AC and BD intersect at O.

To prove. OA = OC and OB = OD.

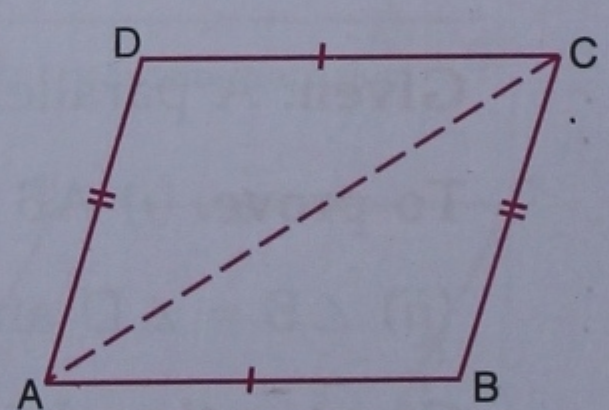
Proof. *Statements*	*Reasons*
In ∆s OAB and OCD	
1. ∠BAC = ∠ACD	1. Alt. ∠s are equal, since AB ∥ DC.
2. ∠ABD = ∠BDC	2. Alt. ∠s are equal, since AB ∥ DC.
3. AB = CD	3. Opp. sides of a ∥gm are equal.
4. ∆OAB ≅ ∆OCD	4. A.S.A. (Axiom of congruency)
∴ OA = OC and OB = OD	'c.p.c.t.'
Q.E.D.	

Theorem 16. *If each pair of opposite sides of a quadrilateral are equal, then it is a parallelogram.*

Given. A quadrilateral ABCD in which AB = DC and BC = AD.

To prove. ABCD is a parallelogram.

Construction. Join AC.

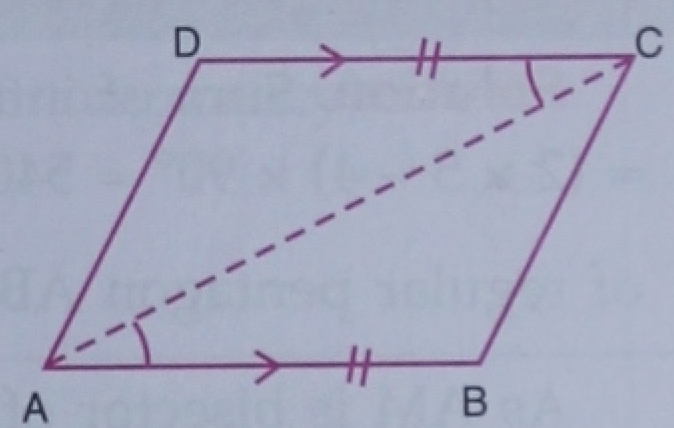


Proof. *Statements*	*Reasons*
In Δs ABC and CDA	
1. AB = DC	1. Given.
2. BC = AD	2. Given.
3. AC = AC	3. Common.
4. Δ ABC ≅ Δ CDA	4. S.S.S. (Axiom of congruency)
5. ∠BAC = ∠ACD	5. c.p.c.t.
⇒ AB ∥ DC	alt. ∠s are equal.
6. ∠ACB = ∠CAD	6. c.p.c.t.
⇒ BC ∥ AD	alt. ∠s are equal.
Hence ABCD is a parallelogram.	
Q.E.D.	

Corollary. *Every rhombus is a parallelogram.*

[In a rhombus, all sides are equal, so opposite sides are equal. Therefore, every rhombus is a parallelogram.]

Theorem 17. *If a pair of opposite sides of a quadrilateral are equal and parallel, it is a parallelogram.*



Given. A quadrilateral ABCD in which AB ∥ DC and AB = DC.

To prove. ABCD is a parallelogram.

Construction. Join AC.

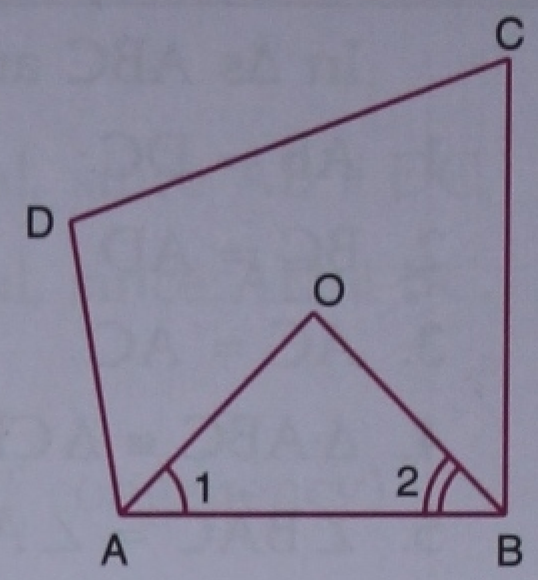
Proof. *Statements*	*Reasons*
In Δs ABC and CDA	
1. ∠BAC = ∠ACD	1. Alt. ∠s are equal, since AB ∥ DC.
2. AB = DC	2. Given.
3. AC = AC	3. Common.
4. Δ ABC ≅ Δ CDA	4. S.A.S. (Axiom of congruency)
5. ∠ACB = ∠CAD	5. 'c.p.c.t.'
6. AD ∥ BC	6. AC cuts AD and BC, and alt. ∠s are equal.
Hence ABCD is a parallelogram.	**Q.E.D.**

ILLUSTRATIVE EXAMPLES

Example 1. *In a quadrilateral ABCD, AO and BO are the bisectors of ∠A and ∠B respectively. Prove that* $\angle AOB = \frac{1}{2}(\angle C + \angle D)$.

Solution. Given ABCD is a qudrilateral, OA and OB are the bisectors of ∠A and ∠B respectively. Mark the angles as shown in the figure given below.

As OA and OB are bisectors of $\angle A$ and $\angle B$ respectively,



$$\angle 1 = \frac{1}{2}\angle A \text{ and } \angle 2 = \frac{1}{2}\angle B \qquad ...(i)$$

$$\angle AOB + \angle 1 + \angle 2 = 180°$$

(sum of angles in ΔOAB)

$\Rightarrow \quad \angle AOB = 180° - (\angle 1 + \angle 2)$

$\Rightarrow \quad \angle AOB = 180° - \frac{1}{2}(\angle A + \angle B) \qquad$ (using (*i*))

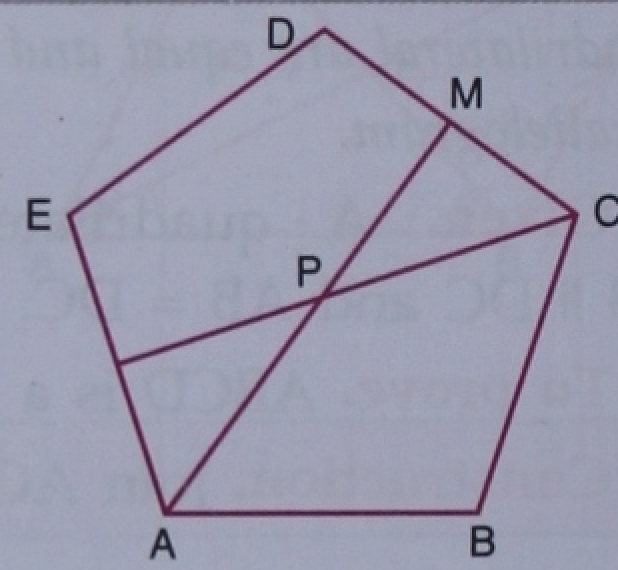
$\Rightarrow \quad \angle AOB = 180° - \frac{1}{2}(360° - (\angle C + \angle D))$

[$\because$ sum of angles in a qudrilateral is 360°, so $\angle A + \angle B + \angle C + \angle D = 360°$

$\Rightarrow \angle A + \angle B = 360° - (\angle C + \angle D)$]

$\Rightarrow \quad \angle AOB = \frac{1}{2}(\angle C + \angle D)$.

Example 2. *ABCDE is a regular pentagon and bisector of $\angle EAB$ meets CD at M. If bisector of $\angle BCD$ meets AM at P, find $\angle CPM$.*



Solution. Sum of interior angles of pentagon $= (2 \times 5 - 4) \times 90° = 540°$, so each interior angle of regular pentagon ABCDE $= \frac{540°}{5} = 108°$.

As AM is bisector of $\angle A$,

$\therefore \quad \angle MAB = \frac{1}{2}\angle A = \frac{1}{2} \times 108° = 54°$.

Since the sum of angles in a quadrilateral is 360°, so in quadrilateral ABCM,

$\angle MAB + \angle B + \angle C + \angle AMC = 360°$

$\Rightarrow \quad 54° + 108° + 108° + \angle AMC = 360°$

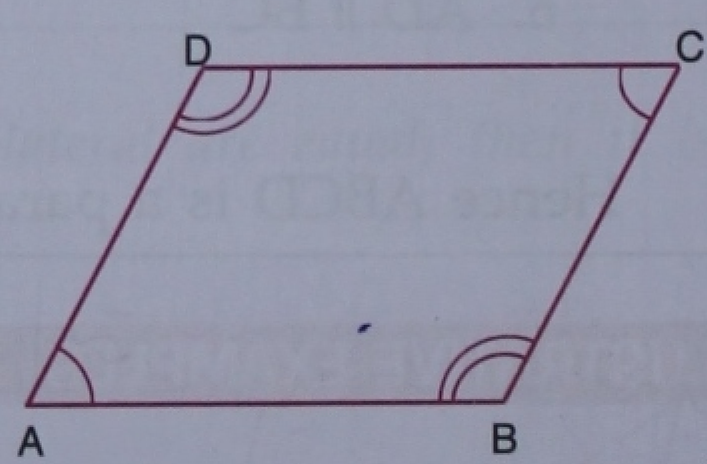
$\Rightarrow \quad 270° + \angle AMC = 360° \Rightarrow \angle AMC = 90°$ *i.e.* $\angle PMC = 90°$

As CP is bisector $\angle C$, $\angle PCM = \frac{1}{2}\angle C = \frac{1}{2} \times 108° = 54°$.

In ΔPCM, $\angle CPM + \angle PMC + \angle PCM = 180° \qquad (\because$ sum of angles in a $\Delta = 180°)$

$\Rightarrow \quad \angle CPM + 90° + 54° = 180° \Rightarrow \angle CPM = 36°$.

Example 3. *If opposite angles of a quadrilateral are equal, it is a parallelogram.*



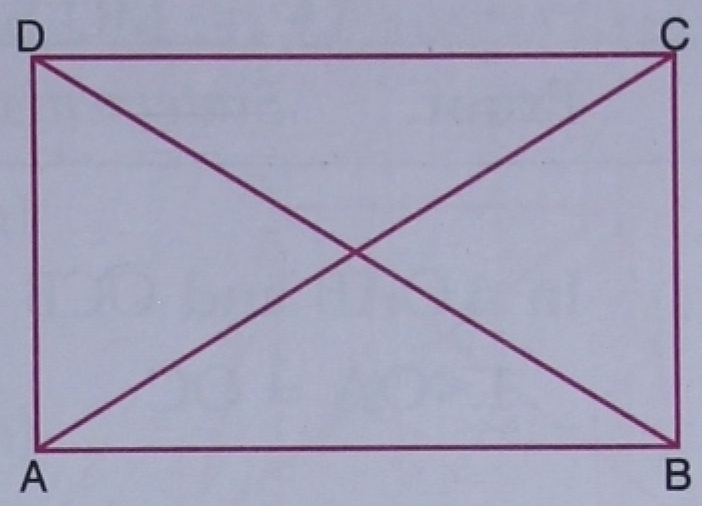
Given. A quadrilateral ABCD in which $\angle A = \angle C$ and $\angle B = \angle D$.

To prove. ABCD is a parallelogram.

Proof. *Statements*	*Reasons*
1. $\angle A + \angle B + \angle C + \angle D = 360°$	1. Sum of angles of a quad.
2. $2\angle A + 2\angle B = 360°$	2. $\angle C = \angle A$ and $\angle D = \angle B$ given.
3. $\angle A + \angle B = 180°$	3. Dividing by 2.

4. AD ∥ BC Similarly AB ∥ DC. Hence, ABCD is a parallelogram. **Q.E.D.**	4. Co-int. ∠s are supplementary.

Example 4. *Prove that the diagonals of a rectangle are equal.*

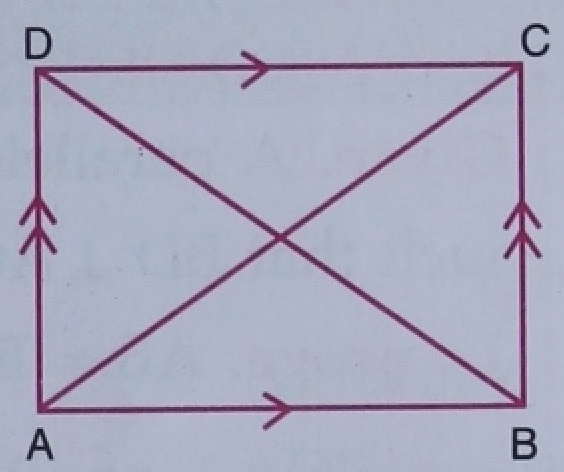


Given. ABCD is a rectangle.

To prove. AC = BD.

Proof. *Statements*	*Reasons*
In Δs ABC and BAD	
1. AB = AB	1. Common.
2. BC = AD	2. Opp. sides of rectangle.
3. ∠ABC = ∠BAD	3. Each angle of a rectangle is 90°.
4. Δ ABC ≅ Δ BAD	4. S.A.S. (Axiom of congruency).
5. AC = BD	5. 'c.p.c.t.'
Q.E.D.	

Example 5. *If the diagonals of a parallelogram are equal, then the parallelogram is a rectangle.*

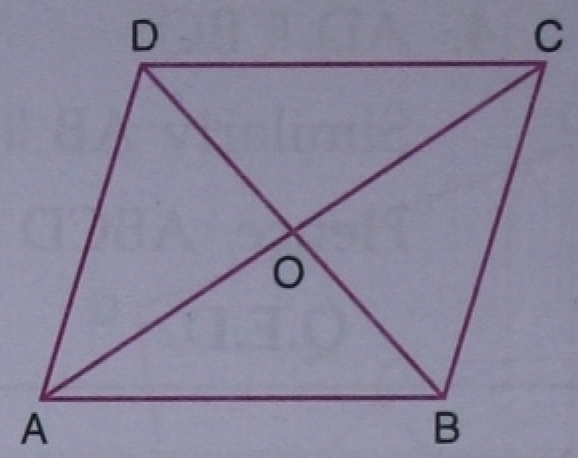


Given. A parallelogram ABCD in which AC = BD.

To prove. ∠A = 90°.

Proof. *Statements*	*Reasons*
In Δs ABC and BAD	
1. AB = AB	1. Common.
2. BC = AD	2. Opp. sides of ∥gm.
3. AC = BD	3. Given.
4. Δ ABC ≅ Δ BAD	4. S.S.S. (Axiom of congruency).
5. ∠B = ∠A	5. 'c.p.c.t.'
6. ∠A + ∠B = 180°	6. AD ∥ BC, ∠A and ∠B are co-int.
7. 2 ∠A = 180°	7. From 5 and 6.
⇒ ∠A = 90°	
Q.E.D.	

Example 6. *Prove that the diagonals of a rhombus intersect at right angles.*

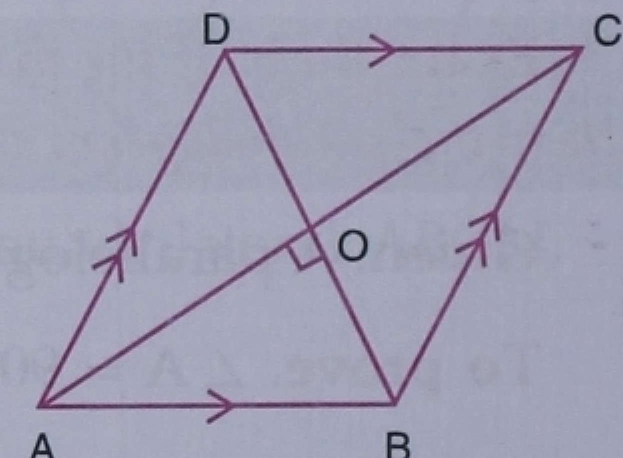


Given. A rhombus ABCD such that its diagonals AC and BD intersect at O.

To prove. AC ⊥ BD.

Proof. *Statements*	*Reasons*
In ∆OAB and OCB	
1. OA = OC	1. Every rhombus is a parallelogram and in a parallelogram, diagonals bisect each other.
2. AB = BC	2. Sides of rhombus ABCD.
3. OB = OB	3. Common.
4. ∆OAB ≅ ∆OCB	4. S.S.S. (Axiom of congruency)
5. ∠AOB = ∠BOC	5. 'c.p.c.t.'
6. ∠AOB + ∠BOC = 180°	6. AOC is a straigth line.
7. 2∠AOB = 180°	7. From 5 and 6.
⇒ ∠AOB = 90°	
⇒ AC ⊥ BD	
Q.E.D.	

Example 7. *If the diagonals of a parallelogram intersect at right angles, then it is a rhombus.*



Given. A parallelogram ABCD such that BD ⊥ AC.

To prove. AB = BC.

Proof. *Statements*	*Reasons*
In ∆s OAB and OCB	
1. OA = OC	1. Diagonals of a ∥gm bisect each other.
2. ∠AOB = ∠BOC	2. Each being a right angle.
3. OB = OB	3. Common.
4. ∆OAB ≅ ∆OCB	4. S.A.S. (Axiom of congruency).
5. AB = BC	5. 'c.p.c.t.'
Q.E.D.	

Remark

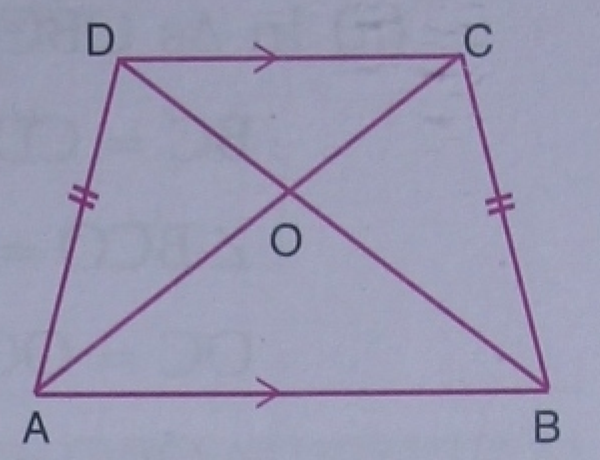
From the above example, it follows that rhombus is a special parallelogram whose diagonals bisect each other at right angles.

Example 8. *In the adjoining figure, ABCD is an isosceles trapezium and its diagonals meet at O. Prove that :*

(i) ∠A = ∠B and ∠C = ∠D.

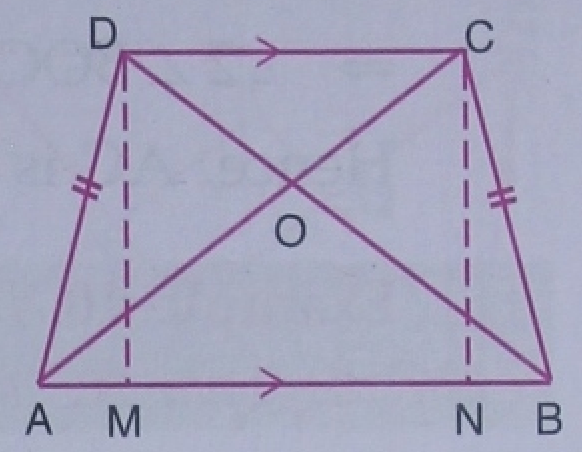
(ii) AC = BD.

(iii) OA = OB and OC = OD.



Solution. (*i*) From C and D, draw perpendiculars CN and DM on AB respectively.

In Δs AMD and BNC



AD = BC (given)

∠AMD = ∠CNB

(∵ DM ⊥ AB and CN ⊥ AB, by construction)

MD = CN (distance between parellel lines)

∴ ΔAMD ≅ ΔBNC (R.H.S. Axiom of congruency)

∴ ∠A = ∠B (c.p.c.t.)

Also ∠A + ∠D = 180° and ∠B + ∠C = 180°) (∵ AB ∥ DC, sum of co-int. ∠s = 180°)

⇒ ∠A + ∠D = ∠B + ∠C

⇒ ∠D = ∠C (∵ ∠A = ∠B, proved above)

(*ii*) In Δs ABD and BAC

∠A = ∠B (proved above)

AD = BC (given)

AB = AB (common)

∴ ΔABD ≅ ΔBAC (S.A.S. axiom of congruency)

∴ AC = BD (c.p.c.t.)

(*iii*) In Δs OAD and OBC,

AD = BC (given)

∠AOD = ∠BOC (vert. opp. ∠s)

∠ADO = ∠BCO (∵ ΔABD ≅ ΔBAC, so ∠ADB = ∠ACB)

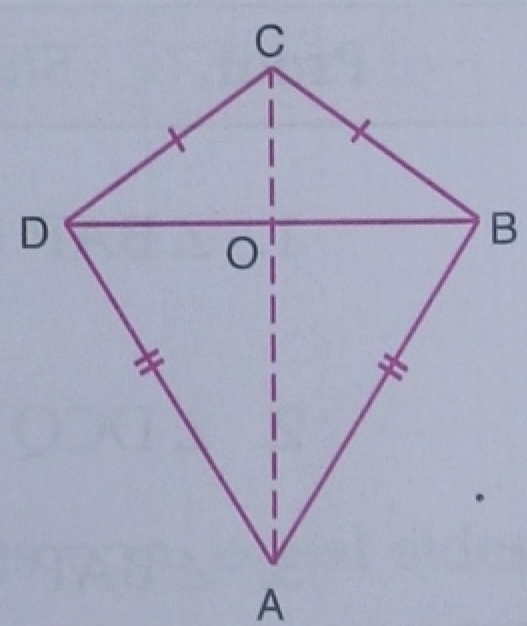
∴ ΔOAD ≅ ΔOBC (A.A.S. axiom of congruency)

∴ OA = OB and OC = OD. (c.p.c.t.)

Example 9. *In the adjoining figure, ABCD is a kite in which AB = AD and BC = CD. Prove that :*

(i) AC is a bisector of ∠A and of ∠C.

(ii) AC is perpendicular bisector of BD.



Solution. (*i*) In Δs ABC and ADC

AB = AD (given)

BC = CD (given)

CA = CA (common)

∴ ΔABC ≅ ΔADC (S.S.S. Axiom of congruency)

∠BAC = ∠CAD and ∠BCA = ∠ACD. (c.p.c.t.)

Hence, AC is bisector of ∠A and of ∠C.

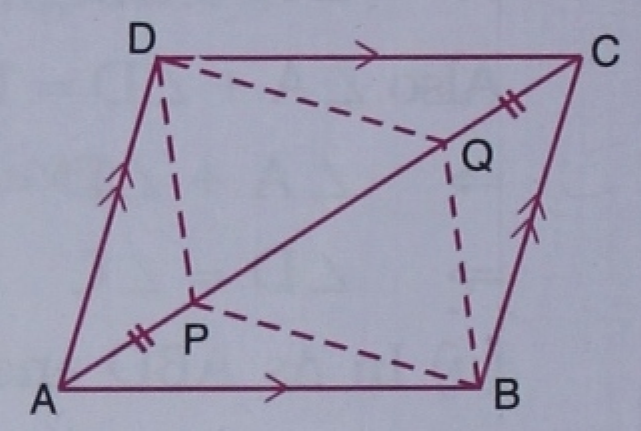
(*ii*) In ∆s OBC and ODC

BC = CD	(given)
∠BCO = ∠OCD	(proved above)
OC = OC	(common)
∴ ∆OBC ≅ ∆ODC	(S.A.S. Axiom of congruency)
∴ OB = OD and ∠BOC = ∠COD	(c.p.c.t.)
But ∠BOC + ∠COD = 180°	(linear pair)

⇒ 2∠BOC = 180° ⇒ ∠BOC = 90°.

Hence, AC is perpendicular bisector of BD.

Example 10. *In the adjoining figure, ABCD is a parallelogram. P and Q are points on the diagonal AC such that AP = CQ. Prove that PBQD is a parallelogram.*



Solution. In ∆s ABP and CDQ

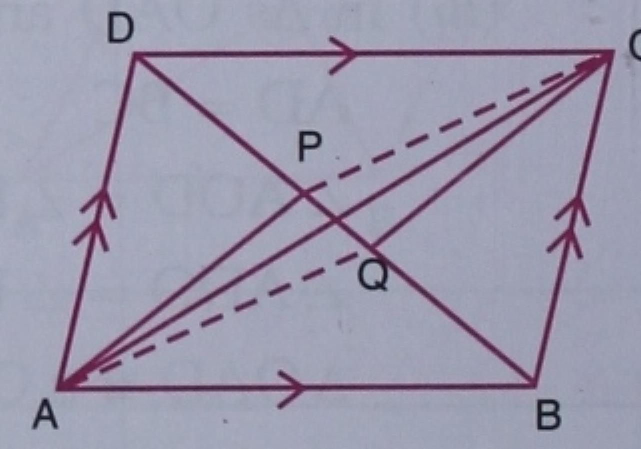
AP = CQ	(given)
AB = CD	(opp. sides of ‖gm ABCD).
∠PAB = ∠QCD	(alt. ∠s, AB ‖ CD)
∴ ∆ABP ≅ ∆CDQ	
∴ BP = QD	(c.p.c.t.)

Similarly, ∆APD ≅ ∆CQB

∴ DP = QB (c.p.c.t.)

Thus, in quadrilateral PBQD, BP = QD and PD = QB *i.e.* each pair of opposite sides are equal, therefore, PBQD is a parallelogram. (Theorem 16).

Example 11. *ABCD is a parallelogram. If the bisectors of ∠A and ∠C meet the diagonal BD at P and Q respectively, prove that the quadrilateral PCQA is a parallelogram.*



Given. ABCD is a ‖gm, AP bisects ∠A and CQ bisects ∠C.

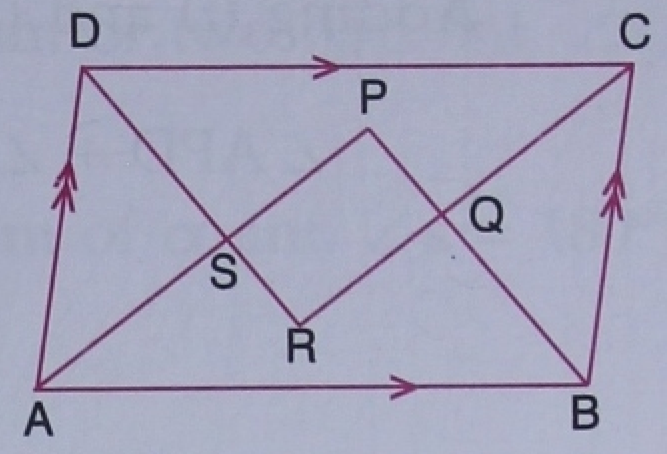
To prove. AP ‖ QC and PC ‖ AQ.

Construction. Join AC.

Proof. *Statements*	*Reasons*
1. $\angle BAP = \frac{1}{2}\angle A$	1. AP is bisector of ∠A.
2. $\angle DCQ = \frac{1}{2}\angle C$	2. CQ is bisector of ∠C.
3. ∠BAP = ∠DCQ	3. ∠A = ∠C, since ABCD is a ‖gm.
4. ∠BAC = ∠DCA	4. Alt. ∠s, since AB ‖ DC.
5. ∠BAP – ∠BAC = ∠DCQ – ∠DCA	5. Subtracting 4 from 3.
6. ∠CAP = ∠ACQ	6. From figure.

7. AP ∥ QC Similarly, PC ∥ AQ. Hence PCQA is a parallelogram. **Q.E.D.**	7. Alt. ∠s are equal.

Example 12. *Prove that the bisectors of the angles of a parallelogram form a rectangle.*



Solution. Let ABCD be a parallelogram and let P, Q, R and S be the points of intersection of the bisectors of ∠A and ∠B, ∠B and ∠C, ∠C and ∠D, and ∠D and ∠A respectively.

We need to show that PQRS is a rectangle.

As ABCD is a ∥ gm, AD ∥ BC and AB is a transvesal.

∴ $\angle A + \angle B = 180°$ (sum of co-int. ∠s = 180°)

⇒ $\frac{1}{2}\angle A + \frac{1}{2}\angle B = 90°$

⇒ $\angle PAB + \angle PBA = \angle 90°$ (∵ AP is bisector of ∠A and BP is bisector of ∠B)

In ΔPAB, $\angle APB + \angle PAB + \angle PBA = 180°$ (sum of angles in a Δ)

⇒ $\angle APB + 90° = 180°$ ⇒ $\angle APB = 90°$ ⇒ $\angle SPQ = 90°$.

Similarly, $\angle PQR = 90°$, $\angle QRS = 90°$ and $\angle RSP = 90°$.

So, PQRS is a quadrilateral in which each angle is 90°.

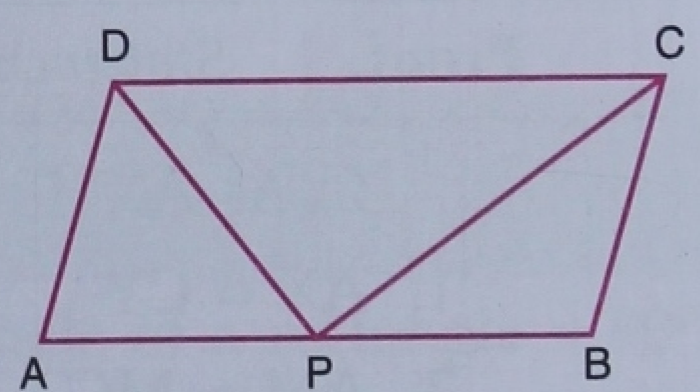
Now, $\angle SPQ = \angle QRS$ (each = 90°)

and $\angle PQR = \angle RSP$ (each = 90°)

Thus, PQRS is a quadrilateral in which both pairs of opposite angles are equal, therefore, PQRS is a parallelogram. Also, in this parallelogram one angle (in fact all angles) is 90°.

Therefore, PQRS is a rectangle.

Example 13. *In the adjoining figure, ABCD is a parallelogram. If AB = 2AD and P is mid-point of AB, prove that ∠DPC = 90°.*



Solution. Given P is mid-point of AB

⇒ $AP = PB = \frac{1}{2}AB$.

Also $AB = 2AD$ ⇒ $AD = \frac{1}{2}AB$.

∴ $AP = AD$.

In Δ APD, AP = AD

⇒ $\angle APD = \angle ADP$ (angles opp. equal sides)

But $\angle A + \angle APD + \angle ADP = 180°$ (sum of angles in a Δ = 180°)

⇒ $\angle A + \angle APD + \angle APD = 180°$ (∵ ∠ ADP = ∠ APD)

⇒ $2\angle APD = 180° - \angle A$ ⇒ $\angle APD = \frac{180° - \angle A}{2}$ ...(*i*)

As PB = AP and BC = AD (opp. sides of ∥gm ABCD)

$\Rightarrow$ PB = BC.

In ΔBPC, PB = BC $\Rightarrow \angle CPB = \angle BCP$.

But $\angle B + \angle CPB + \angle BCP = 180°$

$\Rightarrow \angle B + \angle CPB + \angle CPB = 180°$

$\Rightarrow 2\angle CPB = 180° - \angle B \Rightarrow \angle CPB = \frac{180° - \angle B}{2}$ ...(*ii*)

Adding (*i*) and (*ii*), we get

$$\angle APD + \angle CPB = 180° - \frac{1}{2}(\angle A + \angle B)$$

$$= 180° - \frac{1}{2}(180°)$$

($\because$ ABCD is a ∥gm, AD ∥ BC, so $\angle A + \angle B = 180°$)

$\Rightarrow \angle APD + \angle CPB = 90°$ ...(*iii*)

But $\angle APD + \angle DPC + \angle CPB = 180°$ ($\because$ APB is a st. line)

$\Rightarrow (\angle APD + \angle CPB) + \angle DPC = 180°$

$\Rightarrow 90° + \angle DPC = 180°$

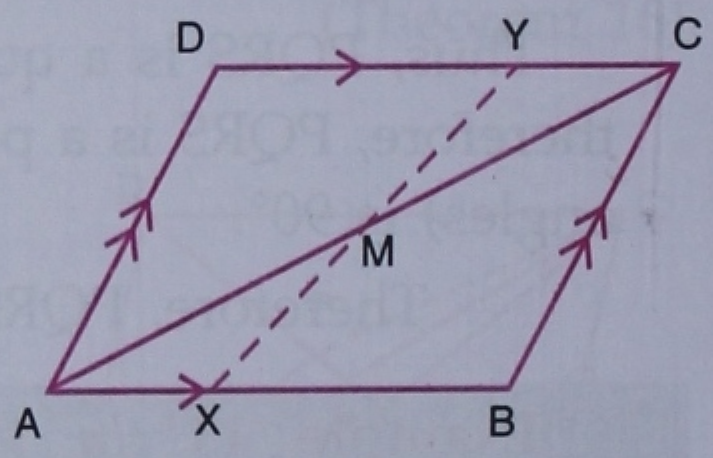
$\Rightarrow \angle DPC = 90°$ (using (*iii*))

Example 14. *In the parallelogram ABCD, M is mid-point of AC, and X, Y are points on AB and DC respectively such that AX = CY. Prove that*

(i) triangle AXM is congruent to triangle CYM.

(ii) XMY is a straight line.

Given. ABCD is a ∥gm, M is mid-point of AC, X, Y are points on AB, CD such that AX = CY.



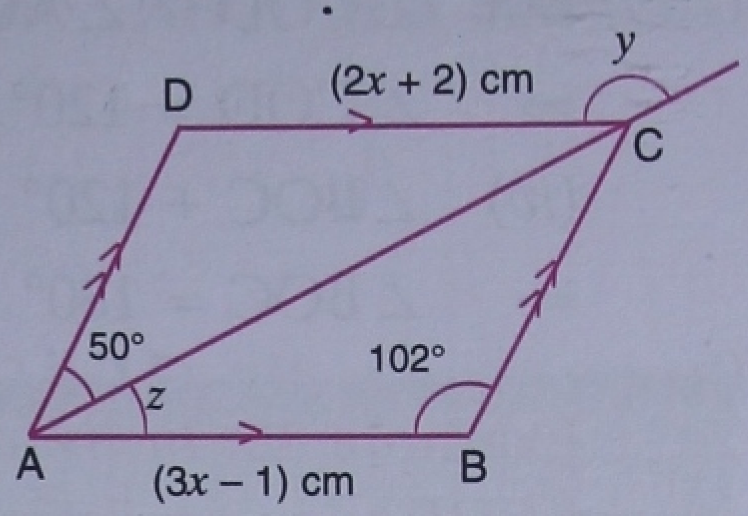
To prove. (*i*) $\Delta AXM \cong \Delta CYM$

(*ii*) XMY is a straight line.

Construction. Join XM and MY.

Proof. *Statements*	*Reasons*
In Δs AXM and CYM	
1. AX = CY	1. Given.
2. AM = MC	2. M is mid-point of AC.
3. $\angle XAM = \angle MCY$	3. Alt. $\angle$s, since AB ∥ DC.
$\therefore$ (*i*) $\Delta AXM \cong \Delta CYM$	S.A.S. (Axiom of congruency).
4. $\angle CMY = \angle AMX$	4. 'c.p.c.t.'
5. $\angle XMC = \angle XAM + \angle AXM$	5. Ext. $\angle$ = sum of two opp. int. $\angle$s.
6. $\angle CMY + \angle XMC = \angle AMX + \angle XAM + \angle AXM$	6. Adding 4 and 5.
7. $\angle CMY + \angle XMC = 180°$	7. Sum of $\angle$s of a Δ = 180°.
(*ii*) XMY is a straight line	Sum of adj. $\angle$s = 180°.
Q.E.D.	

Example 15. *In the adjoining figure, ABCD is parallelogram. Find the values of x, y and z.*



Solution. Given ABCD is a parallelogram,

$3x - 1 = 2x + 2$ (opp. sides are equal)

$\Rightarrow \quad x = 3.$

$\angle D = \angle B = 102°$ (opp. ∠s are equal)

For ΔACD, $y = 50° + \angle D$ (∵ ext. ∠ = sum of two opp. int. ∠s)

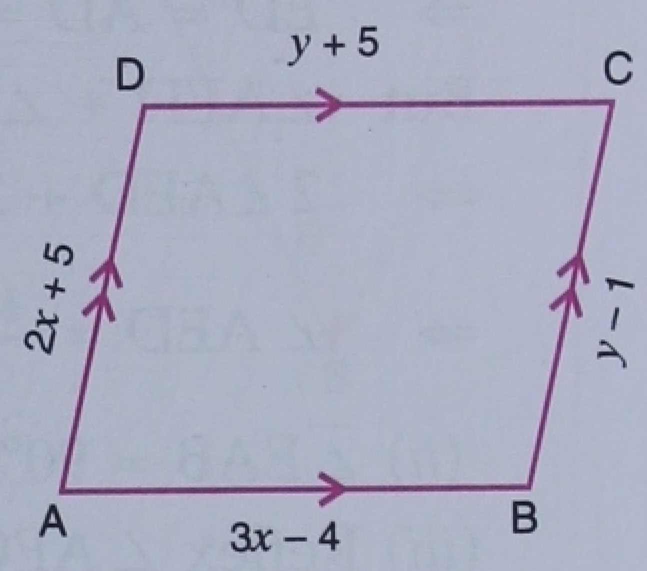
$\Rightarrow \quad y = 50° + 102° = 152°$

$\angle DAB + 102° = 180°$ (AD ∥ BC, sum of co-int. ∠s = 180°)

$\Rightarrow \quad \angle DAB = 180° - 102° = 78°.$

From figure, $z = \angle DAB - \angle DAC = 78° - 50° = 28°.$

Example 16. *In the adjoining figure, ABCD is a parallelogram. Find the ratio of AB : BC. All measurements are in centimetres.*



Solution. Given ABCD is a parallelogram,

$3x - 4 = y + 5$ (opp. sides are equal)

$\Rightarrow \quad 3x - y - 9 = 0$ ...(*i*)

and $2x + 5 = y - 1$ (opp. sides are equal)

$\Rightarrow \quad 2x - y + 6 = 0$ ...(*ii*)

On subtracting (*ii*) from (*i*), we get

$x - 15 = 0 \Rightarrow x = 15.$

On substituting this value of x in (*i*), we get

$3 \times 15 - y - 9 = 0 \Rightarrow 36 - y = 0 \Rightarrow y = 36.$

$\therefore \quad AB = 3x - 4 = 3 \times 15 - 4 = 41$

and $BC = y - 1 = 36 - 1 = 35.$

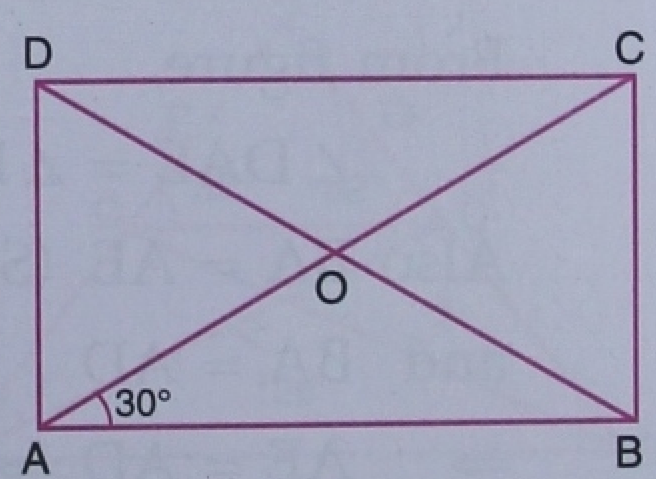
Hence AB : BC = 41 : 35.

Example 17. *In a rectangle ABCD, diagonals intersect at O. If ∠OAB = 30°, find*
(i) ∠ACB (ii) ∠ABO (iii) ∠COD (iv) ∠BOC.

Solution. (*i*) $\angle ABC = 90°$ (each angle of a rectangle = 90°)

$\angle ACB + 30° + 90° = 180°$ (sum of angles in ΔABC)

$\Rightarrow \quad \angle ACB = 180° - 30° - 90° = 60°.$



(*ii*) $AC = BD$ (diagonals are equal)

$\Rightarrow \quad 2AO = 2BO$ (diagonals bisect each other)

$\Rightarrow \quad AO = BO$

$\Rightarrow \quad \angle ABO = \angle OAB$

(angles opp. equal sides in ΔOAB)

$\Rightarrow \quad \angle ABO = 30°$ (∵ ∠OAB = 30° given)

(*iii*) $\angle AOB + 30° + 30° = 180°$ (sum of angles in ΔOAB)

$\Rightarrow \quad \angle AOB = 180° - 30° - 30° = 120°.$

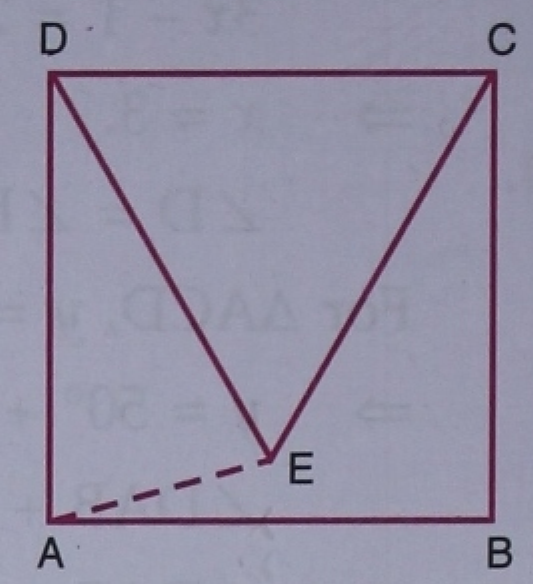
But ∠COD = ∠AOB (vert. opp. ∠s)

⇒ ∠COD = 120°.

(*iv*) ∠BOC + 120° = 180° (linear pair)

⇒ ∠BOC = 180° – 120° = 60°.

Example 18. *In the adjoining figure, ABCD is a square and CDE is an equilateral triangle. Find*
(i) ∠AED (ii) ∠EAB (iii) reflex ∠AEC.



Solution. (*i*) From figure, ∠ADE = 90° – 60°

(∵ each angle in a square = 90° and each angle in an equilateral triangle = 60°)

⇒ ∠ADE = 30°.

ED = DC (sides of equilateral triangle)

AD = DC (sides of square)

⇒ ED = AD ⇒ ∠AED = ∠EAD (angles opp. equal sides in ΔAED)

But ∠AED + ∠EAD + ∠ADE = 180° (sum of angles in ΔAED)

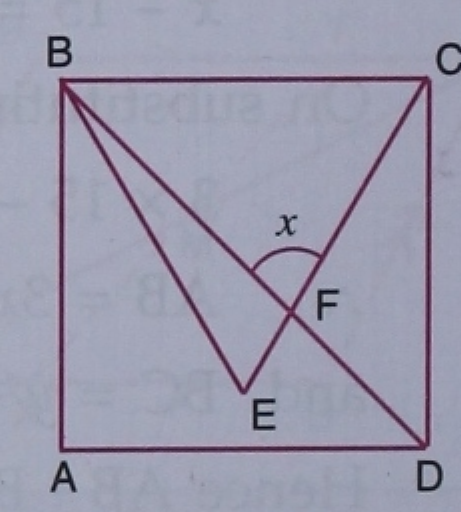
⇒ 2∠AED + 30° = 180°

⇒ ∠AED = $\frac{180° - 30°}{2}$ = 75°.

(*ii*) ∠EAB = 90° – 75° = 15° (∵ ∠EAD = ∠AED = 75°)

(*iii*) Reflex ∠AEC = 360° – 75° – 60° = 225°.

Example 19. *BEC is an equilateral triangle in the square ABCD. Find the value of x in the figure.*



Solution. Since ABCD is a square and BD is a diagonal,

∴ ∠DBC = 45°.

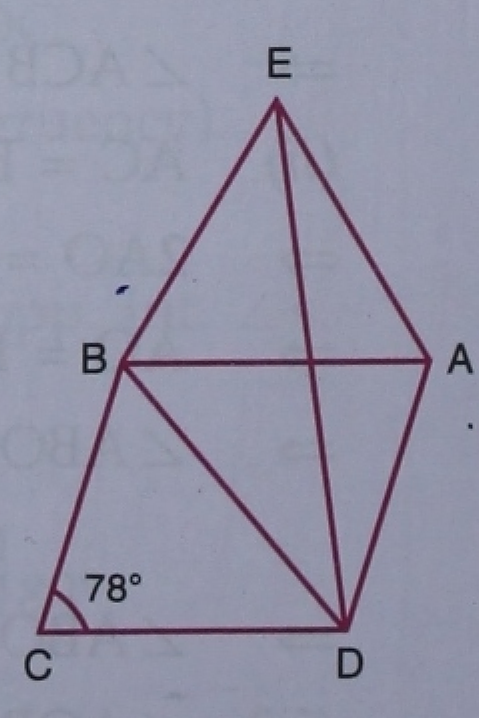
As BEC is an equilateral triangle,

∠BCE = 60°.

In ΔBFC, x + 45° + 60° = 180°

⇒ x = 180° – 45° – 60° = 75°.

Example 20. *In the figure, ABCD is a rhombus and ABE is an equilateral triangle. E and D lie on opposite sides of AB. If ∠BCD = 78°, calculate ∠ADE and ∠BDE.*



Solution. Since ABCD is a rhombus, ∠DAB = ∠BCD = 78°.

As ABE is an equilateral triangle, ∠BAE = 60°.

From figure,

∠DAE = ∠DAB + ∠BAE = 78° + 60° = 138°.

Also BA = AE (Since ABE is equilateral triangle)

and BA = AD (∵ ABCD is a rhombus)

⇒ AE = AD ⇒ ∠ADE = ∠AED

(∵ angles opp. equal sides in ΔAED)

∴ ∠ADE = $\frac{1}{2}$ (180° – 138°) = $\frac{1}{2}$ × 42° = 21°.

In ΔBCD, $BC = CD$ (∵ ABCD is a rhombus)

$\Rightarrow \quad \angle CBD = \angle CDB.$

$\therefore \quad \angle CBD = \frac{1}{2}(180^\circ - 78^\circ) = \frac{1}{2}\, 102^\circ = 51^\circ.$

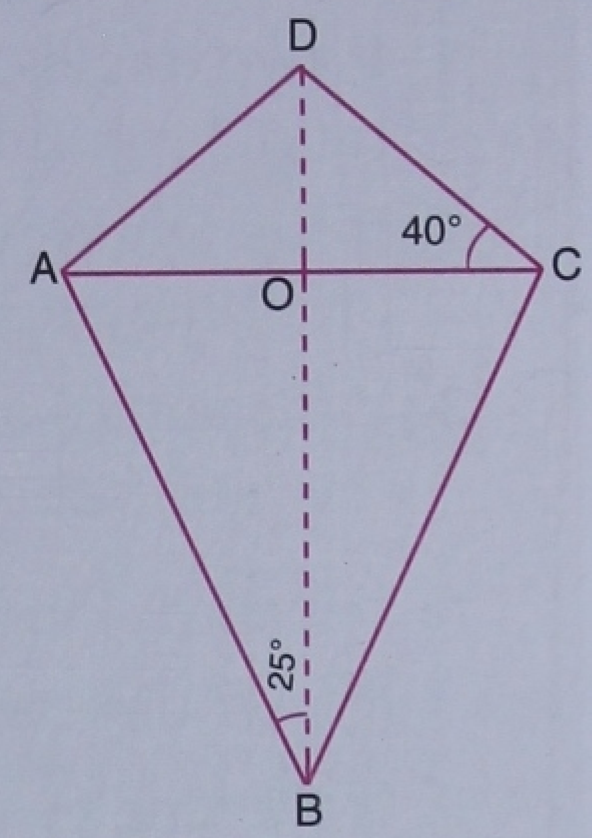
But $\angle BDA = \angle CBD$ (BC ∥ AD, alt. ∠s are equal)

$\Rightarrow \quad \angle BDA = 51^\circ.$

From figure, $\angle BDE = \angle BDA - \angle EDA = 51^\circ - 21^\circ = 30^\circ.$

Example 21. *In the adjoining kite, diagonals intersect at O. If $\angle ABO = 25^\circ$ and $\angle OCD = 40^\circ$, find*

(i) $\angle ABC$ (ii) $\angle ADC$ (iii) $\angle BAD$.



Solution. (*i*) Since the diagonal BD bisects $\angle ABC$, $\angle ABC = 2\angle ABO = 2 \times 25^\circ = 50^\circ.$

(*ii*) $\angle DOC = 90^\circ$

(diagonals intersect at right angles)

$\angle ODC + 40^\circ + 90^\circ = 180^\circ$

(sum of angles in ΔOCD)

$\Rightarrow \quad \angle ODC = 180^\circ - 40^\circ - 90^\circ = 50^\circ.$

Since the diagonal BD bisects $\angle ADC$,

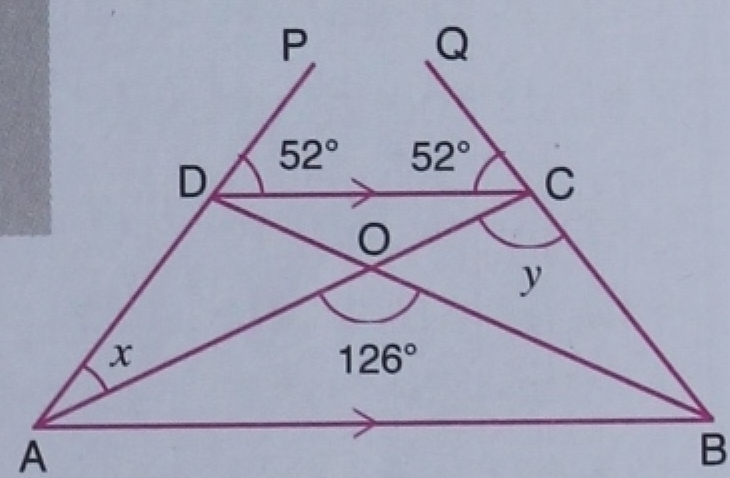
$\angle ADC = 2\angle ODC = 2 \times 50^\circ = 100^\circ.$

(*iii*) Since the diagonal BD bisects $\angle ADC$, $\angle ADB = \angle ODC = 50^\circ.$

$\angle BAD + 50^\circ + 25^\circ = 180^\circ$ (sum of angles in ΔABD)

$\Rightarrow \quad \angle BAD = 180^\circ - 50^\circ - 25^\circ = 105^\circ.$

Example 22. *In the adjoining figure, ABCD is a trapezium. If $\angle AOB = 126^\circ$ and $\angle PDC = \angle QCD = 52^\circ$, find the values of x and y.*



Solution. Produce AP and BQ to meet at R.

In ΔRDC, $\angle RDC = \angle RCD$ (each angle = 52°)

$\therefore \quad DR = CR$

(sides opp. equal angles are equal)

$\angle RAB = \angle RDC$ (corres. ∠s, AB ∥ DC).

Similarly, $\angle RBA = \angle RCD$.

$\therefore \quad \angle RAB = \angle RBA$

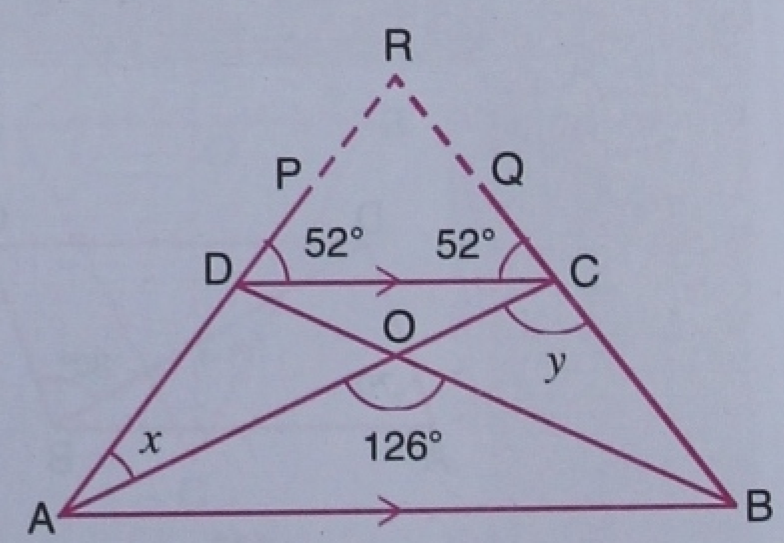
$\Rightarrow \quad AR = RB.$

$\therefore \quad AD = AR - DR = RB - CR = BC$

$\Rightarrow$ ABCD is an isosceles trapezium.

$\therefore \quad OA = OB$ (Example 8)

$\Rightarrow \quad \angle OAB = \angle OBA.$



$\therefore \quad \angle OAB = \frac{180^\circ - 126^\circ}{2} = 27^\circ.$

$\angle DAC = \angle DAB - \angle OAB = 52° - 27° = 25°$

$\therefore \quad x = 25°.$

$\angle ACB + \angle CAB + \angle ABC = 180°$ (sum of angle in ΔABC)

$\Rightarrow \quad y + 27° + 52° = 180° \Rightarrow y = 180° - 27° - 52°$

$\Rightarrow \quad y = 101°.$

Exercise 16.2

1. If two angles of a quadrilateral are 40° and 110° and the other two are in the ratio 3 : 4, find these angles.
2. If the angles of a quadrilateral are in the ratio 3 : 5 : 6 : 6, find the measures of all the angles of the quadrilateral.
3. If the angles of a quadrilateral, taken in order, are in the ratio 1 : 2 : 3 : 4, prove that it is a trapezium.
4. ABCD is parallelogram. If the diagonal AC bisects $\angle A$, then prove that :

   (*i*) AC bisects $\angle C$ (*ii*) ABCD is a rhombus (*iii*) $AC \perp BD$.
5. If the angles of a quadrilateral are equal, prove that it is a rectangle.

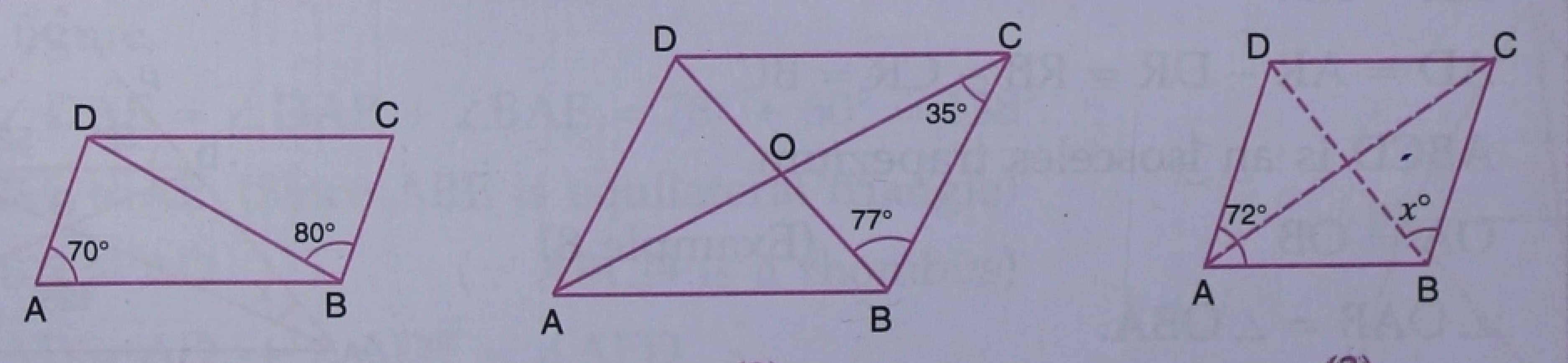
Prove the following (6 to 11) :

6. If the diagonals of a quadrilateral bisect each other, it is a parallelogram.
7. (*i*) The diagonals of a square are equal.

   (*ii*) The diagonals of a square bisect each other and are at right angles.
8. If the diagonals of a rectangle intersect each other at right angles, it is a square.
9. A diagonal of a square makes an angle of 45° with a side of the square.
10. If the diagonals of a rhombus are equal, it is a square.
11. A diagonal of a rhombus bisects the angles at vertices.
12. If an angle of a parallelogram is two-thirds of its adjacent angle, find the angles of the parallelogram.
13. (*a*) In figure (1) given below, ABCD is a parallelogram in which $\angle DAB = 70°$, $\angle DBC = 80°$. Calculate angles CDB and ADB.

    (*b*) In figure (2) given below, ABCD is a parallelogram. Find the angles of the ΔAOD.

    (*c*) In figure (3) given below, ABCD is a rhombus. Find the value of x.

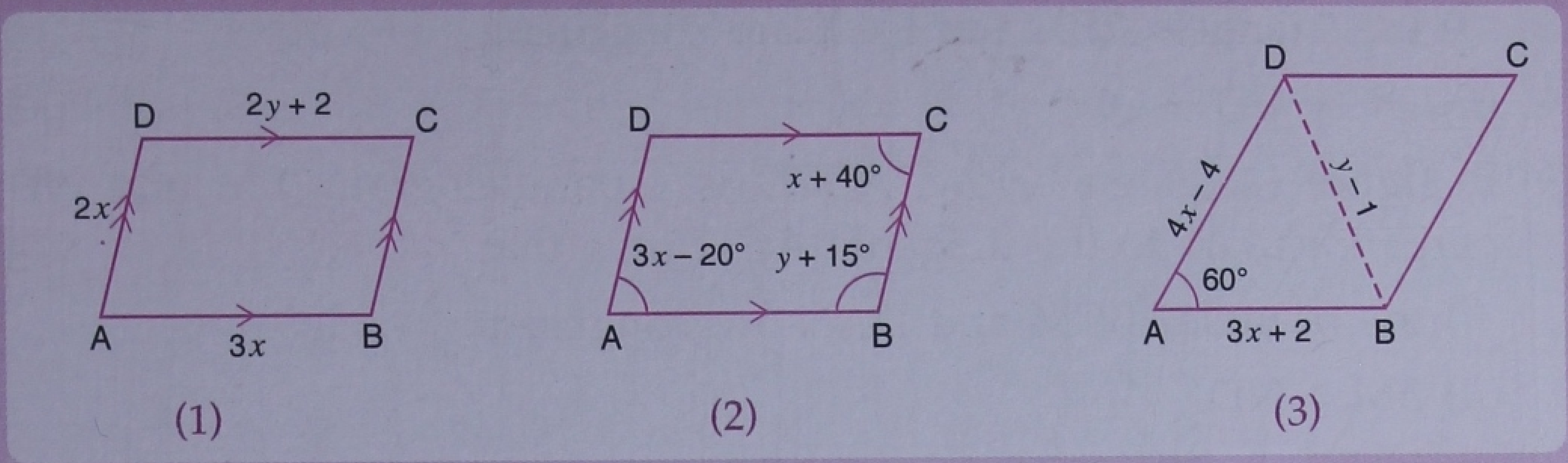


(1) (2) (3)

14. (*a*) In figure (1) given below, ABCD is a parallelogram with perimeter 40. Find the values of x and y.

(*b*) In figure (2) given below, ABCD is a parallelogram. Find the values of x and y.

(*c*) In figure (3) given below, ABCD is a rhombus. Find x and y.

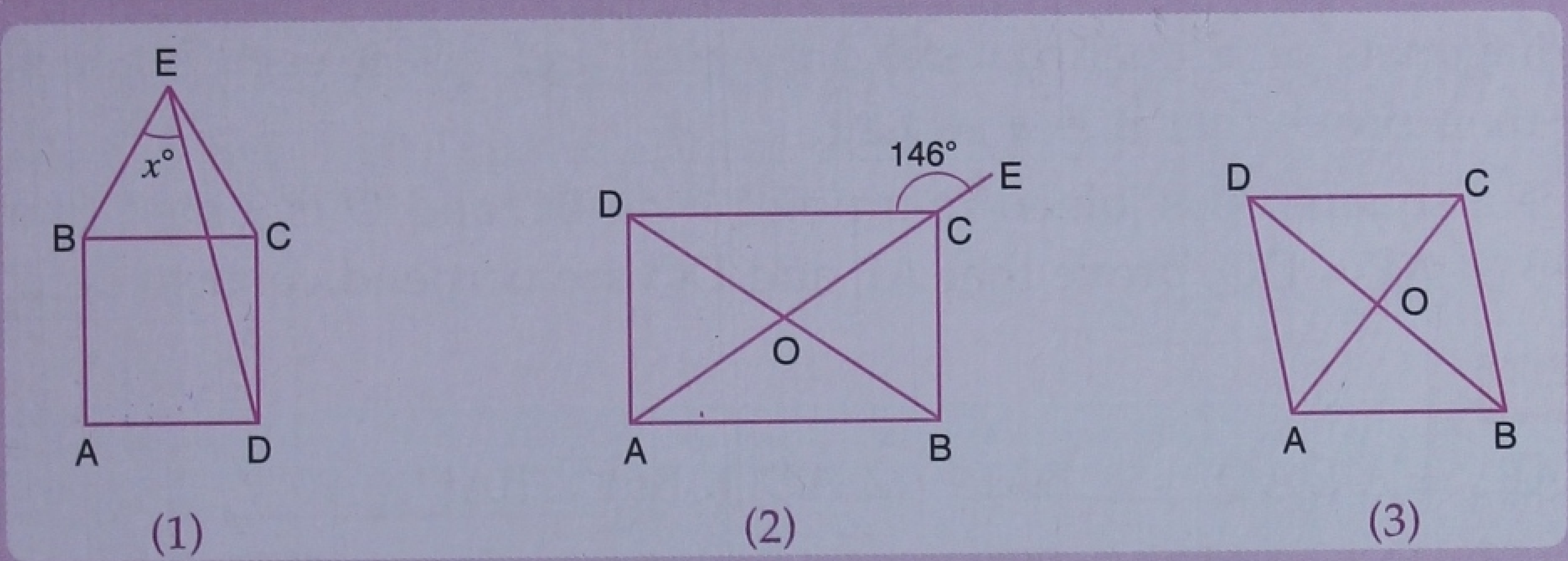


15. The diagonals AC and BD of a rectangle ABCD intersect each other at P. If $\angle ABD = 50°$, find $\angle DPC$.

16. (*a*) In figure (1) given below, equilateral triangle EBC surmounts square ABCD. Find angle BED represented by x.

(*b*) In figure (2) given below, ABCD is a rectangle and diagonals intersect at O. AC is produced to E. If $\angle ECD = 146°$, find the angles of the ΔAOB.

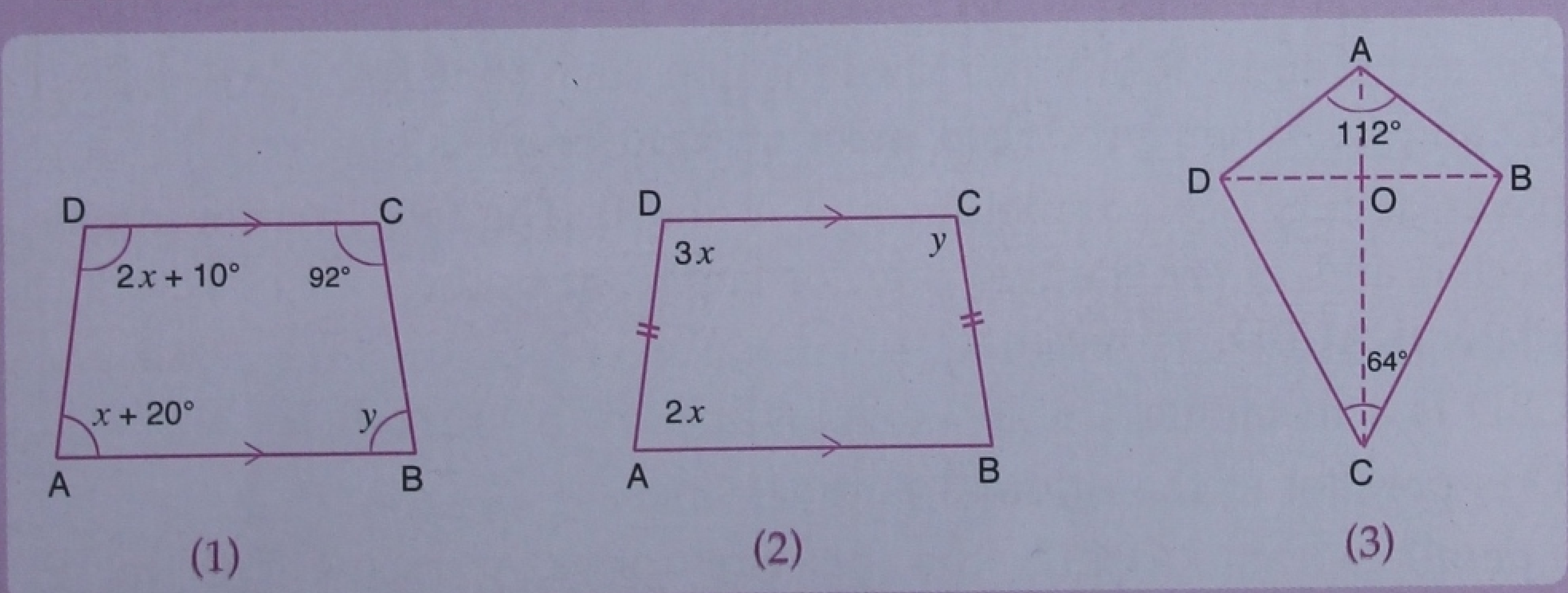
(*c*) In figure (3) given below, ABCD is a rhombus and diagonals intersect at O. If $\angle OAB : \angle OBA = 3 : 2$, find the angles of the ΔAOD.



17. (*a*) In figure (1) given below, ABCD is a trapezium. Find the values of x and y.

(*b*) In figure (2) given below, ABCD is an isosceles trapezium. Find the values of x and y.

(*c*) In figure (3) given below, ABCD is a kite and diagonals intersect at O. If $\angle DAB = 112°$ and $\angle DCB = 64°$, find $\angle ODC$ and $\angle OBA$.



18. The diagonals of a parallelogram intersect at M. P is a point on AB and PM (produced) meets the side DC in point Q. Prove that M is mid-point of PQ.

19. (*a*) In figure (1) given below, ABCD is a parallelogram and X is mid-point of BC. The line AX produced meets DC produced at Q. The parallelogram ABPQ is completed. Prove that

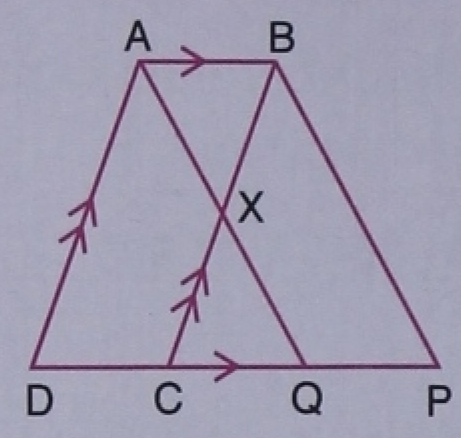
(*i*) the triangles ABX and QCX are congruent.

(*ii*) DC = CQ = QP

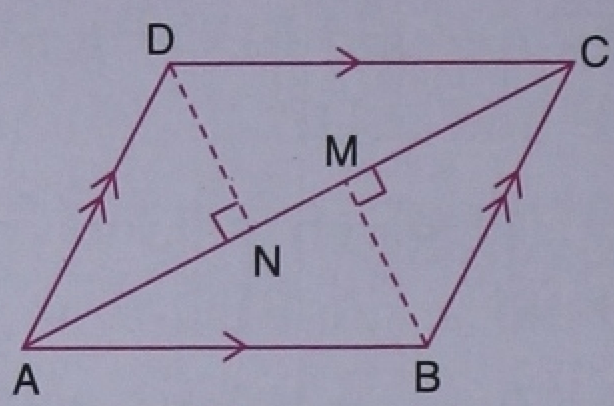
(*b*) In figure (2) given below, ABCD is a parallelogram. BM and ND are perpendiculars to the diagonal AC. Prove that

(*i*) the triangles BCM and DAN are congruent.

(*ii*) BM = ND.



(1)



(2)

20. Prove that bisectors of any two adjacent angles of a parallelogram are at right angles.

21. Prove that bisectors of any two opposite angles of a parallelogram are parallel.

22. If the diagonals of a quadrilateral are equal and bisect each other at right angles, then prove that it is a square.

23. ABCD is a square. A is joined to a point P on BC and D is joined to a point Q on AB. If AP = DQ, prove that AP and DQ are perpendicular to each other.

Hint

Δ ABP $\cong$ Δ DAQ $\Rightarrow \angle$BAP = $\angle$ADQ. But $\angle$BAD = 90°

$\Rightarrow \angle$PAD + $\angle$ADQ = 90°.

24. If P and Q are points of trisection of the diagonal BD of a parallelogram ABCD, prove that CQ ∥ AP.

Hint

Δ ABP $\cong$ Δ CDQ.

25. ABCD is a rhombus. RABS is a straight line such that RA = AB = BS. Prove that RD and SC when produced meet at right angles.

26. A transversal cuts two parallel lines at A and B. The two interior angles at A are bisected and so are the two interior angles at B; the four bisectors form a quadrilateral ACBD. Prove that

(*i*) ACBD is a rectangle.

(*ii*) CD is parallel to the original parallel lines.

27. In a parallelogram ABCD, the bisector of $\angle$A meets DC in E and AB = 2AD. Prove that

(*i*) BE bisects $\angle$B

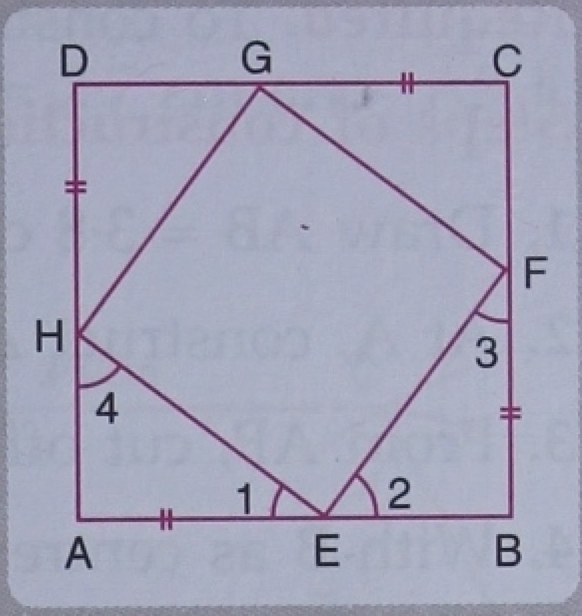
(*ii*) $\angle$AEB = a right angle.

28. ABCD is a parallelogram, bisectors of angles A and B meet at E which lies on DC. Prove that AB = 2AD.

29. ABCD is a square and the diagonals intersect at O. If P is a point on AB such that AO = AP, prove that $3\angle POB = \angle AOP$.

30. ABCD is a squre. E, F, G and H are points on the sides AB, BC, CD and DA respectively such that AE = BF = CG = DH. Prove that EFGH is a square.

Hint



Given AE = BF = CG = DH

$\Rightarrow$ EB = FC = GD = HA.

In Δs AEH and BFE,

AE = BF, AH = EB,

$\angle A = \angle B$ (each $\angle = 90°$)

$\therefore \Delta AEH \cong \Delta BFE$

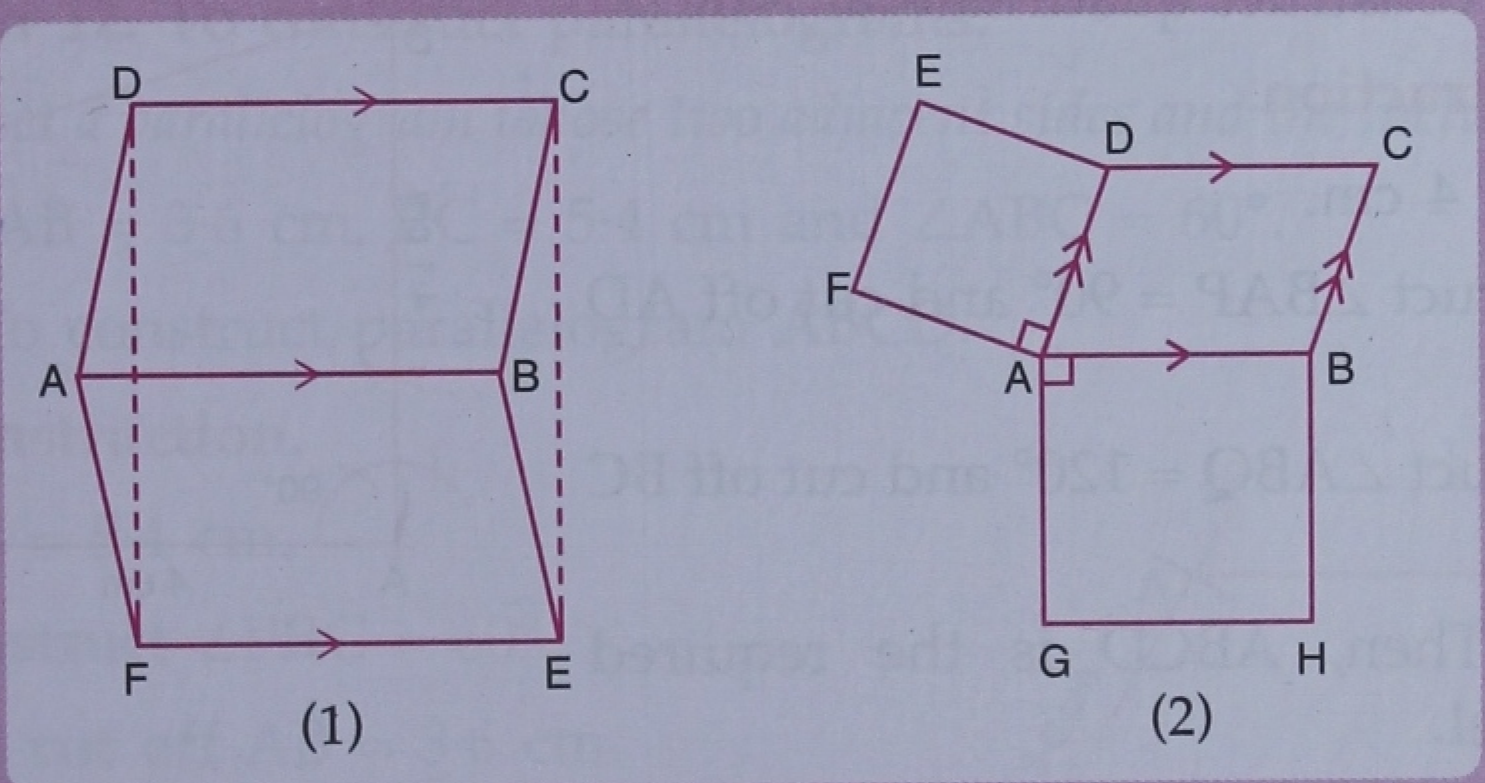
$\Rightarrow$ EH = EF and $\angle 4 = \angle 2$.

But $\angle 1 + \angle 4 = 90° \Rightarrow \angle 1 + \angle 2 = 90°$ $(\because \angle 4 = \angle 2)$

$\Rightarrow \angle HEF = 90°$.

31. (*a*) In the figure (1) given below, ABCD and ABEF are parallelograms. Prove that (*i*) CDFE is a parallelogram (*ii*) FD = EC (*iii*) $\Delta AFD \cong \Delta BEC$.

(*b*) In the figure (2) given below, ABCD is a parallelogram, ADEF and AGHB are two squares. Prove that FG = AC.



Hint

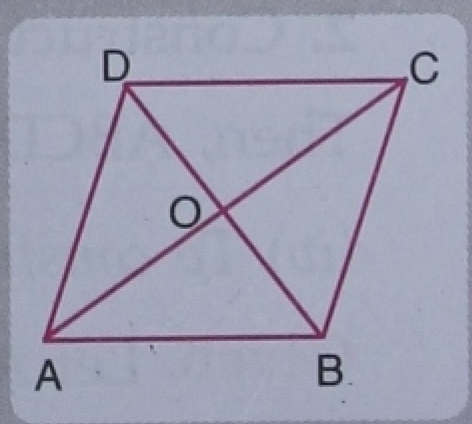
(*a*) AB ∥ DC, AB ∥ FE ⇒ DC ∥ FE, AB = DC, AB = FE ⇒ DC = FE.

(*b*) $\Delta AFG \cong \Delta BCA$, for, AF = BC, AG = AB,

$\angle FAG = 360° - 90° - 90° - \angle A = 180° - A = \angle B$.

32. ABCD is a rhombus in which $\angle A = 60°$. Find the ratio AC : BD.

Hint



As $\angle A = 60°$, ABD is an equilateral triangle. Let AB $= a$, then BD $= a \Rightarrow$ OB $= \frac{a}{2}$. In Δ AOB, $\angle AOB = 90°$.

By Pythagoras th., $AO^2 = AB^2 - OB^2 = a^2 - \left(\frac{1}{2}a\right)^2 = \frac{3}{4}a^2$

$\Rightarrow AO = \frac{\sqrt{3}}{2}a \Rightarrow AC = \sqrt{3}\,a$.

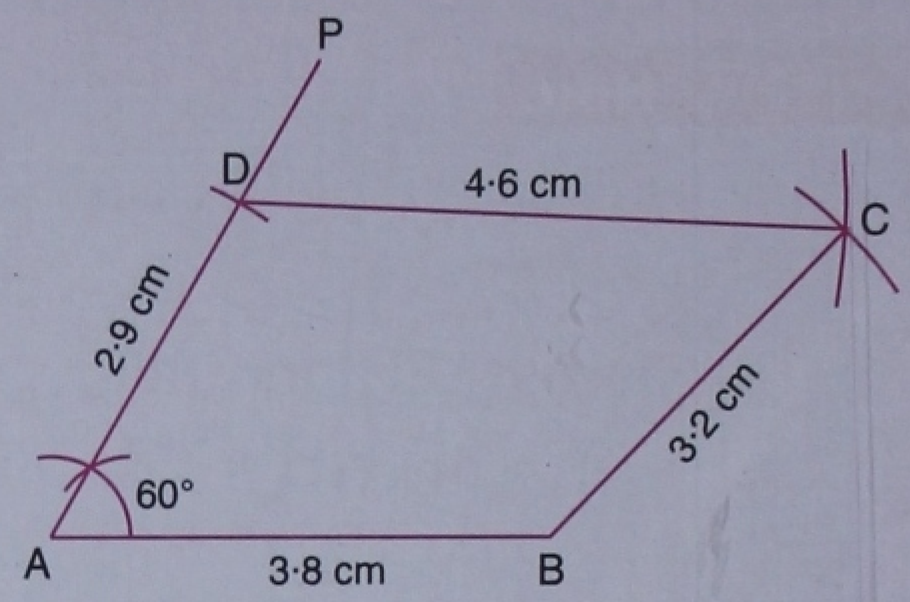
16.3 CONSTRUCTION OF QUADRILATERALS

Quadrilaterals are constructed by splitting the figure into suitable triangles. Students are advised to draw a rough *free hand* sketch before constructing the actual figure.

Construction 10. To construct quadrilaterals.

(*i*) *To construct a quadrilateral when its four sides and one angle are given.*

Given. Let AB = 3·8 cm, BC = 3·2 cm, CD = 4·6 cm, DA = 2·9 cm and ∠BAD = 60°.



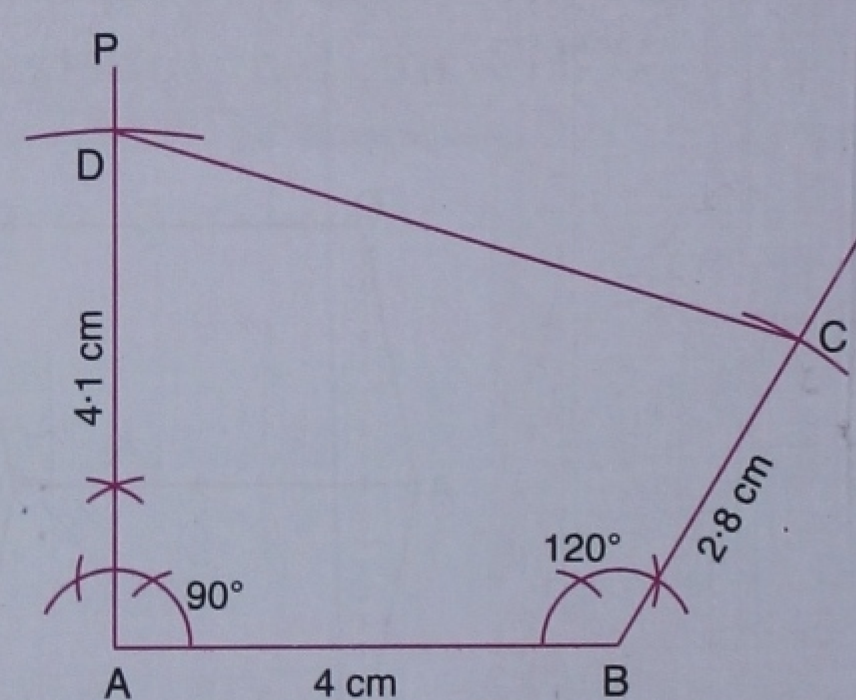
Required. To construct quad. ABCD.

Steps of construction.

1. Draw AB = 3·8 cm.
2. At A, construct ∠BAP = 60°.
3. From AP, cut off AD = 2·9 cm.
4. With B as centre and radius 3·2 cm, draw an arc.
5. With D as centre and radius 4·6 cm, draw an arc to meet the previous arc at C.
6. Join BC and DC. Then, ABCD is the required quadrilateral.

(*ii*) *To construct a quadrilateral when its three sides and two angles are given.*

Given. Let AB = 4 cm, BC = 2·8 cm, AD = 4·1 cm, ∠A = 90° and ∠B = 120°.



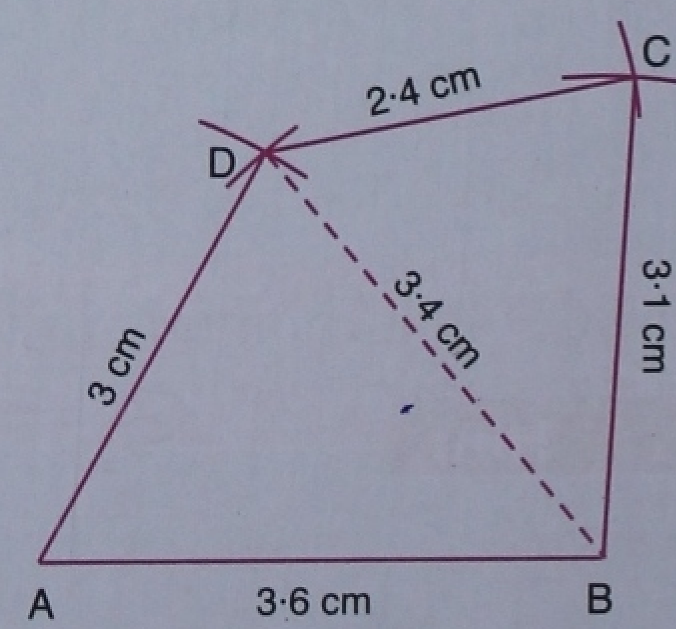
Required. To construct quadrilateral ABCD.

Steps of construction.

1. Draw AB = 4 cm.
2. At A, construct ∠BAP = 90° and cut off AD = 4·1 cm.
3. At B, construct ∠ABQ = 120° and cut off BC = 2·8 cm.
4. Join CD. Then, ABCD is the required quadrilateral.

(*iii*) *To construct a quadrilateral whose four sides and one diagonal are given.*

Given. Let AB = 3·6 cm, BC = 3·1 cm, CD = 2·4 cm, DA = 3 cm and BD = 3·4 cm.



Required. To construct quad. ABCD.

Steps of construction.

1. Construct Δ ABD.
2. Construct Δ BCD.

Then, ABCD is the required quadrilateral.

(*iv*) *To construct a quadrilateral whose three sides and two diagonals are given.*

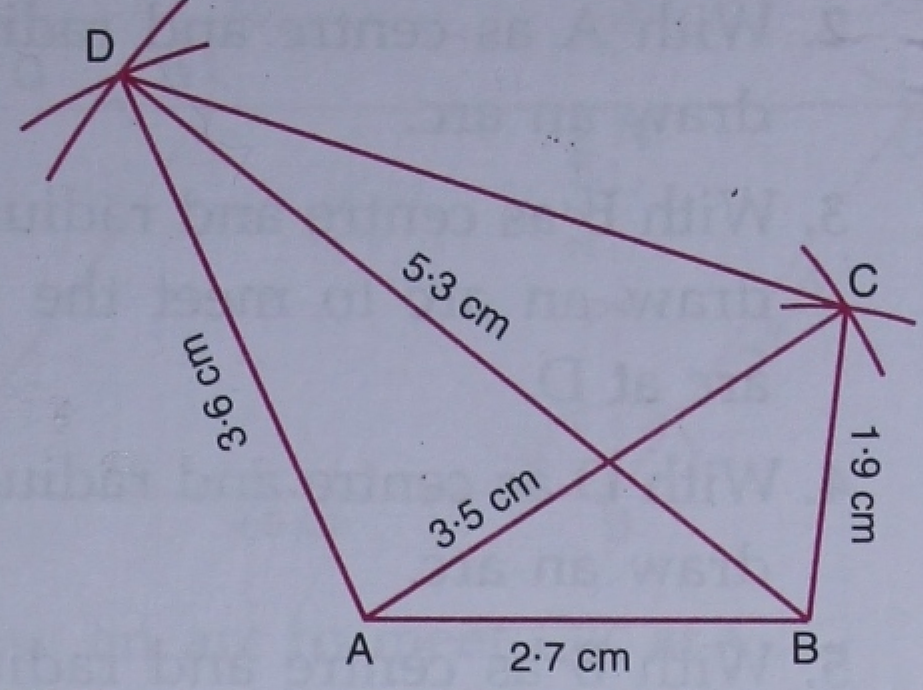
Given. Let AB = 2·7 cm, BC = 1·9 cm, AD = 3·6 cm, AC = 3·5 cm and BD = 5·3 cm.

Required. To construct quad. ABCD.

Steps of construction.

1. Construct Δ ABC.
2. Construct Δ ABD.
3. Join CD.

Then, ABCD is the required quadrilateral.



Construction 11. To construct trapezium.

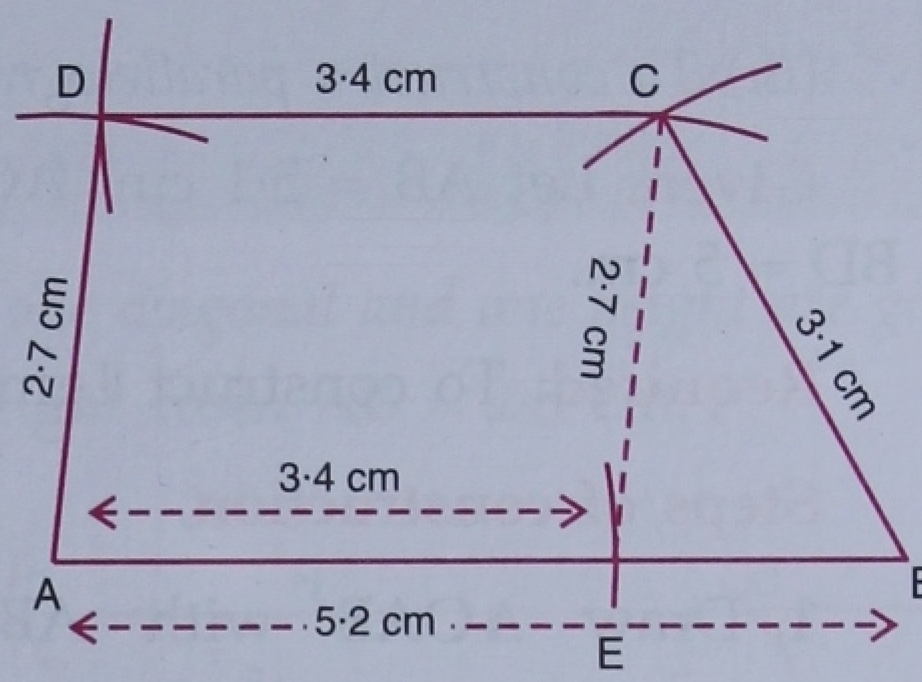
To construct a trapezium whose four sides are given.

Given. Let AB = 5·2 cm, BC = 3·1 cm, CD = 3·4 cm, DA = 2·7 cm and AB ∥ DC.

Required. To construct trapezium ABCD.

Steps of construction.

1. Draw AB = 5·2 cm.
2. From AB, cut off AE = 3·4 cm.
3. Draw Δ EBC with EC = 2·7 cm and BC = 3·1 cm.
4. With A as centre and radius = 2·7 cm, draw an arc.
5. With C as centre and radius 3·4 cm, draw an arc to meet the previous arc at D.
6. Join AD and DC. Then, ABCD is the required trapezium.



Construction 12. To construct parallelograms.

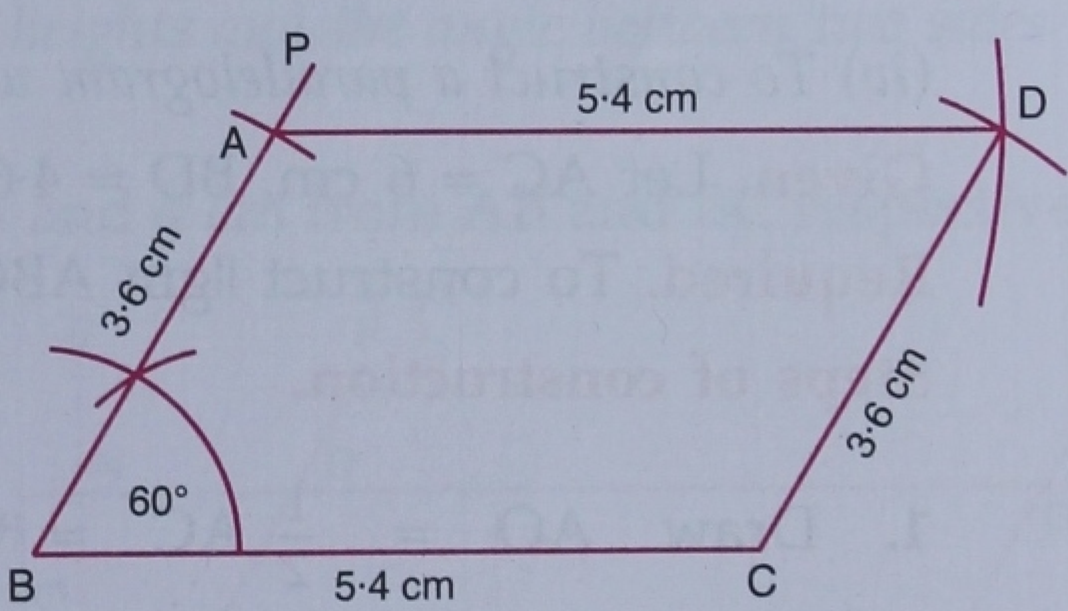
(*i*) *To construct a parallelogram whose two adjacent sides and the included angle are given.*

Given. Let AB = 3·6 cm, BC = 5·4 cm and ∠ABC = 60°.

Required. To construct parallelogram ABCD.

Steps of construction.

1. Draw BC = 5·4 cm.
2. At B, construct ∠PBC = 60°.
3. From BP, cut off AB = 3·6 cm.
4. With A as centre and radius 5·4 cm, draw an arc.
5. With C as centre and radius 3·6 cm, draw an arc to meet the previous arc at D.
6. Join AD and CD. Then, ABCD is the required parallelogram.



(*ii*) *To construct a parallelogram whose two adjacent sides and one diagonal are given.*

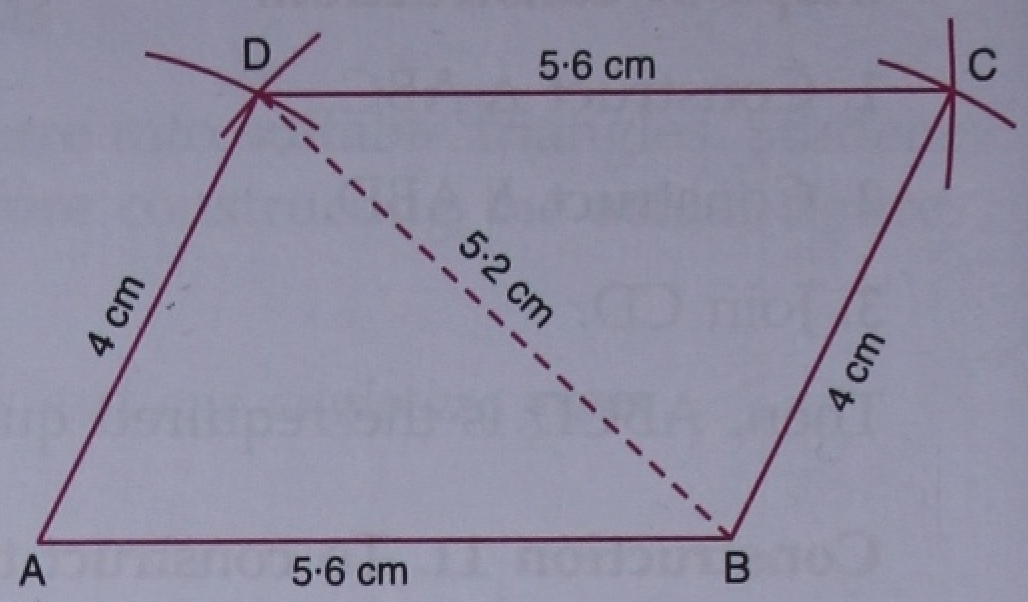
Given. Let AB = 5·6 cm, AD = 4 cm and BD = 5·2 cm.

Required. To construct parallelogram ABCD.

Steps of construction.

1. Draw AB = 5·6 cm.

2. With A as centre and radius 4 cm, draw an arc.
3. With B as centre and radius 5·2 cm, draw an arc to meet the previous arc at D.
4. With D as centre and radius 5·6 cm, draw an arc.
5. With B as centre and radius 4 cm, draw an arc to meet the previous arc at C.



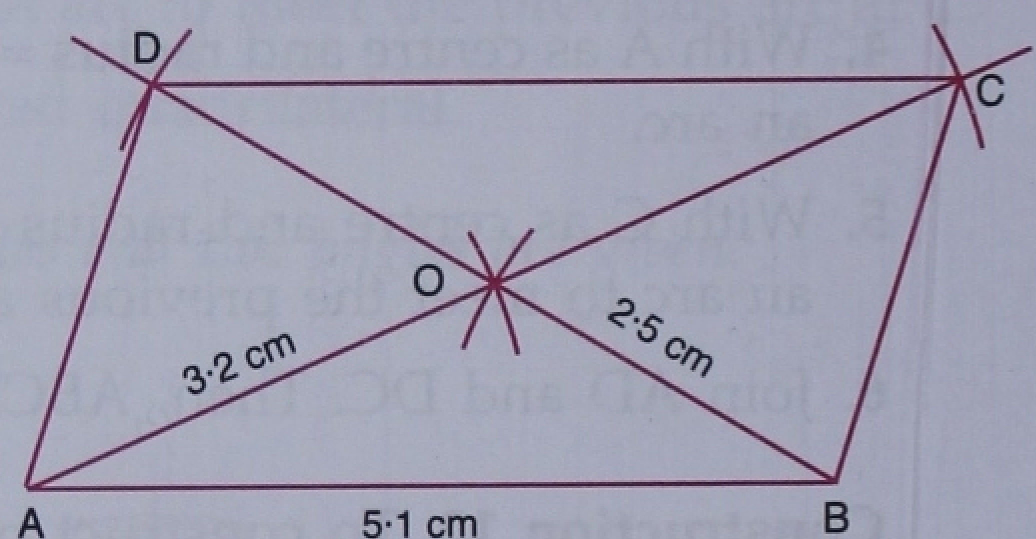
6. Join AD, BC and DC. Then ABCD is the required parallelogram.

(*iii*) *To construct a parallelogram whose one side and both diagonals are given.*

Given. Let AB = 5·1 cm, AC = 6·4 cm and BD = 5 cm.

Required. To construct ‖gm ABCD.

Steps of construction.



1. Draw ΔOAB with AB = 5·1 cm, $OA = \frac{1}{2} \cdot AC = \left(\frac{1}{2} \times 6{\cdot}4\right)$ cm = 3·2 cm, and $OB = \frac{1}{2} \cdot BD = \left(\frac{1}{2} \times 5\right)$ cm = 2·5 cm

(∵ Diagonals of a ‖gm bisect each other)

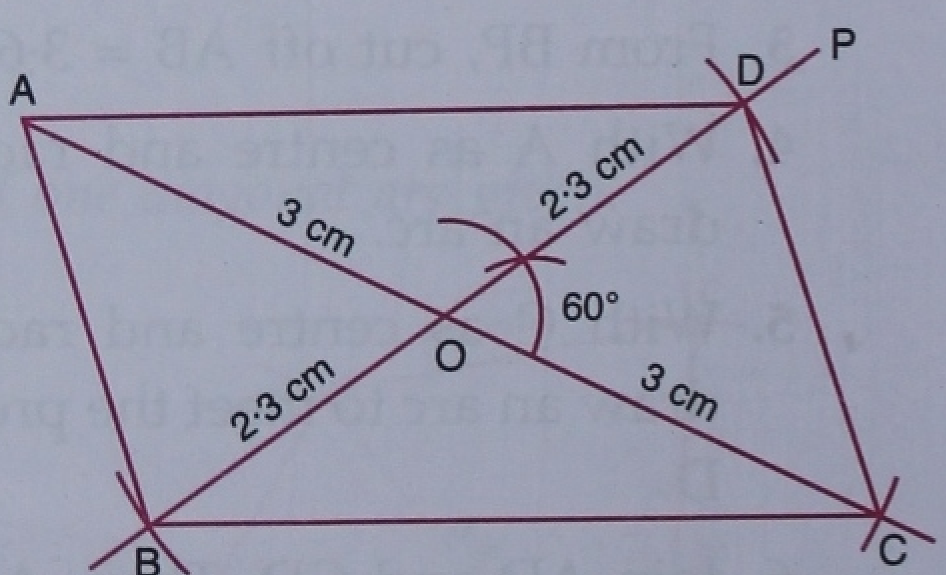
2. Produce AO to C such that OC = OA.
3. Produce BO to D such that OD = OB.
4. Join CD. Then, ABCD is the required parallelogram.

(*iv*) *To construct a parallelogram whose two diagonals and the included angle are given.*

Given. Let AC = 6 cm, BD = 4·6 cm and ∠COD = 60°.

Required. To construct ‖gm ABCD.

Steps of construction.



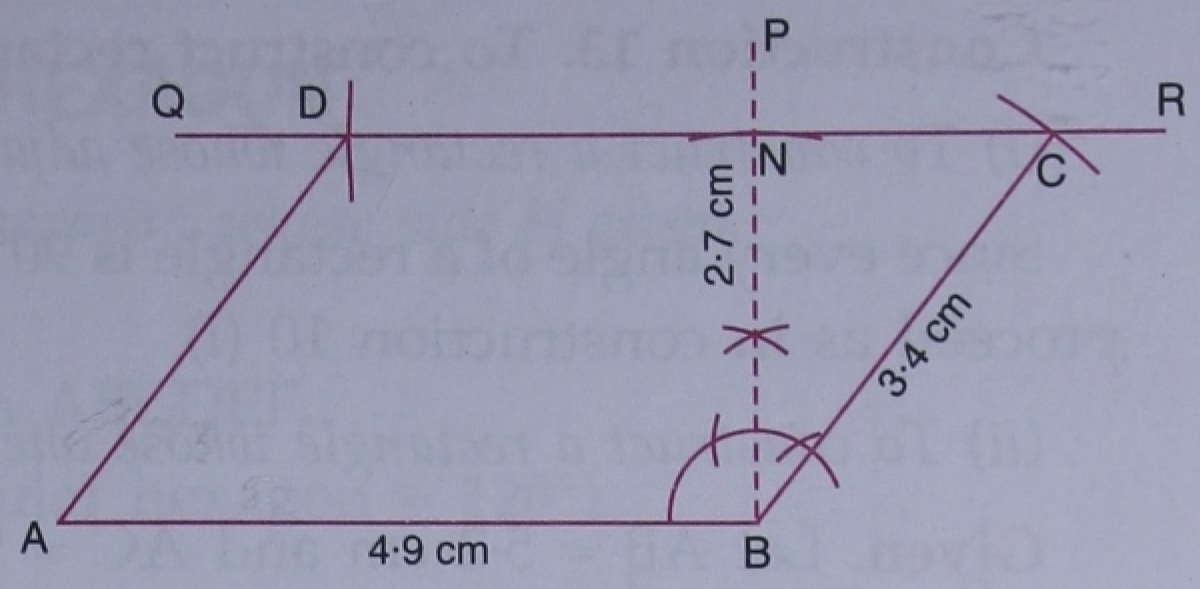
1. Draw $AO = \frac{1}{2} AC = \left(\frac{1}{2} \times 6\right)$ cm = 3 cm, and produce AO to C such that OC = OA.
2. At O, construct ∠COP = 60°.
3. From OP, cut off $OD = \frac{1}{2} BD = \left(\frac{1}{2} \times 4{\cdot}6\right)$ cm = 2·3 cm, produce DO to B such that OB = OD.
4. Join AB, BC, CD and DA. Then, ABCD is the required parallelogram.

(*v*) *To construct a parallelogram whose two adjacent sides and one height are given.*

Given. Let AB = 4·9 cm, BC = 3·4 cm and height from AB = 2·7 cm.

Required. To construct ∥gm ABCD.

Steps of construction.



1. Draw AB = 4·9 cm.
2. At B, draw BP ⊥ AB. From BP, cut off BN = 2·7 cm.
3. Through N, draw a st. line QR parallel to AB.
4. With B as centre and radius = 3·4 cm, draw an arc to meet QR at C.
5. With C as centre and radius 4·9 cm, draw an arc to meet QR at D.
6. Join AD and BC. Then, ABCD is the required parallelogram.

Note

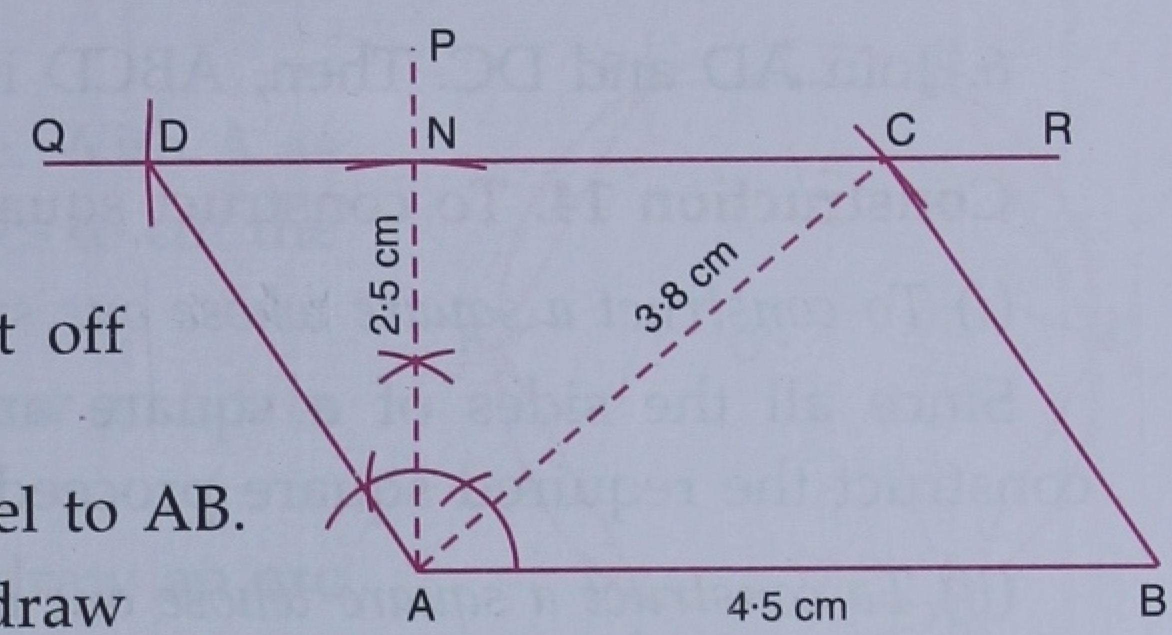
With the given data, two parallelograms are possible. However, if ∠A is acute, then only one parallelogram can be constructed.

(vi) *To construct a parallelogram whose one side, one diagonal and one height are given.*

Given. Let AB = 4·5 cm, AC = 3·8 cm and height from AB = 2·5 cm.

Required. To construct ∥gm ABCD.

Steps of construction.



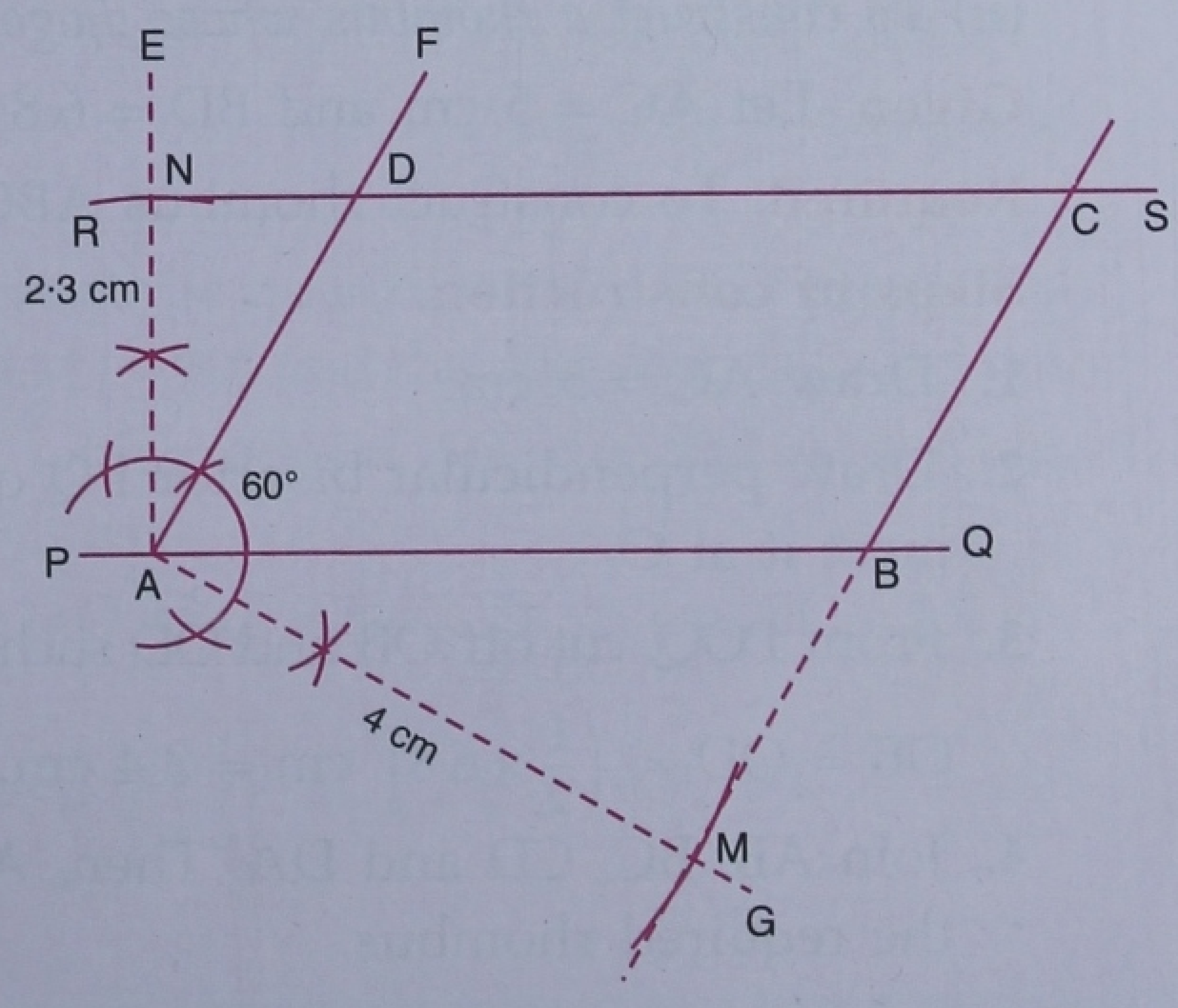
1. Draw AB = 4·5 cm.
2. At A, draw AP ⊥ AB. From AP, cut off AN = 2·5 cm.
3. Through N, draw a st. line QR parallel to AB.
4. With A as centre and radius = 3·8 cm, draw an arc to meet QR at C.
5. With C as centre and radius = 4·5 cm, draw an arc to meet QR at D.
6. Join BC and AD. Then, ABCD is the required parallelogram.

(vii) *To construct a parallelogram whose both heights and the angle between two sides are given.*

Given. Let ∠BAD = 60°, heights be 2·3 cm and 4 cm from AB and BC respectively.

Required. To construct ∥gm ABCD.

Steps of construction.



1. Draw a st. line PQ, take a point A on it.
2. At A, construct ∠QAF = 60°.
3. At A, draw AE ⊥ PQ, from AE cut off AN = 2·3 cm.
4. Through N draw a st. line parallel to PQ to meet AF at D.
5. At A, draw AG ⊥ AD, from AG cut off AM = 4 cm.
6. Through M, draw a st. line parallel to AD to meet AQ at B and ND at C. Then, ABCD is the required parallelogram.

Construction 13. To construct rectangles.

(*i*) *To construct a rectangle whose adjacent sides are given.*

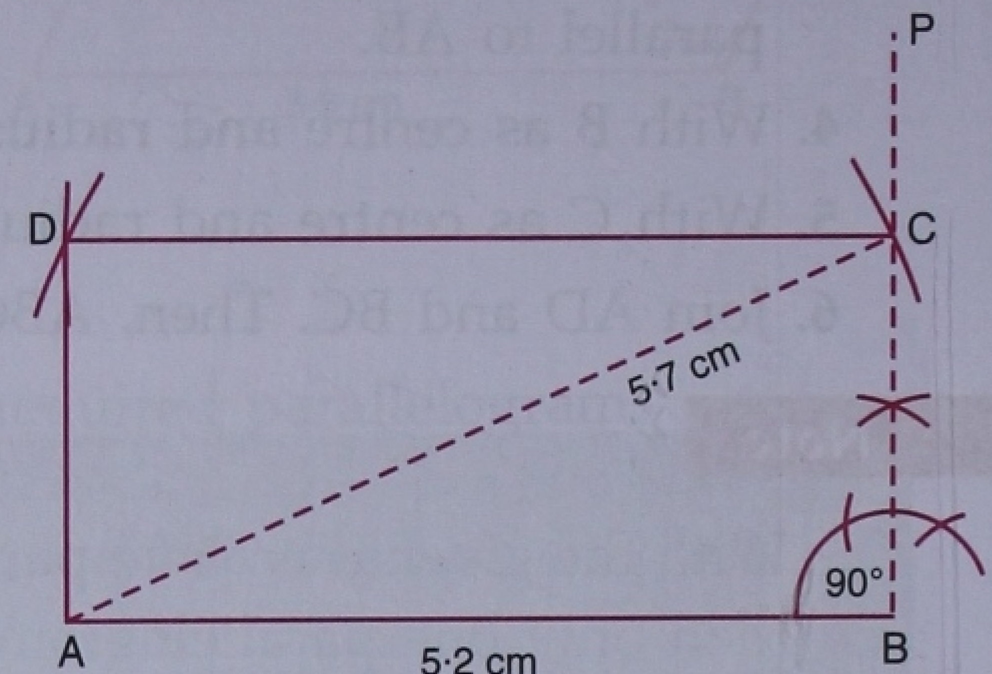
Since every angle of a rectangle is 90°, therefore, to construct the required rectangle proceed as in construction 10 (*i*).

(*ii*) *To construct a rectangle whose one side and one diagonal are given.*

Given. Let AB = 5·2 cm and AC = 5·7 cm.

Required. To construct rectangle ABCD.

Steps of construction.



1. Draw AB = 5·2 cm.
2. At B, draw BP ⊥ AB.
3. With A as centre and radius = 5·7 cm, draw an arc to meet BP at C.
4. With C as centre and radius = 5·2 cm, draw an arc.
5. With B as centre and radius = 5·7 cm, draw an arc to meet the previous arc at D.
6. Join AD and DC. Then, ABCD is the required rectangle.

Construction 14. To construct squares.

(*i*) *To construct a square whose one side is given.*

Since all the sides of a square are equal and each angle is 90°, therefore, to construct the required square proceed as in construction 10 (*i*).

(*ii*) *To construct a square whose one diagonal is given.*

Since both the diagonals of a square are equal and they intersect at right angles, therefore, to construct the required square proceed as in construction 12 (*iv*).

Construction 15. To construct rhombi.

(*i*) *To construct a rhombus whose one side and an angle are given.*

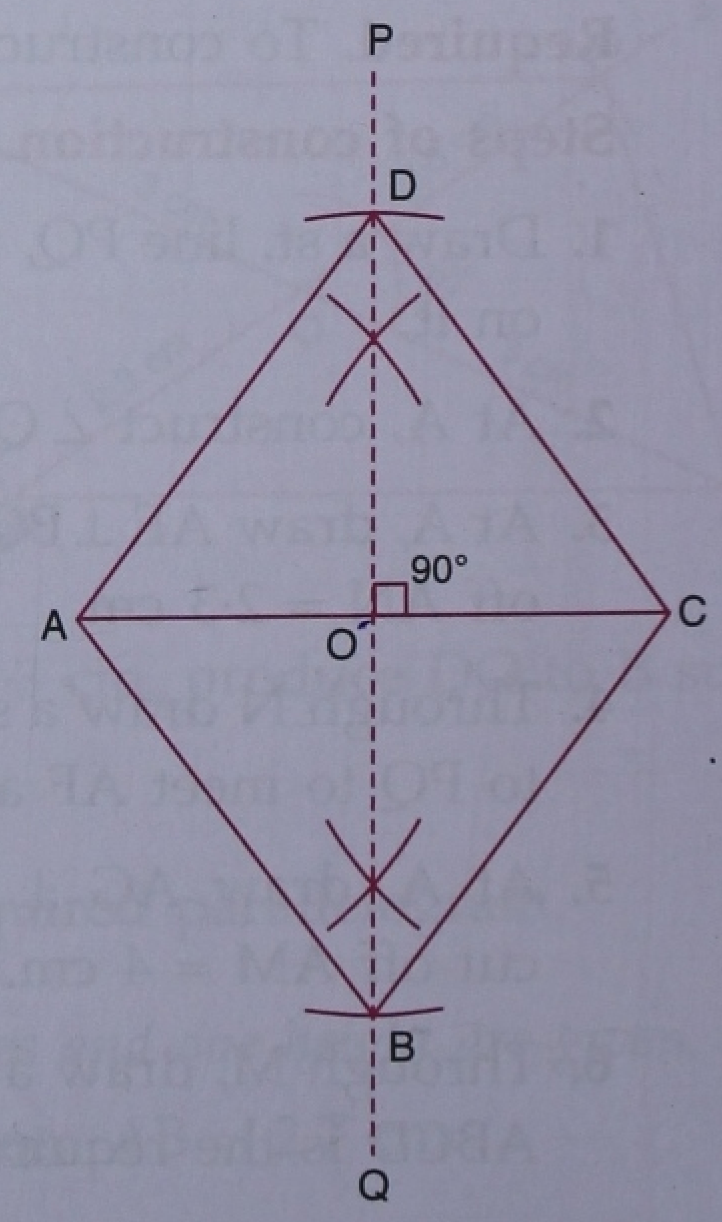
Since all the sides of a rhombus are equal, therefore, to construct the required rhombus proceed as in construction 10 (*i*).

(*ii*) *To construct a rhombus whose diagonals are given.*

Given. Let AC = 5 cm and BD = 6·8 cm.

Required. To construct rhombus ABCD.

Steps of construction.



1. Draw AC = 5 cm.
2. Draw perpendicular bisector PQ of AC to meet it at O.
3. From POQ, cut off OB and OD such that
   $OB = OD = \left(\frac{1}{2} \times 6{\cdot}8\right)$ cm = 3·4 cm.
4. Join AB, BC, CD and DA. Then, ABCD is the required rhombus.

16.4 CONSTRUCTION OF REGULAR HEXAGON

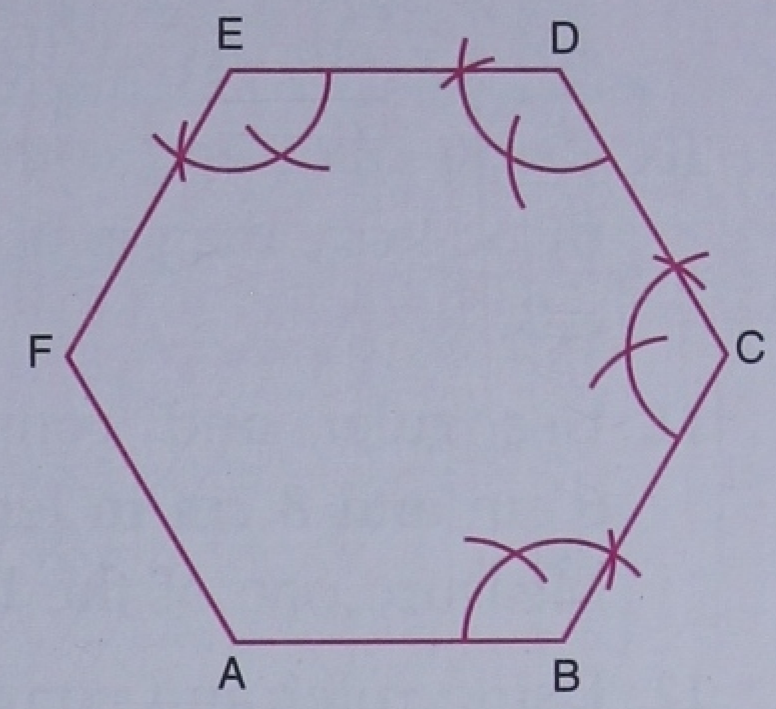
Construction 16. *To construct a regular hexagon whose side is given.*

Given. Let AB = 2 cm.

Required. To construct regular hexagon ABCDEF.

Method I. (Each interior angle of a regular hexagon = 120°)

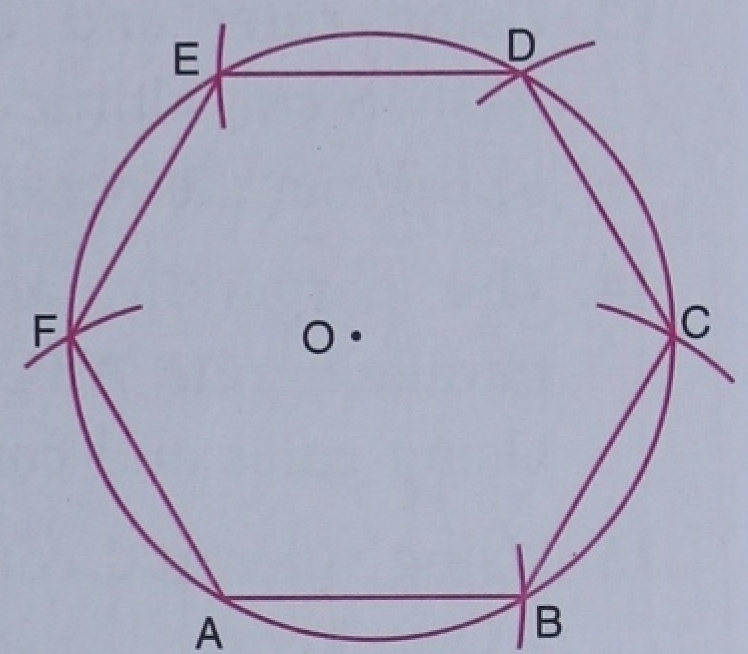
Steps of construction.



1. Draw AB = 2 cm.
2. At B, draw ∠ABC = 120° and BC = 2 cm.
3. At C, draw ∠BCD = 120° and CD = 2 cm.
4. At D, draw ∠CDE = 120° and DE = 2 cm.
5. At E, draw ∠DEF = 120° and EF = 2 cm.
6. Join FA. Then, ABCDEF is the required regular hexagon.

Method II. (The length of side of regular hexagon = radius of its *circumcircle*).

Steps of construction.



1. Draw a circle of radius 2 cm.
2. Take any point A on the circumference. With A as centre and radius = 2 cm, draw two arcs to cut the circle at B and F.
3. With B as centre and radius = 2 cm, draw an arc to cut the circle at C.
4. With C as centre and radius = 2 cm, draw an arc to cut the circle at D.
5. With D as centre and radius = 2 cm, draw an arc to cut the circle at E.
6. Join AB, BC, CD, DE, EF and FA. Then, ABCDEF is the required regular hexagon.

Exercise 16.3

1. Using ruler and compasses only, construct the quadrilateral ABCD in which ∠BAD = 45°, AD = AB = 6 cm, BC = 3·6 cm. CD = 5 cm. Measure ∠BCD.
2. Draw a quadrilateral ABCD with AB = 6 cm, BC = 4 cm, CD = 4 cm and ∠ABC = ∠BCD = 90°.
3. Using ruler and compasses only, construct the quadrilateral ABCD given that AB = 5 cm, BC = 2·5 cm, CD = 6 cm, ∠BAD = 90° and the diagonal AC = 5·5 cm.
4. Construct a quadrilateral ABCD in which AB = 3·3 cm, BC = 4·9 cm, CD = 5·8 cm, DA = 4 cm and BD = 5·3 cm.
5. Construct a trapezium ABCD in which AD ∥ BC, AB = CD = 3 cm, BC = 5·2 cm and AD = 4 cm.
6. Construct a trapezium ABCD in which AD ∥ BC, ∠B = 60°, AB = 5 cm, BC = 6·2 cm and CD = 4·8 cm.

Hint

Draw BC = 6·2 cm. At B, construct ∠CBP = 60°. From BP, cut off AB = 5 cm. Through A, draw a st. line parallel to BC.

7. Using ruler and compasses only, construct a parallelogram ABCD with AB = 5·1 cm, BC = 7 cm and ∠ ABC = 75°.
8. Using ruler and compasses only, construct a parallelogram ABCD in which AB = 4·6 cm, BC = 3·2 cm and AC = 6·1 cm.
9. Using ruler and compasses, construct a parallelogram ABCD given that AB = 4 cm, AC = 10 cm, BD = 6 cm. Measure BC.
10. Using ruler and compasses only, construct a parallelogram ABCD such that BC = 4 cm, diagonal AC = 8·6 cm and diagonal BD = 4·4 cm. Measure the side AB.
11. Use ruler and compasses to construct a parallelogram with diagonals 6 cm and 8 cm in length having given the acute angle between them is 60°. Measure one of the longer sides.
12. Using ruler and compasses only, draw a parallelogram whose diagonals are 4 cm and 6 cm long and contain an angle of 75°. Measure and write down the length of one of the shorter sides of the parallelogram.
13. Using ruler and compasses only, construct a parallelogram ABCD with AB = 6 cm, altitude = 3·5 cm and side BC = 4 cm. Measure the acute angles of the parallelogram.
14. The perpendicular distances between the pairs of opposite sides of a parallelogram ABCD are 3 cm and 4 cm and one of its angles measures 60°. Using ruler and compasses only, construct ABCD.
15. Using ruler and compasses, construct a rectangle ABCD with AB = 5 cm and AD = 3 cm.
16. Using ruler and compasses only, construct a rectangle each of whose diagonals measures 6 cm and the diagonals intersect at an angle of 45°.
17. Using ruler and compasses only, construct a square having a diagonal of length 5 cm. Measure its sides correct to the nearest millimetre.
18. Using ruler and compasses only, construct a rhombus ABCD, given that AB = 5 cm, AC = 6 cm. Measure ∠BAD.
19. Using ruler and compasses only, construct rhombus ABCD with sides of length 4 cm and diagonal AC of length 5 cm. Measure ∠ABC.
20. Construct a rhombus PQRS whose diagonals PR, QS are 8 cm and 6 cm respectively.
21. Construct a rhombus ABCD of side 4·6 cm and ∠BCD = 135°, by using ruler and compasses only.
22. Construct a trapezium in which AB ∥ CD, AB = 4·6 cm, ∠ABC = 90°, ∠DAB = 120° and the distance between parallel sides is 2·9 cm.
23. Construct a trapezium ABCD when one of parallel sides AB = 4·8 cm, height = 2·6 cm, BC = 3·1 cm and AD = 3·6 cm.

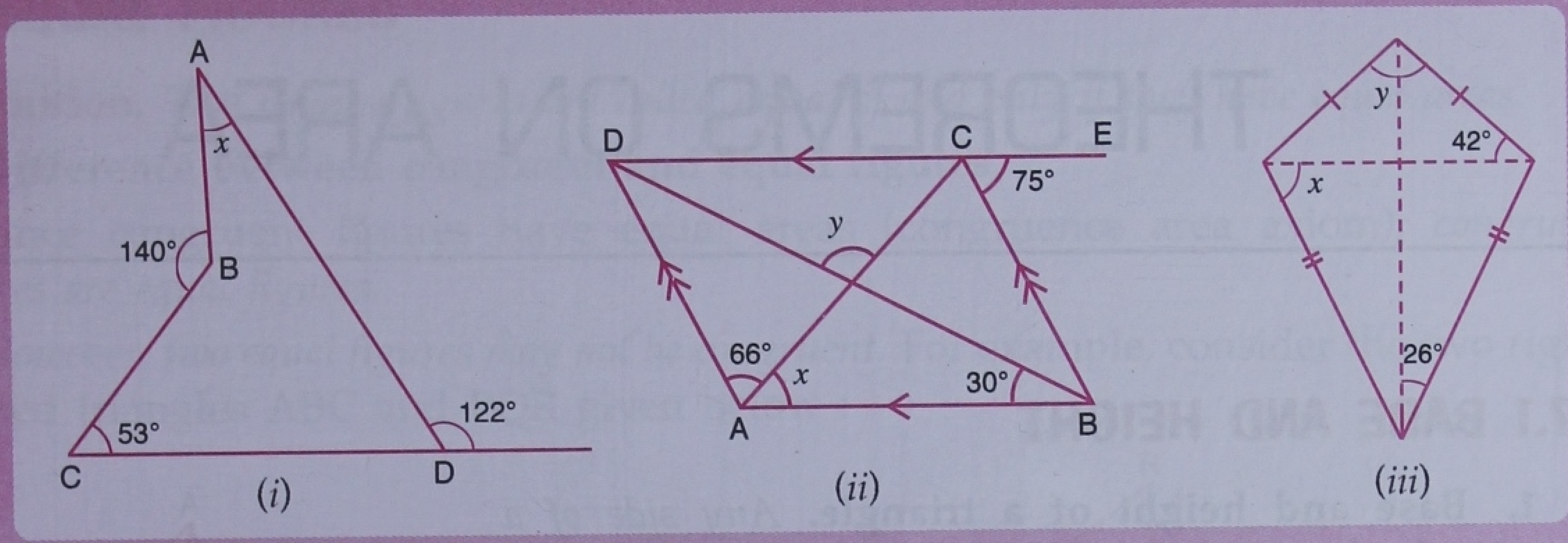
Hint

Draw a line PQ parallel to AB at a distance = 2·6 cm.

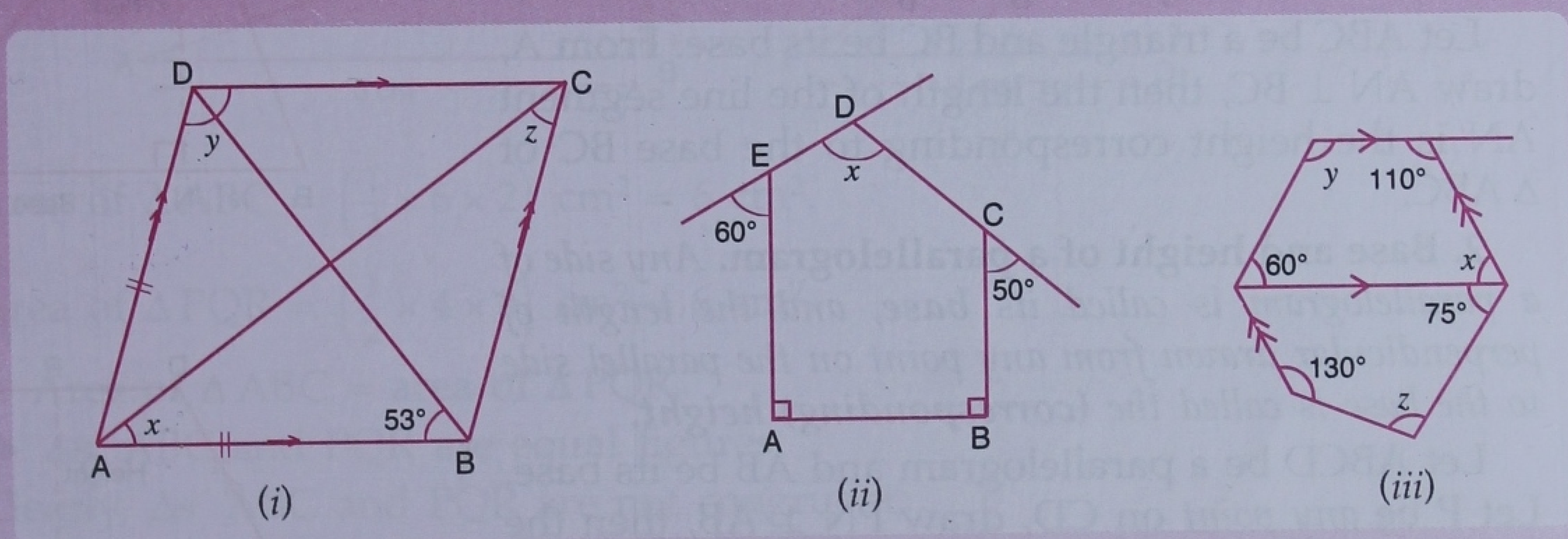
24. Construct a regular hexagon of side 2·5 cm.

CHAPTER TEST

1. The interior angles of a polygon add up to 4320°. How many sides does the polygon have?
2. If the ratio of an interior angle to the exterior angle of a regular polygon is 5 : 1, find the number of sides.
3. In a pentagon ABCDE, BC ∥ ED and ∠B : ∠A : ∠E = 3 : 4 : 5. Find ∠A.
4. Prove that the quadrilateral obtained by joining the mid-points of an isosceles trapezium is a rhombus.
5. Find the size of each lettered angle in the following figures :



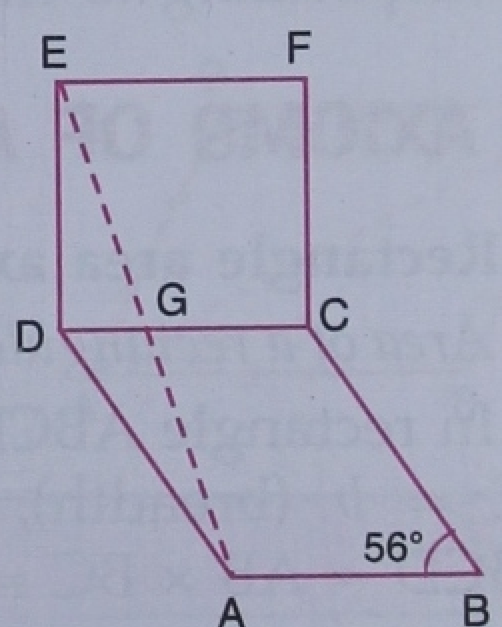
6. Find the size of each lettered angle in the following figures :



7. In the adjoining figure, ABCD is a rhombus and DCFE is a square. If ∠ABC = 56°, find

   (i) ∠DAG (ii) ∠FEG

   (iii) ∠GAC (iv) ∠AGC.



8. If one angle of a rhombus is 60° and the length of a side is 8 cm, find the lengths of its diagonals.
9. Using ruler and compasses only, construct a parallelogram ABCD with AB = 5 cm, AD = 2·5 cm and ∠BAD = 45°. If the bisector of ∠BAD meets DC at E, prove that ∠AEB is a right angle.