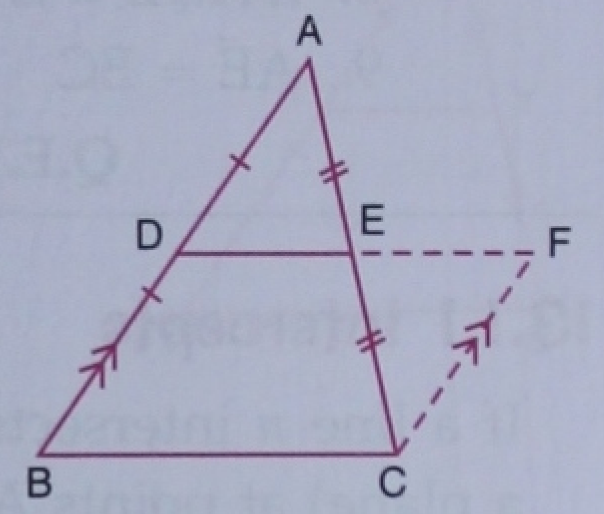
13

MID-POINT THEOREM

13.1 MID-POINT THEOREM

Theorem 6. *The line segment joining mid-points of any two sides of a triangle is parallel to the third side and is equal to half of it.*



Given. A triangle ABC, D and E are mid-points of AB and AC respectively.

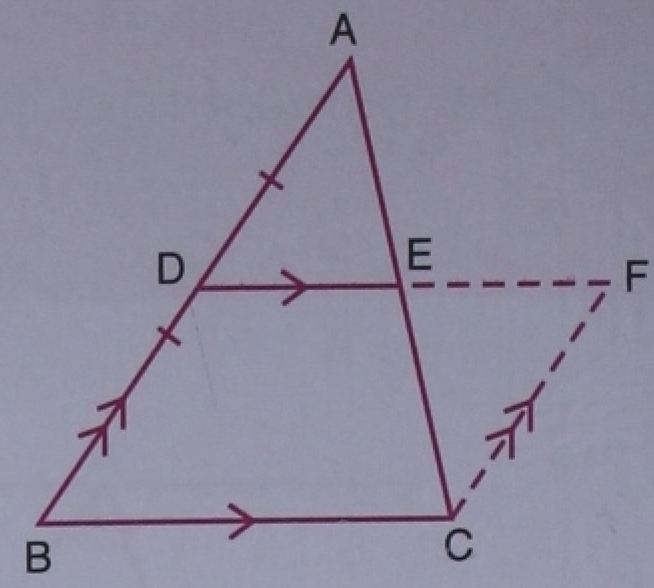
To prove. DE ∥ BC and DE = $\frac{1}{2}$ BC.

Construction. Through C, draw a st. line parallel to BA to meet DE produced at F.

Proof. *Statements*	*Reasons*
In Δs ADE and ECF	
1. AE = EC	1. E is mid-point of AC (given).
2. ∠DEA = ∠CEF	2. Vert. opp. ∠s.
3. ∠DAE = ∠ECF	3. Alt ∠s, since BA ∥ CF.
4. Δ ADE ≅ Δ ECF	4. A.S.A. (Axiom of congruency).
5. DE = EF and AD = CF	5. 'c.p.c.t.'
6. AD = BD	6. D is mid-point of AB (given).
7. BD = CF	7. From 5 and 6.
8. DBCF is a parallelogram	8. Since BD ∥ CF and BD = CF (from 7)
9. DE ∥ BC and DF = BC	9. Since DBCF is a ∥gm.
10. DE = $\frac{1}{2}$ DF	10. From 5, DE = EF.
11. DE = $\frac{1}{2}$ BC	11. Since DF = BC (from 9).
Hence DE ∥ BC and DE = $\frac{1}{2}$ BC.	
Q.E.D.	

The above theorem is known as mid-point theorem.

Theorem 7. *(Converse of mid-point theorem) The line segment drawn through the mid-point of one side of a triangle parallel to another side bisects the third side.*



Given. A triangle ABC, D mid-point of AB and DE is parallel to BC.

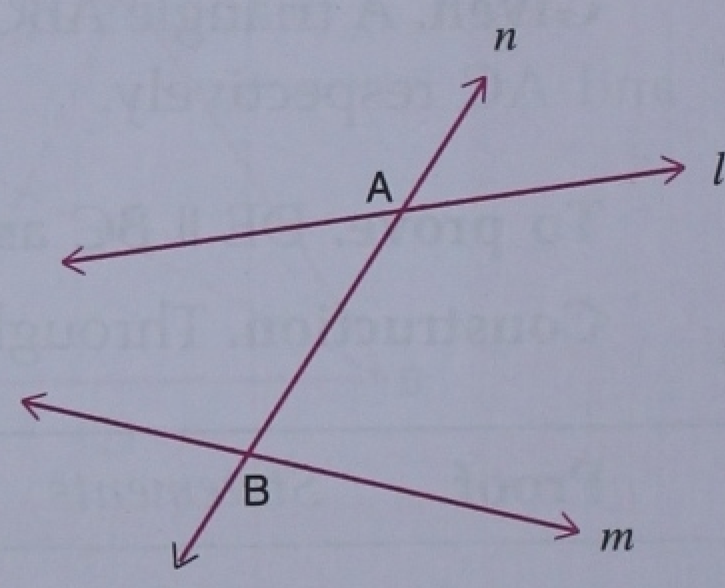
To prove. AE = EC.

Construction. Through C, draw a st. line parallel to BA to meet DE produced at F.

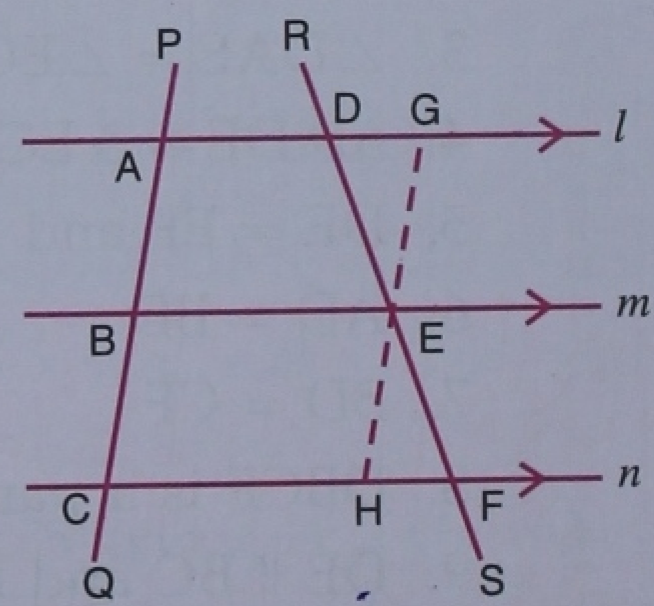
Proof. *Statements*	*Reasons*
1. DBCF is a parallelogram	1. DE ∥ BC (given), BD ∥ CF (const.).
2. BD = CF	2. Opp. sides of a ∥gm are equal.
3. BD = DA	3. D is mid-point of AB (given).
4. DA = CF	4. From 2 and 3.
In Δs ADE and ECF	
5. ∠DAE = ∠ECF	5. Alt. ∠s, since BA ∥ CF.
6. ∠DEA = ∠CEF	6. Vert. opp. ∠s.
7. DA = CF	7. From 4.
8. Δ ADE ≅ Δ ECF	8. A.A.S. (Axiom of congruency).
9. AE = EC	9. 'c.p.c.t.'
Q.E.D.	

13.1.1 Intercepts

If a line n intersects two st. lines l and m (drawn in a plane) at points A and B respectively (as shown in the adjoining diagram), then the line segment AB is called the intercept made on n by the lines l and m.



Theorem 8. *If a transversal makes equal intercepts on three (or more) parallel lines, then any other line cutting them also makes equal intercepts.*



Given. Three parallel lines l, m and n. A transversal PQ intersects them at points A, B and C respectively such that AB = BC, and any other line RS cuts them at points D, E and F respectively.

To prove. DE = EF.

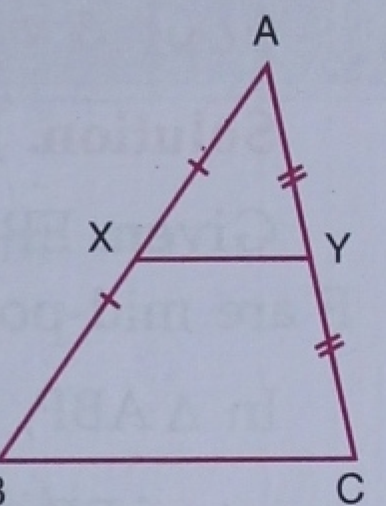
Construction. Through E, draw st. line parallel to PQ to meet line l at G and line n at H.

Proof. *Statements*	*Reasons*
1. ABEG is a parallelogram	1. AG ∥ BE (given), BA ∥ EG (const.)
2. EG = AB	2. ABEG is a ∥gm, opp. sides equal.
3. BCHE is a parallelogram	3. BE ∥ CH (given), CB ∥ HE (const.)

4. HE = BC	4. BCHE is a ∥gm, opp. sides equal.
5. EG = HE	5. From 2 and 4, also AB = BC (given).
In Δs DEG and EHF	
6. EG = HE	6. Proved above.
7. ∠EGD = ∠EHF	7. Alt. ∠s, AG ∥ CH (given).
8. ∠DEG = ∠HEF	8. Vert. opp. ∠s.
9. ΔDEG ≅ ΔEHF	9. A.S.A. (Axiom of congruency).
10. DE = EF	10. 'c.p.c.t.'.
Q.E.D.	

ILLUSTRATIVE EXAMPLES

Example 1. *In the adjoining figure, X and Y are mid-points of AB and AC respectively. Given that BC = 6 cm, AB = 5·4 cm and AC = 5 cm, calculate the perimeter of trapezium XYCB.*



Solution. Since X is mid-point of AB and Y is mid-point of AC, therefore, XY is parallel to BC and

$$XY = \frac{1}{2}\ BC = \left(\frac{1}{2} \times 6\right)\ cm = 3\ cm.$$

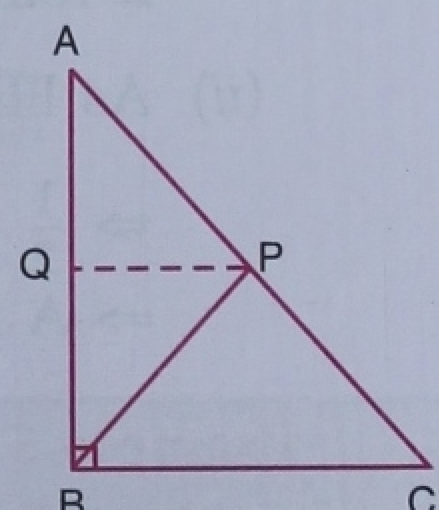
Also $XB = \frac{1}{2}\ AB = \left(\frac{1}{2} \times 5{\cdot}4\right)\ cm = 2{\cdot}7\ cm$ and

$$YC = \frac{1}{2}\ AC = \left(\frac{1}{2} \times 5\right)\ cm = 2{\cdot}5\ cm.$$

∴ Perimeter of trapezium XYCB = XY + YC + BC + XB

= (3 + 2·5 + 6 + 2·7) cm = 14·2 cm.

Example 2. *In a right angled ΔABC, ∠B = 90° and P is mid-point of AC. Prove that* $BP = \frac{1}{2}\ AC$.



Given. In ΔABC, ∠B = 90° and P is mid-point of AC.

To prove. $BP = \frac{1}{2}\ AC$.

Construction. Through P, draw a st. line parallel to CB to meet AB in Q.

Proof. Since PQ ∥ CB (construction) and P is mid-point of AC, therefore, Q bisects AB (theorem 7).

In Δs AQP and BQP

(*i*) AQ = QB [∵ Q is mid-point of AB]

(*ii*) ∠AQP = ∠BQP [each = 90°, since BC ⊥ AB and QP ∥ BC ⇒ QP ⊥ AB]

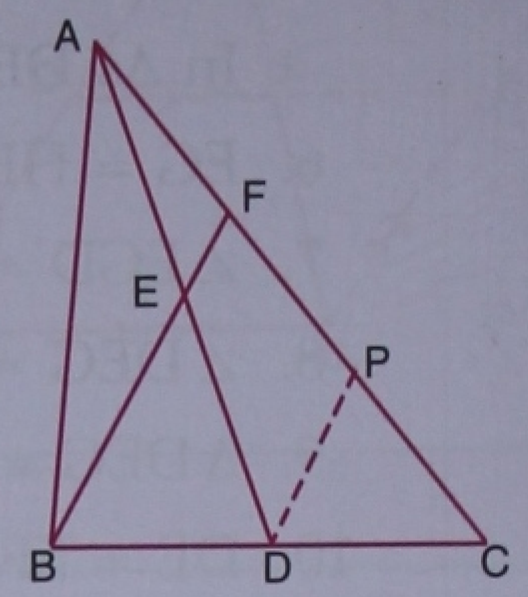
(*iii*) Side QP is common. (S.A.S. axiom of congruency)

∴ ΔAQP ≅ ΔBQP.

∴ BP = AP [c.p.c.t.]

⇒ $BP = \frac{1}{2}\ AC$ [∵ P is mid-point of AC]

Example 3. *In a triangle ABC, AD is a median and E is mid-point of AD. BE is joined and produced to meet AC at F. Prove that AF =* $\frac{1}{3}$ *AC.*



Solution. Let P be mid-point of FC and join PD.

In ΔBCF, D is mid-point of BC and P is mid-point of FC, therefore, DP is parallel to BF (theorem 6) *i.e.* DP ∥ EF. In ΔADP, E is mid-point of AD and DP ∥ EF, therefore, F is mid-point of AP (theorem 6)

i.e. AF = FP ...(*i*)

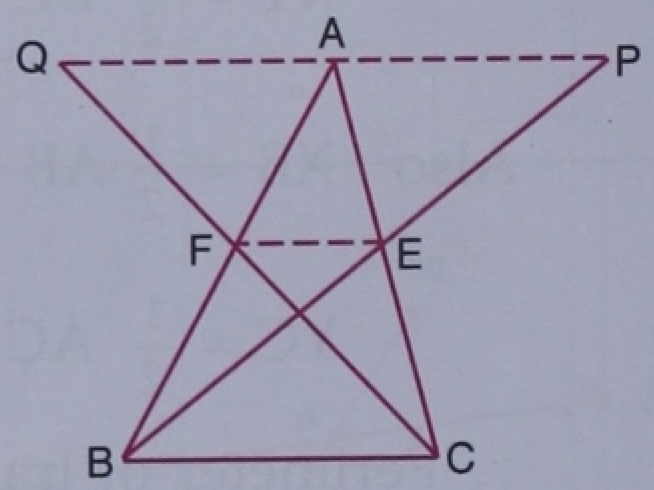
But FP = PC ...(*ii*) (∵ P is mid-point of FC)

From (*i*) and (*ii*), we get

AF = FP = PC ⇒ AF = $\frac{1}{3}$AC.

Example 4. *In ΔABC, the medians BE and CF are produced to points P and Q respectively such that EP = BE and FQ = CF. Prove that :*
(i) Q, A and P are collinear (ii) A is mid-point of QP.

Solution. Join AP, AQ and FE.



Given. EP = BE and FQ = CF, therefore, E and F are mid-points of BP and CQ respectively.

In ΔABP, F is mid-point of AB and E is mid-point of BP, therefore, FE ∥ AP and FE = $\frac{1}{2}$ AP.

In ΔACQ, E is mid-point of AC and F is mid-point of CQ, therefore, FE ∥ QA and FE = $\frac{1}{2}$ QA.

(*i*) Thus FE ∥ AP and FE ∥ QA, therefore, QA and AP lie along the same straight line (because both QA and AP pass through the same point A and are parallel to the straight line EF).

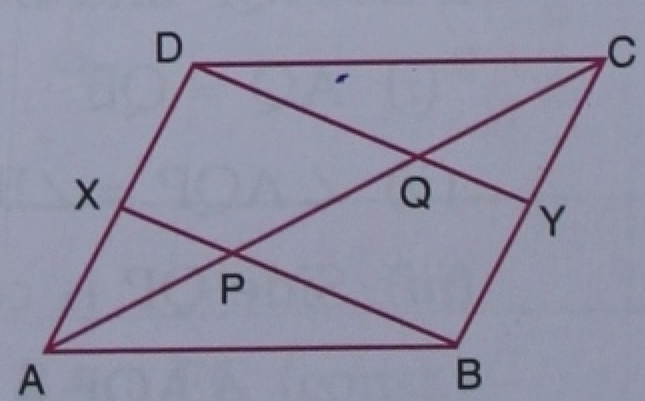
It follows that Q, A and P are collinear.

(*ii*) As FE = $\frac{1}{2}$ AP and FE = $\frac{1}{2}$ QA

⇒ $\frac{1}{2}$ AP = $\frac{1}{2}$ QA ⇒ AP = QA

⇒ A is mid-point of QP.

Example 5. *ABCD is a parallelogram. X, Y are mid-points of AD, BC respectively. Show that AP = PQ = QC.*



Solution. XD = $\frac{1}{2}$ AD [∵ X is mid-point of AD]

and BY = $\frac{1}{2}$ BC [∵ Y is mid-point of BC]

⇒ XD = BY

[∵ AD = BC, being opp. side of a ∥gm]

Also XD ∥ BY [∵ AD ∥ BC]

∴ XBYD is a parallelogram [opp. sides are equal and parallel]

⇒ XB ∥ DY.

In Δ AQD, X is mid-point of AD and XP ∥ DQ,

∴ P bisects AQ ⇒ AP = PQ ...(*i*)

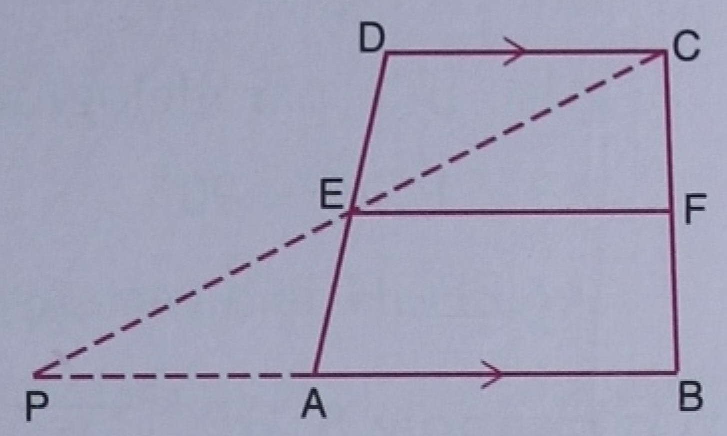
In Δ BCP, Y is mid-point of BC and QY ∥ PB,

∴ Q bisects PC ⇒ PQ = QC ...(*ii*)

From (*i*) and (*ii*), we get AP = PQ = QC, as required.

Example 6. *If E, F are mid-points of non-parallel sides AD, BC respectively of a trapezium ABCD, prove that*

(i) EF is parallel to AB (ii) EF = $\frac{1}{2}$ *(AB + CD).*



Given. ABCD is a trapezium in which AB ∥ DC. E and F are mid-points of non-parallel sides AD and BE respectively.

To prove. (*i*) EF ∥ AB

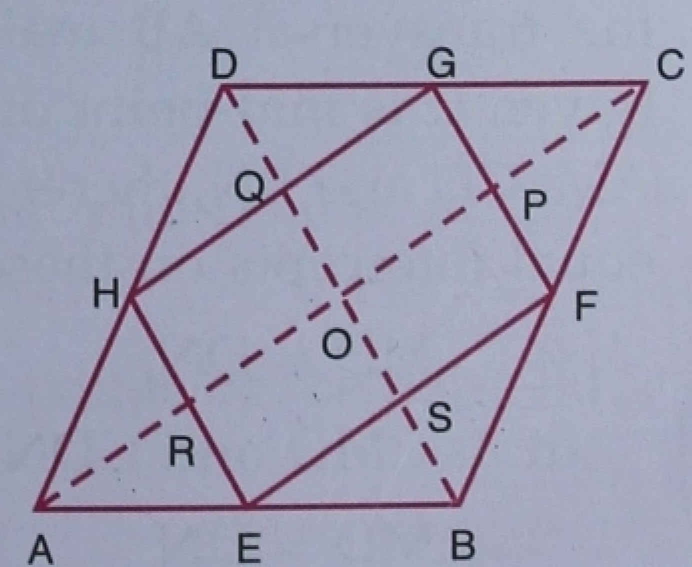
(*ii*) EF = $\frac{1}{2}$ (AB + DC).

Construction. Join CE and produce it to meet BA (produced) at P.

Proof. *Statements*	*Reasons*
In Δs DEC and EPA	
1. EA = DE	1. E is mid-point of AD (given).
2. ∠DEC = ∠AEP	2. Vert. opp. ∠s
3. ∠DCE = ∠EPA	3. Alt. ∠s, PA ∥ DC.
4. Δ DEC ≅ Δ EPA	4. A.A.S (axiom of congruency).
5. PE = EC	5. c.p.c.t.
6. PA = DC	6. c.p.c.t.
In ΔCPB	
7. E is mid-point of PC	7. PE = EC (from 5)
8. F is mid-point of BC	8. Given
9. (*i*) EF ∥ PB *i.e.* EF ∥ AB	9. Theorem 6
10. EF = $\frac{1}{2}$ PB	10. Theorem 6
11. EF = $\frac{1}{2}$ (PA + AB)	11. From figure, PB = PA + AB
12. (*ii*) EF = $\frac{1}{2}$ (AB + DC) **Q.E.D.**	12. PA = DC (from 6)

Example 7. *ABCD is a rhombus. E, F, G and H are mid-points of AB, BC, CD and DA respectively. Prove that EFGH is a rectangle.*

Solution. Join AC and BD. Mark the points as shown in the figure. In Δ ABC, E and F are mid-points of AB and BC respectively,



∴ EF ∥ AC and EF = $\frac{1}{2}$ AC ...(*i*)

In Δ ACD, G and H are mid-points of CD and DA respectively,

∴ HG ∥ AC and HG = $\frac{1}{2}$ AC ...(*ii*)

From (*i*) and (*ii*), we get

EF ∥ HG and EF = HG,

∴ EFGH is a parallelogram [∵ opp. sides are equal and parallel]

In Δ CBD, F and G are mid-points of BC and CD respectively,

∴ FG ∥ BD.

Thus QG ∥ OP and PG ∥ OQ,

∴ QOPG is also a parallelogram.

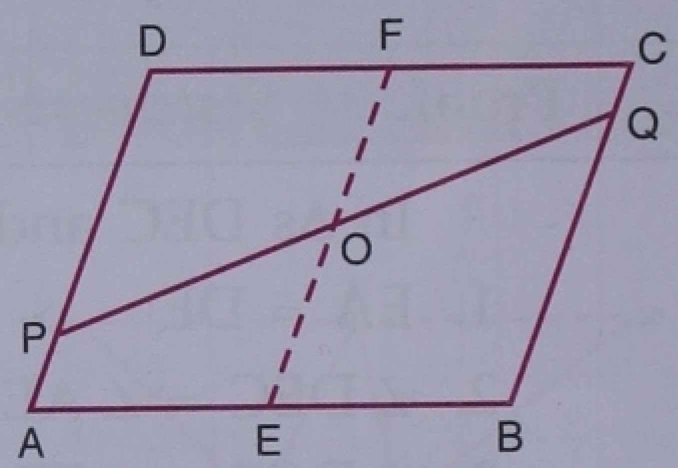
But ∠PQQ = 90° (∵ diagonals of a rhombus intersect at right angles)

Also in a parallelogram opposite angles are equal

∴ ∠PGQ = 90°

∴ EFGH is a rectangle, as required.

Example 8. *ABCD is a parallelogram. E and F are mid-points of the sides AB and CD respectively. PQ is any st. line that meets AD, EF and BC in points P, O and Q respectively. Prove that PO = OQ.*



Solution. AE = $\frac{1}{2}$ AB (∵ E is mid-point of AB)

and DF = $\frac{1}{2}$ DC (∵ F is mid-point of CD)

⇒ AE = DF (∵ AB = DC, opp. sides of ∥gm)

Also AE ∥ DF (AB ∥ DC, ABCD is a ∥gm)

⇒ AEFD is a ∥gm (∵ opp. sides of quad. AEFD are equal and parallel)

⇒ AD ∥ EF

Also AD ∥ BC (∵ ABCD is a ∥gm)

⇒ AD, EF and BC are parallel lines.

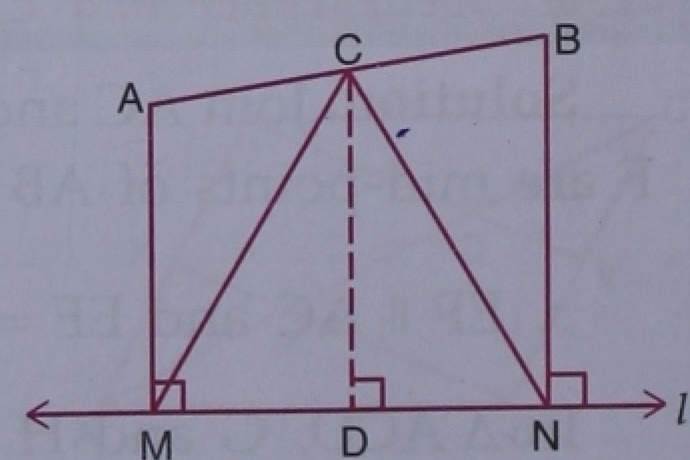
Now, as the transversal AB makes equal intercepts AE = EB on the three parallel lines AD, EF and BC, therefore, the transversal PQ also makes equal intercepts on these parallel lines (theorem 8)

⇒ PO = OQ, as required.

Example 9. *Points A and B are on the same side of a line l. AM and BN are perpendiculars to the line l. If C is the mid-points of AB, prove that CM = CN.*

Solution. From C, draw CD perpendicular to the line *l*.

Since AM, CD and BN are perpendiculars to the same line *l*, AM, CD and BN are parallel lines. Now, as the transversal AB makes equal intercepts AC = CB (given C is mid-point of AB) on the three parallel lines AM, CD and BN, therefore, the transversal *l* also makes equal intercepts on these parallel lines



⇒ MD = DN.

In Δs CMD and CDN

MD = DN (proved above)

$\angle CDM = \angle CDN$ (each angle = 90°, CD is perpendicular to l)

and CD is common.

$\because$ $\Delta CMD \cong \Delta CDN$

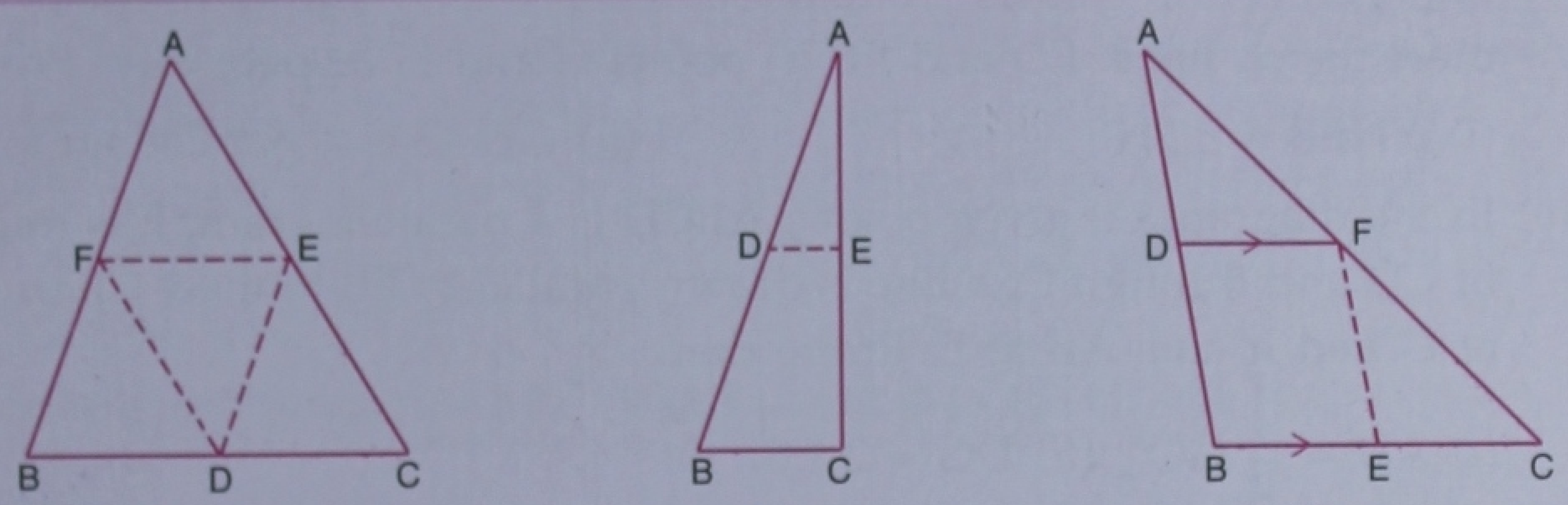
$\Rightarrow$ CM = CN (c.p.c.t.)

Exercise 13

1. (*a*) In the figure (1) given below, D, E and F are mid-points of the sides BC, CA and AB respectively of ΔABC. If AB = 6 cm, BC = 4·8 cm and CA = 5·6 cm, find the perimeter of
   (*i*) the trapezium FBCE (*ii*) the triangle DEF.

   (*b*) In the figure (2) given below, D and E are mid-points of the sides AB and AC respectively. If BC = 5·6 cm and $\angle B = 72°$, compute
   (*i*) DE (*ii*) $\angle ADE$.

   (*c*) In the figure (3) given below, D and E are mid-points of AB, BC respectively and DF ∥ BC. Prove that DBEF is a parallelogram. Calculate AC if AF = 2·6 cm.



(1) (2) (3)

2. Prove that the four triangles formed by joining in pairs the mid-points of the sides of a triangle are congruent to each other.
3. If D, E and F are mid-points of sides AB, BC and CA respectively of an equilateral triangle ABC, prove that ΔDEF is itself an equilateral triangle.
4. If D, E and F are mid-points of the sides AB, BC and CA respectively of an isosceles triangle ABC, prove that ΔDEF is also isosceles.
5. Prove that the figure formed by joining the mid-points of the adjacent sides of a quadrilateral is a parallelogram.
6. Prove that the straight lines joining the mid-points of the opposite sides of a quadrilateral bisect each other.

Hint

Show them as the diagonals of a parallelogram.

7. Show that the quadrilateral formed by joining the mid-points of the consecutive sides of a rectangle is a rhombus.
8. Show that the quadrilateral formed by joining the mid-points of the adjacent sides of a square, is also a square.

9. The diagonals AC and DB of a parallelogram intersect at O. If P is the mid-point of AD, prove that

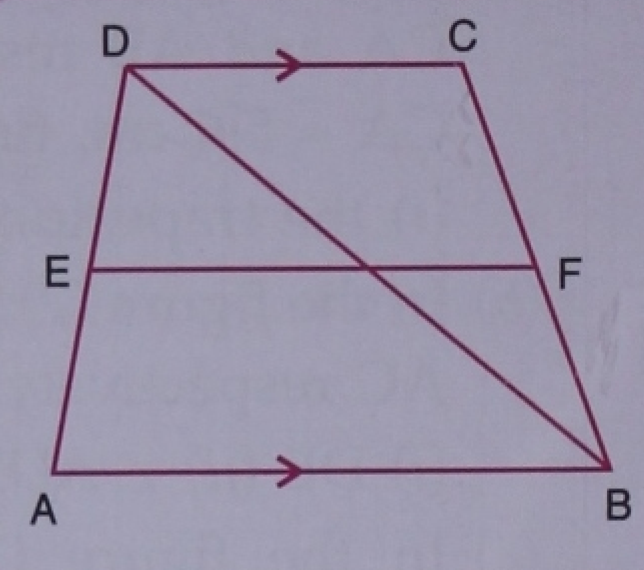
(*i*) PO ∥ AB (*ii*) PO = $\frac{1}{2}$ CD.

10. The diagonals of a parallelogram ABCD intersect at O. Through O, a straight line is drawn parallel to AB to meet AD in P and BC in Q. Prove that

(*i*) P and Q are mid-points of AD and BC respectively.

(*ii*) area of Δ OAB = $\frac{1}{4}$ area of parallelogram ABCD.

11. In the adjoining figure, ABCD is a trapezium in which AB ∥ CD, BD is a diagonal and E is mid-point of AD. A line is drawn through E parallel to AB meeting BC at F. Show that F is mid-point of BC.



12. (*a*) In the diagram (1) given below, ABCD is a parallelogram. E and F are mid-points of the sides AB and CD respectively. The st. lines AF and BF meet the st. lines ED and EC in points G and H respectively. Prove that

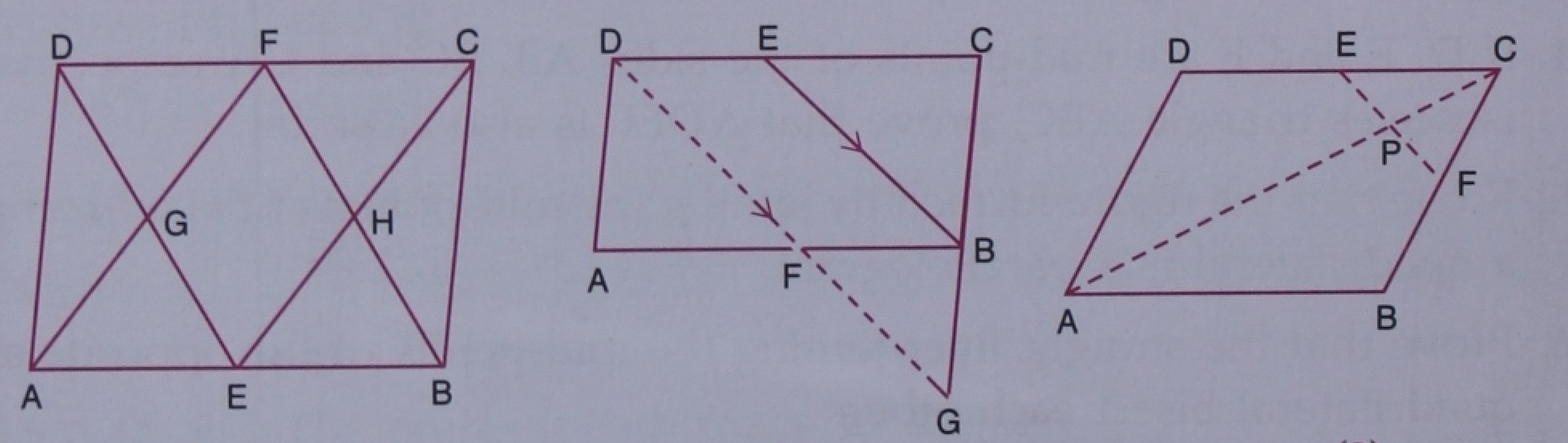
(*i*) Δ HEB ≅ Δ FHC (*ii*) GEHF is a parallelogram.

(*b*) In the diagram (2) given below, ABCD is a parallelogram. E is mid-point of CD and through D, a line is drawn parallel to EB to meet CB produced at G and it cuts AB at F. Prove that

(*i*) AD = $\frac{1}{2}$ GC (*ii*) DG = 2 EB.

(*c*) In the diagram (3) given below, ABCD is a parallelogram. E is mid-point of CD and P is a point on AC such that PC = $\frac{1}{4}$ AC. EP produced meets BC at F. Prove that

(*i*) F is mid-point of BC (*ii*) 2EF = BD.



(1) (2) (3)

13. ABC is an isosceles triangle with AB = AC. D, E and F are mid-points of the sides BC, AB and AC respectively. Prove that the line segment AD is perpendicular to EF and is bisected by it.

Hint

Δ ABD ≅ Δ ADC.

14. (*a*) In the quadrilateral (1) given below, AB ∥ DC, E and F are mid-points of AD and BD respectively. Prove that

(*i*) G is mid-point of BC (*ii*) EG = $\frac{1}{2}$ (AB + DC).

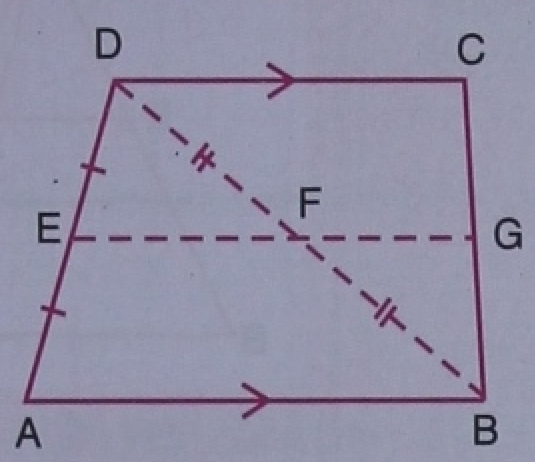
(*b*) In the quadrilateral (2) given below, AB ∥ DC ∥ EG. If E is mid-point of AD, prove that

(*i*) G is mid-point of BC (*ii*) 2EG = AB + CD.

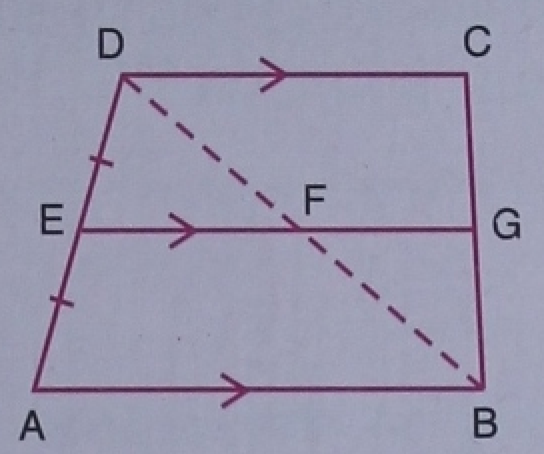
(*c*) In the quadrilateral (3) given below, AB ∥ DC. E and F are mid-points of non-parallel sides AD and BC respectively. Calculate :

(*i*) EF if AB = 6 cm and DC = 4 cm

(*ii*) AB if DC = 8 cm and EF = 9 cm.



(1)



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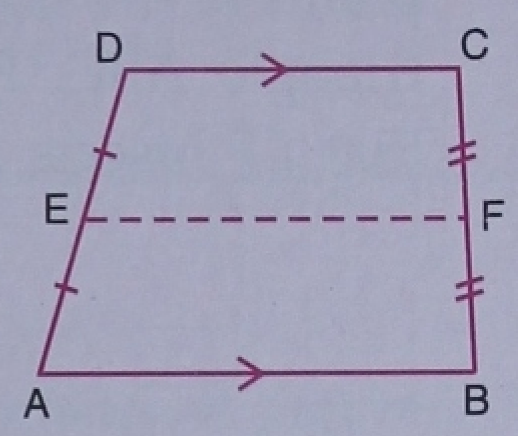


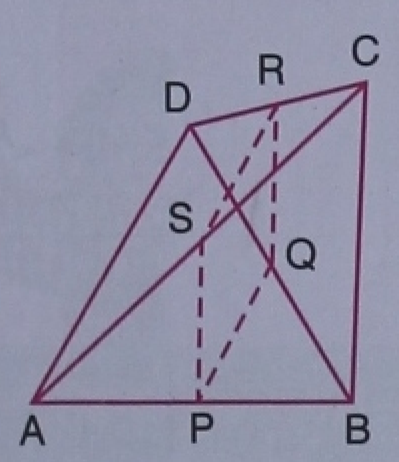
(3)

15. (*a*) In the quadrilateral (1) given below, AD = BC. P, Q, R and S are mid-points of AB, BD, CD and AC respectively. Prove that PQRS is a rhombus.

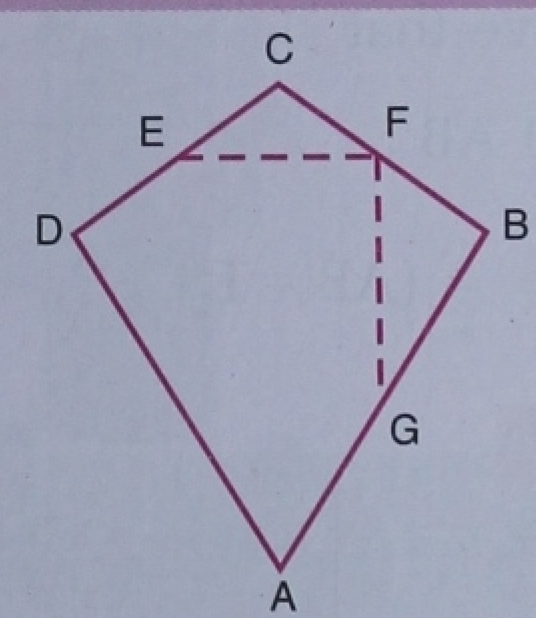
(*b*) In the diagram (2) given below, ABCD is a kite in which BC = CD, AB = AD. E, F, G are mid-points of CD, BC and AB respectively. Prove that

(*i*) ∠ EFG = 90°.

(*ii*) the line drawn through G and parallel to FE bisects DA.



(1)



(2)

Hint

(*a*) PQ ∥ AD and PQ = $\frac{1}{2}$ AD, SR ∥ AD and SR = $\frac{1}{2}$ AD.

(*b*) Diagonals of a kite intersect at right angles.

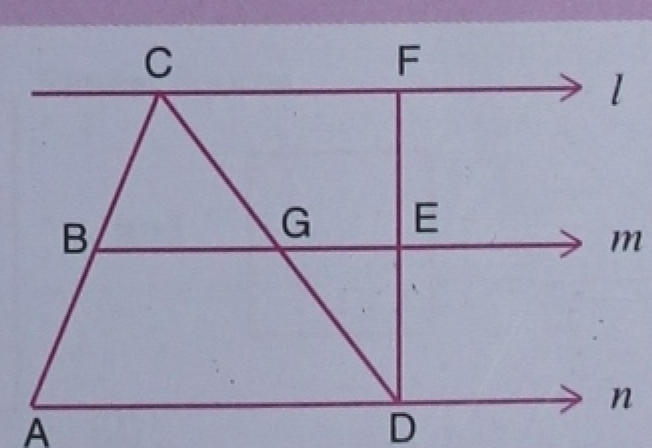
16. In the adjoining diagram, the st. lines *l*, *m* and *n* are parallel to each other, and G is mid-point of CD. Calculate :

(*i*) BG if AD = 6 cm

(*ii*) CF if GE = 2·3 cm

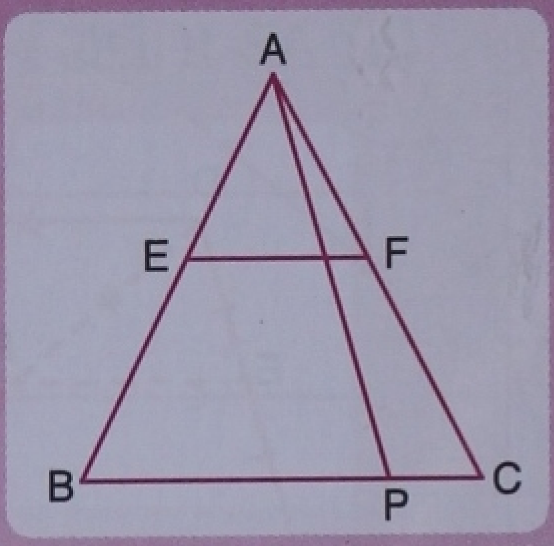
(*iii*) AB if BC = 2·4 cm

(*iv*) ED if FD = 4·4 cm.



CHAPTER TEST

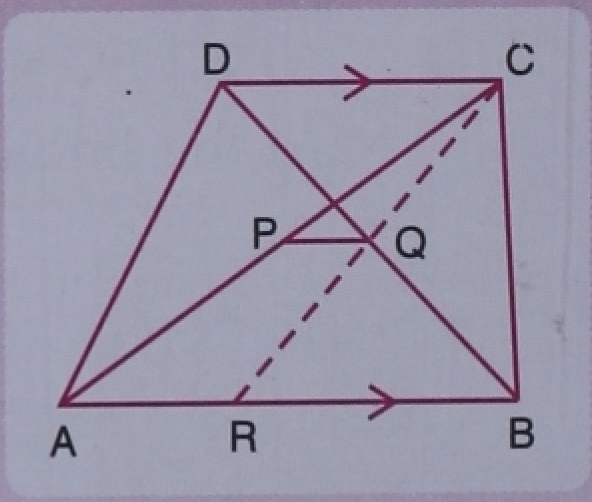
1. ABCD is a rhombus with P, Q and R as mid-points of AB, BC and CD respectively. Prove that $PQ \perp QR$.
2. The diagonals of a quadrilateral ABCD are perpendicular. Show that the quadrilateral formed by joining the mid-points of its adjacent sides is a rectangle.
3. If D, E, F are mid-points of the sides BC, CA and AB respectively of a ΔABC, prove that AD and FE bisect each other.
4. In the adjoining figure, E and F are mid-points of the sides AB and AC respectively of triangle ABC. P is any point on BC. Prove that EF bisects AP.



5. In ΔABC, D and E are mid-points of the sides AB and AC respectively. Through E, a straight line is drawn parallel to AB to meet BC at F. Prove that BDEF is a parallelogram. If AB = 8 cm and BC = 9 cm, find the perimeter of the parallelogram BDEF. **[Ans.** 17 cm.]
6. In the adjoining figure, ABCD is a trapezium. P and Q are mid-points of AC and BD respectively. CQ is joined to meet AB at R. Prove that

   (*i*) $PQ \parallel AB$

   (*ii*) $PQ = \frac{1}{2}(AB - DC)$.



Hint

$\Delta CDQ \cong \Delta QRB \Rightarrow CQ = RQ$ and $DC = RB$. In ΔACR, P and Q are mid-points of AC and RC, therefore, $PQ \parallel AR$ and $PQ = \frac{1}{2} AR$.