11

LOGARITHMS

11.1 LOGARITHMS

Logarithms were developed for making complicated calculations simple. However, with the advent of computers and hand calculators, doing calculations with the use of logarithms is no longer necessary. But still, logarithmic and exponential equations and functions are very common in mathematics.

To learn the concept of *logarithm*, consider the equality $2^3 = 8$, another way of writing this is $\log_2 8 = 3$.

It is read as "*logarithm* (abbreviated 'log') of 8 to the *base* 2 is equal to 3". Thus, $2^3 = 8$ is equivalent to $\log_2 8 = 3$. In general, we have :

Definition. *If a is any positive real number (except 1), n is any rational number and $a^n = b$, then n is called* ***logarithm*** *of b to the base a. It is written as $\log_a b$ (read as log of b to the base a). Thus,*

$a^n = b$ if and only if $\log_a b = n$.

$a^n = b$ is called the ***exponential form*** and $\log_a b = n$ is called the ***logarithmic form***.

For example,

(*i*) $3^2 = 9$, $\therefore$ $\log_3 9 = 2$.

(*ii*) $5^4 = 625$, $\therefore$ $\log_5 625 = 4$.

(*iii*) $7^0 = 1$, $\therefore$ $\log_7 1 = 0$.

(*iv*) $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$, $\therefore$ $\log_2 \frac{1}{8} = -3$.

(*v*) $(10)^{-2} = \frac{1}{100} = \cdot 01$, $\therefore$ $\log_{10}(\cdot 01) = -2$.

Remarks.

- Since a is any positive real number (except 1), a^n is always a positive real number for every rational number n *i.e.* b is always a positive real number, therefore, logarithm of only positive real numbers are defined.
- Since $a^0 = 1$, $\log_a 1 = 0$ and $a^1 = a$, $\log_a a = 1$.

  Thus, remember that

  (*i*) $\log_a 1 = 0$ (*ii*) $\log_a a = 1$

  where a is any positive real number (except 1).

- If $\log_a x = \log_a y = n$ (say), then $x = a^n$ and $y = a^n$, so $x = y$.
  Thus, remember that
  $\log_a x = \log_a y \Rightarrow x = y.$
- Logarithms to the base 10 are called **common logarithms.**
- If no base is given, the base is always taken as 10.
  For example, $\log 2 = \log_{10} 2$.

ILLUSTRATIVE EXAMPLES

Example 1. *Convert the following to logarithmic form :*

(i) $(10)^4 = 10000$ *(ii)* $3^{-5} = x$ *(iii)* $(0{\cdot}3)^3 = 0{\cdot}027$.

Solution. (*i*) $(10)^4 = 10000 \Rightarrow \log_{10} 10000 = 4$.

(*ii*) $3^{-5} = x \Rightarrow \log_3 x = -5$.

(*iii*) $(0{\cdot}3)^3 = 0{\cdot}027 \Rightarrow \log_{0\cdot3}(0{\cdot}027) = 3$.

Example 2. *Convert the following to exponential form :*

(i) $\log_3 81 = 4$ *(ii)* $\log_8 32 = \frac{5}{3}$ *(iii)* $\log_{10}(0{\cdot}1) = -1$.

Solution. (*i*) $\log_3 81 = 4 \Rightarrow 3^4 = 81$.

(*ii*) $\log_8 32 = \frac{5}{3} \Rightarrow (8)^{5/3} = 32$.

(*iii*) $\log_{10}(0{\cdot}1) = -1 \Rightarrow (10)^{-1} = 0{\cdot}1$.

Example 3. *Find the value of the following (by converting to exponential form) :*

(i) $\log_2 16$ *(ii)* $\log_{16} 2$ *(iii)* $\log_3 \frac{1}{3}$ *(iv)* $\log_{\sqrt{2}} 8$ *(v)* $\log_5(0{\cdot}008)$.

Solution. (*i*) Let $\log_2 16 = x \Rightarrow 2^x = 16 \Rightarrow 2^x = (2)^4 \Rightarrow x = 4$,

$\therefore \quad \log_2 16 = 4.$

(*ii*) Let $\log_{16} 2 = x \Rightarrow 16^x = 2 \Rightarrow (2^4)^x = 2$

$\Rightarrow \quad 2^{4x} = 2^1 \Rightarrow 4x = 1 \Rightarrow x = \frac{1}{4}$,

$\therefore \quad \log_{16} 2 = \frac{1}{4}.$

(*iii*) Let $\log_3 \frac{1}{3} = x \Rightarrow 3^x = \frac{1}{3} \Rightarrow 3^x = (3)^{-1} \Rightarrow x = -1$,

$\therefore \quad \log_3 \frac{1}{3} = -1.$

(*iv*) Let $\log_{\sqrt{2}} 8 = x \Rightarrow (\sqrt{2})^x = 8 \Rightarrow (2^{1/2})^x = 2^3$

$\Rightarrow \quad 2^{x/2} = 2^3 \Rightarrow \frac{x}{2} = 3 \Rightarrow x = 6.$

(*v*) Let $\log_5(0{\cdot}008) = x \Rightarrow 5^x = 0{\cdot}008$

$\Rightarrow \quad 5^x = \frac{8}{1000} \Rightarrow 5^x = \frac{1}{125} \Rightarrow 5^x = (5)^{-3} \Rightarrow x = -3$,

$\therefore \quad \log_5(0{\cdot}008) = -3.$

Example 4. *Find the value of x in each of the following :*

(i) $\log_2 x = 5$ (ii) $\log_4 x = 2\cdot5$ (iii) $\log_{64} x = \frac{2}{3}$ (iv) $\log_{\sqrt{3}} x = 4.$

Solution. (i) $\log_2 x = 5 \Rightarrow x = 2^5 \Rightarrow x = 32.$

(ii) $\log_4 x = 2\cdot5 \Rightarrow x = 4^{2\cdot5} \Rightarrow x = (2^2)^{5/2}$

$\Rightarrow x = 2^{2\times\frac{5}{2}} \Rightarrow x = 2^5 \Rightarrow x = 32.$

(iii) $\log_{64} x = \frac{2}{3} \Rightarrow x = (64)^{2/3} \Rightarrow x = (4^3)^{2/3}$

$\Rightarrow x = 4^{3\times\frac{2}{3}} \Rightarrow x = 4^2 \Rightarrow x = 16.$

(iv) $\log_{\sqrt{3}} x = 4 \Rightarrow x = (\sqrt{3})^4 \Rightarrow x = (3^{1/2})^4$

$\Rightarrow x = 3^2 \Rightarrow x = 9.$

Example 5. *Solve for x :*

(i) $\log_x 243 = -5$ (ii) $\log_x 16 = 2$ (iii) $\log_9 27 = 2x + 3$

(iv) $\log (3x - 2) = 2$ (v) $\log_x 64 = \frac{3}{2}$ (vi) $\log_2 (x^2 - 4) = 5.$

Solution. (i) Given $\log_x 243 = -5 \Rightarrow x^{-5} = 243$

$\Rightarrow x^{-5} = 3^5 \Rightarrow \left(\frac{1}{x}\right)^5 = 3^5$ $\left[\because a^{-n} = \left(\frac{1}{a}\right)^n\right]$

$\Rightarrow \frac{1}{x} = 3 \Rightarrow x = \frac{1}{3}.$

(ii) Given $\log_x 16 = 2 \Rightarrow x^2 = 16 \Rightarrow x = \pm 4.$

But the base of a logarithm cannot be negative, so $x = -4$ is rejected.

$\therefore$ The solution of the given equation is $x = 4.$

(iii) Given $\log_9 27 = 2x + 3 \Rightarrow 9^{2x+3} = 27 \Rightarrow (3^2)^{2x+3} = 3^3$

$\Rightarrow 3^{2(2x+3)} = 3^3 \Rightarrow 2(2x + 3) = 3 \Rightarrow 4x + 6 = 3$

$\Rightarrow 4x = -3 \Rightarrow x = -\frac{3}{4}.$

(iv) Given $\log (3x - 2) = 2 \Rightarrow \log_{10}(3x - 2) = 2$

[If no base is given, we take it as 10.]

$\Rightarrow 3x - 2 = 10^2 \Rightarrow 3x - 2 = 100$

$\Rightarrow 3x = 102 \Rightarrow x = 34.$

(v) Given $\log_x 64 = \frac{3}{2} \Rightarrow x^{3/2} = 64$

$\Rightarrow x = (64)^{2/3} = (2^6)^{2/3} = 2^{6\times 2/3} = 2^4$

$\Rightarrow x = 16.$

(vi) Given $\log_2 (x^2 - 4) = 5 \Rightarrow x^2 - 4 = 2^5$

$\Rightarrow x^2 - 4 = 32 \Rightarrow x^2 = 36 \Rightarrow x = \pm 6.$

Example 6. *Given $\log_{10} x = a$, $\log_{10} y = b$,*

(i) write down 10^{a-1} in terms of x. *(ii) write down 10^{2b} in terms of y.*

(iii) if $\log_{10} P = 2a - b$, express P in terms of x and y.

Solution. Given $\log_{10} x = a \Rightarrow 10^a = x$, and $\log_{10} y = b \Rightarrow 10^b = y.$

(*i*) $10^{a-1} = 10^a \cdot 10^{-1} = 10^a \cdot \frac{1}{10} = \frac{x}{10}$.

(*ii*) $10^{2b} = (10^b)^2 = y^2$.

(*iii*) $\log_{10} P = 2a - b \Rightarrow 10^{2a-b} = P$,

$\therefore\ P = 10^{2a} \cdot 10^{-b} = (10^a)^2 \cdot \frac{1}{10^b} = \frac{x^2}{y}$.

Example 7. *If $\log_3 x = a$, find 81^{a-1} in terms of x.*

Solution. Given $\log_3 x = a \Rightarrow 3^a = x$ ...(*i*)

$\therefore\quad 81^{a-1} = (3^4)^{a-1} = 3^{4a-4} = 3^{4a} \times 3^{-4}$

$= \frac{(3^a)^4}{3^4} = \frac{x^4}{81}$ [Using (*i*)]

Example 8. *If $\log_2 x = a$ and $\log_3 y = a$, find 12^{2a-1} in terms of x and y.*

Solution. Given $\log_2 x = a$ and $\log_3 y = a$

$\Rightarrow\ 2^a = x$ and $3^a = y$ ...(*i*)

$\therefore\quad 12^{2a-1} = (2^2 \times 3)^{2a-1} = 2^{2(2a-1)} \times 3^{2a-1}$

$= 2^{4a-2} \times 3^{2a-1} = 2^{4a} \times 2^{-2} \times 3^{2a} \times 3^{-1}$

$= \frac{(2^a)^4 \times (3^a)^2}{2^2 \times 3^1} = \frac{x^4y^2}{12}$ [Using (*i*)]

Exercise 11.1

1. Convert the following to logarithmic form :

(*i*) $5^2 = 25$ (*ii*) $a^5 = 64$ (*iii*) $7^x = 100$ (*iv*) $9^0 = 1$

(*v*) $6^1 = 6$ (*vi*) $3^{-2} = \frac{1}{9}$ (*vii*) $10^{-2} = 0{\cdot}01$ (*viii*) $(81)^{3/4} = 27$.

2. Convert the following into exponential form :

(*i*) $\log_2 32 = 5$ (*ii*) $\log_3 81 = 4$ (*iii*) $\log_3 \frac{1}{3} = -1$

(*iv*) $\log_8 4 = \frac{2}{3}$ (*v*) $\log_8 32 = \frac{5}{3}$ (*vi*) $\log_{10} (0{\cdot}001) = -3$

(*vii*) $\log_2 0{\cdot}25 = -2$ (*viii*) $\log_a \left(\frac{1}{a}\right) = -1$.

3. By converting to exponential form, find the values of :

(*i*) $\log_2 16$ (*ii*) $\log_5 125$ (*iii*) $\log_4 8$ (*iv*) $\log_9 27$

(*v*) $\log_{10} ({\cdot}01)$ (*vi*) $\log_7 \frac{1}{7}$ (*vii*) $\log_{\cdot 5} 256$ (*viii*) $\log_2 0{\cdot}25$.

4. Solve for *x* the following equations :

(*i*) $\log_3 x = 2$ (*ii*) $\log_x 25 = 2$ (*iii*) $\log_{10} x = -2$

(*iv*) $\log_4 x = \frac{1}{2}$ (*v*) $\log_x 11 = 1$ (*vi*) $\log_x \frac{1}{4} = -1$

(*vii*) $\log_{81} x = \frac{3}{2}$ (*viii*) $\log_9 x = 2{\cdot}5$ (*ix*) $\log_4 x = -1{\cdot}5$

(*x*) $\log_{\sqrt{5}} x = 2$ (*xi*) $\log_x 0{\cdot}001 = -3$ (*xii*) $\log_{\sqrt{3}} (x + 1) = 2$

(*xiii*) $\log_4 (2x + 3) = \frac{3}{2}$ (*xiv*) $\log_{\sqrt[3]{2}} x = 3$ (*xv*) $\log_2 (x^2 - 1) = 3$

(*xvi*) $\log x = -1$ (*xvii*) $\log (2x - 3) = 1$ (*xviii*) $\log x = -2, 0, \frac{1}{3}$.

5. Given $\log_{10} a = b$, express 10^{2b-3} in terms of a.
6. Given $\log_{10} x = a$, $\log_{10} y = b$ and $\log_{10} z = c$,
   (*i*) write down 10^{2a-3} in terms of x.
   (*ii*) write down 10^{3b-1} in terms of y.
   (*iii*) if $\log_{10} P = 2a + \frac{b}{2} - 3c$, express P in terms of x, y and z.
7. If $\log_{10} x = a$ and $\log_{10} y = b$, find the value of xy.
8. Given $\log_{10} a = m$ and $\log_{10} b = n$, express $\frac{a^3}{b^2}$ in terms of m and n.
9. Given $\log_{10} x = 2a$ and $\log_{10} y = \frac{b}{2}$,
   (*i*) write 10^a in terms of x.
   (*ii*) write 10^{2b+1} in terms of y.
   (*iii*) if $\log_{10} P = 3a - 2b$, express P in terms of x and y.
10. If $\log_2 y = x$ and $\log_3 z = x$, find 72^x in terms of y and z.
11. If $\log_2 x = a$ and $\log_5 y = a$, write 100^{2a-1} in terms of x and y.

11.2 THREE STANDARD LAWS OF LOGARITHMS

☆ $\log_a mn = \log_a m + \log_a n$. [Product Law]

The above result is capable of extension *i.e.*

$\log_a (mnp\ldots) = \log_a m + \log_a n + \log_a p + \ldots$

☆ $\log_a \frac{m}{n} = \log_a m - \log_a n$. [Quotient Law]

☆ $\log_a m^n = n \log_a m$. [Power Law]

Deductions

1. $\log_a a^x = x$
2. $a^{\log_a x} = x$.

11.2.1 Base changing formula

$$\log_a m = \frac{\log_b m}{\log_b a},\ m > 0,\ a, b > 0,\ a \neq 1,\ b \neq 1.$$

Deductions

1. $\log_b m = \log_a m \times \log_b a$.
2. $\log_b a \times \log_a b = 1$. (Put $m = b$ in 1)
3. $\log_b a = \frac{1}{\log_a b}$. (Reciprocal formula)

ILLUSTRATIVE EXAMPLES

Example 1. *Express $\log_{10} \frac{a^2c}{\sqrt{b}}$ in terms of $\log_{10} a$, $\log_{10} b$, $\log_{10} c$.*

Solution. $\log_{10} \frac{a^2c}{\sqrt{b}} = \log_{10} a^2c - \log_{10} \sqrt{b}$ [Quotient Law]

$= \log_{10} a^2 + \log_{10} c - \log_{10} (b)^{\frac{1}{2}}$ [Product Law]

$= 2 \log_{10} a + \log_{10} c - \frac{1}{2} \log_{10} b$. [Power Law]

Example 2. *Evaluate : $3 + \log_{10}(10^{-2})$.*

Solution. $3 + \log_{10}(10^{-2}) = 3 + (-2)\log_{10} 10$ [Power Law]

$= 3 + (-2).1$ [$\because \log_{10} 10 = 1$]

$= 3 - 2 = 1.$

Example 3. *Evaluate the following :*

(i) $\frac{\log 125}{\log \sqrt{5}}$ *(ii)* $\log_6 72 - \log_6 2$ *(iii)* $\log_4 8 - \log_8 32$.

Solution. (*i*) $\frac{\log 125}{\log \sqrt{5}} = \frac{\log 5^3}{\log 5^{1/2}} = \frac{3 \log 5}{\frac{1}{2}\log 5} = 6.$

(*ii*) $\log_6 72 - \log_6 2 = \log_6 \frac{72}{2} = \log_6 36 = \log_6 6^2$

$= 2\log_6 6 = 2 \times 1$ ($\because \log_a a = 1$)

$= 2.$

(*iii*) $\log_4 8 - \log_8 32 = \log_4 2^3 - \log_8 2^5 = \log_4 (2^2)^{3/2} - \log_8 (2^3)^{5/3}$

$= \log_4 4^{3/2} - \log_8 8^{5/3} = \frac{3}{2}\log_4 4 - \frac{5}{3}\log_8 8$

$= \frac{3}{2} \times 1 - \frac{5}{3} \times 1$ ($\because \log_a a = 1$)

$= \frac{3}{2} - \frac{5}{3} = \frac{9-10}{6} = -\frac{1}{6}.$

Example 4. *Express as a single logarithm : $2 + \frac{1}{2}\log_{10} 9 - 2\log_{10} 5$.*

Solution. $2 + \frac{1}{2}\log_{10} 9 - 2\log_{10} 5 = 2.1 + \frac{1}{2}\log_{10} 9 - 2\log_{10} 5$

$= 2\log_{10} 10 + \log_{10}(9)^{1/2} - \log_{10}(5)^2$ [$\because \log_{10} 10 = 1$]

$= \log_{10}(10)^2 + \log_{10} 3 - \log_{10} 25$

$= \log_{10} \frac{(10)^2 \times 3}{25} = \log_{10} \frac{100 \times 3}{25} = \log_{10} 12.$

Example 5. *Prove that $16^{\log 3} = 9^{\log 4}$.*

Solution. $16^{\log 3} = 9^{\log 4}$ is true

if $\log 16^{\log 3} = \log 9^{\log 4}$ is true (Taking logs of both sides)

i.e. if $\log 3 \times \log 16 = \log 4 \times \log 9$ is true

i.e. if $\log 3 \times \log 2^4 = \log 2^2 \times \log 3^2$ is ture

i.e. if $\log 3 \times 4\log 2 = 2\log 2 \times 2\log 3$ is true

i.e. if $4\log 3 \times \log 2 = 4\log 2 \times \log 3$ is true, which is true.

Hence $16^{\log 3} = 9^{\log 4}$.

Example 6. *If $\log 7 - \log 2 + \log 16 - 2\log 3 - \log \frac{7}{45} = 1 + \log n$, find n.*

Solution. Given $\log 7 - \log 2 + \log 16 - 2\log 3 - \log \frac{7}{45} = 1 + \log n$

$\Rightarrow \log 7 - \log 2 + \log 16 - \log (3)^2 - \log \frac{7}{45} = \log 10 + \log n$ [$\because \log 10 = 1$]

$$\Rightarrow \log \frac{7 \times 16}{2 \times 3^2 \times \frac{7}{45}} = \log(10 \times n) \Rightarrow \log \frac{7 \times 16 \times 45}{2 \times 9 \times 7} = \log 10n$$

$$\Rightarrow \log 40 = \log 10n \Rightarrow 40 = 10n \Rightarrow n = 4.$$

Example 7. *If $3 \log \sqrt{m} + 2 \log \sqrt[3]{n} - 1 = 0$, find the value of $m^9 n^4$.*

Solution. Given $3 \log \sqrt{m} + 2 \log \sqrt[3]{n} - 1 = 0$

$\Rightarrow \log(\sqrt{m})^3 + \log(\sqrt[3]{n})^2 = 1$

$\Rightarrow \log(m^{3/2} \times n^{2/3}) = \log 10$ [$\because \log 10 = 1$]

$\Rightarrow m^{3/2} . n^{2/3} = 10$, raising both sides to the power 6, we get

$(m^{3/2} . n^{2/3})^6 = 10^6$

$\Rightarrow (m^{3/2})^6 . (n^{2/3})^6 = 10^6 \Rightarrow m^9 n^4 = 10^6.$

Example 8. *Given $2 \log_{10} x + \frac{1}{2} \log_{10} y = 1$, express y in terms of x.*

Solution. Given $2 \log_{10} x + \frac{1}{2} \log_{10} y = 1$

$\Rightarrow \log_{10} x^2 + \log_{10} y^{\frac{1}{2}} = 1$

$\Rightarrow \log_{10} x^2 y^{\frac{1}{2}} = 1 \Rightarrow x^2 y^{\frac{1}{2}} = 10^1$ [Squaring]

$\Rightarrow x^4 y = 10^2 \Rightarrow y = \frac{100}{x^4}.$

Example 9. *Simplify the following :*

(i) $\frac{\log_3 8}{\log_9 16 \ \log_4 10}$ *(ii)* $(\sqrt{x})^{4 \log_x a}$ *(iii)* $\log(\log x^2) - \log(\log x)$

(iv) $\log_b a . \log_c b . \log_a c$ *(v)* $\log_2(\log_2(\log_2 16))$ *(vi)* $3^{-1/2 \log_3 9}$.

Solution. (*i*) $\frac{\log_3 8}{\log_9 16 \ \log_4 10} = \frac{\log_{10} 8}{\log_{10} 3} \cdot \frac{\log_{10} 9 \log_{10} 4}{\log_{10} 16 \log_{10} 10}$ (changing all logs to base 10)

$$= \frac{\log_{10} 2^3 . \log_{10} 3^2 . \log_{10} 2^2}{\log_{10} 3 . \log_{10} 2^4 . 1} = \frac{(3 \log_{10} 2)(2 \log_{10} 3)(2 \log_{10} 2)}{(\log_{10} 3)(4 \log_{10} 2)}$$

$$= 3 \log_{10} 2.$$

(*ii*) $(\sqrt{x})^{4 \log_x a} = x^{1/2 . 4 \log_x a} = x^{2 \log_x a} = x^{\log_x a^2} = a^2.$ $(\because a^{\log_a x} = x)$

(*iii*) $\log(\log x^2) - \log(\log x) = \log(2 \log x) - \log(\log x) = \log\left(\frac{2 \log x}{\log x}\right) = \log 2.$

(*iv*) $\log_b a . \log_c b . \log_a c = (\log_b a . \log_c b) . \log_a c = \log_c a . \log_a c = 1.$

(*v*) $\log_2(\log_2(\log_2 16)) = \log_2(\log_2(\log_2 2^4)) = \log_2(\log_2(4)) = \log_2(\log_2 2^2) = \log_2(2) = 1.$

(*vi*) $3^{-1/2 \log_3 9} = 3^{\log_3 9^{-1/2}} = 9^{-1/2} = \frac{1}{9^{1/2}} = \frac{1}{3}.$

Example 10. *(i) If* $\frac{\log a}{b-c} = \frac{\log b}{c-a} = \frac{\log c}{a-b}$, *prove that* $a^a \cdot b^b \cdot c^c = 1$.

(ii) If $\frac{1}{\log_a n} + \frac{1}{\log_c n} = \frac{2}{\log_b n}$, *prove that* $b^2 = ac$.

(iii) Show that $\frac{\log_a n}{\log_{ab} n} = 1 + \log_a b$.

Solution. (*i*) Let $\frac{\log a}{b-c} = \frac{\log b}{c-a} = \frac{\log c}{a-b} = k$

$\Rightarrow \log a = k(b-c);\ \log b = k(c-a);\ \log c = k(a-b)$

$\Rightarrow a \log a + b \log b + c \log c = ka(b-c) + kb(c-a) + kc(a-b) = 0$

$\Rightarrow \log a^a . b^b . c^c = 0 = \log 1 \Rightarrow a^a . b^b . c^c = 1$.

(*ii*) Given $\frac{1}{\log_a n} + \frac{1}{\log_c n} = \frac{2}{\log_b n}$

$\Rightarrow \log_n a + \log_n c = 2 \log_n b$ (using reciprocal formula)

$\Rightarrow \log_n ac = \log_n b^2$

$\Rightarrow ac = b^2$, as required.

(*iii*) $\frac{\log_a n}{\log_{ab} n} = \frac{1/\log_n a}{1/\log_n ab} = \frac{\log_n ab}{\log_n a} = \log_a ab = \log_a a + \log_a b = 1 + \log_a b$.

Example 11. *If* $a = \log_x yz$, $b = \log_y zx$ *and* $c = \log_z xy$, *then prove that* $\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} = 1$.

Solution. $\frac{1}{1+a} = \frac{1}{1+\log_x yz} = \frac{1}{\log_x x + \log_x yz} = \frac{1}{\log_x xyz} = \log_{xyz} x$.

Similarly, $\frac{1}{1+b} = \log_{xyz} y$ and $\frac{1}{1+c} = \log_{xyz} z$.

$\therefore \frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} = \log_{xyz} x + \log_{xyz} y + \log_{xyz} z$

$= \log_{xyz} xyz = 1$.

Example 12. *Solve for x :*

(i) $\log x = \frac{\log 125}{\log \frac{1}{5}}$ *(ii)* $\log_2 (\log_3 x) = 4$

(iii) $\log_x 15\sqrt{5} = 2 - \log_x 3\sqrt{5}$

(iv) $\log (5x - 4) - \log (x + 1) = \log 4$

(v) $\log (x + 5) + \log (x - 5) = 2 \log 3 + 4 \log 2$.

Solution. (*i*) $\log x = \frac{\log 125}{\log \frac{1}{5}} = \frac{\log (5)^3}{\log 5^{-1}} = \frac{3 \log 5}{(-1) \log 5} = -3$

$\Rightarrow x = 10^{-3} \Rightarrow x = \frac{1}{(10)^3} = \frac{1}{1000} = 0{\cdot}001$.

(*ii*) $\log_2 (\log_3 x) = 4 \Rightarrow \log_3 x = 2^4 = 16 \Rightarrow x = 3^{16}$.

(*iii*) Given $\log_x 15\sqrt{5} = 2 - \log_x 3\sqrt{5} \Rightarrow \log_x 15\sqrt{5} + \log_x 3\sqrt{5} = 2$

$\Rightarrow \log_x (15\sqrt{5} \times 3\sqrt{5}) = 2 \Rightarrow \log_x 225 = 2$

$\Rightarrow$ $\log_x(15)^2 = 2 \Rightarrow 2\log_x 15 = 2$

$\Rightarrow$ $\log_x 15 = 1 \Rightarrow x^1 = 15$

$\Rightarrow$ $x = 15.$

(*iv*) Given $\log(5x - 4) - \log(x + 1) = \log 4$

$\Rightarrow$ $\log \frac{5x-4}{x+1} = \log 4$

$\Rightarrow$ $\frac{5x-4}{x+1} = 4 \Rightarrow 5x - 4 = 4x + 4 \Rightarrow x = 8.$

(*v*) Given $\log(x + 5) + \log(x - 5) = 2\log 3 + 4\log 2$

$\Rightarrow$ $\log(x + 5)(x - 5) = \log 3^2 + \log 2^4$

$\Rightarrow$ $\log(x^2 - 25) = \log(3^2 \times 2^4)$

$\Rightarrow$ $x^2 - 25 = 3^2 \times 2^4 \Rightarrow x^2 - 25 = 144$

$\Rightarrow$ $x^2 = 169 \Rightarrow x = \pm 13.$

When $x = -13$, then $x + 5$ and $x - 5$ are both negative and the logarithm of a negative number is not defined, so $x = -13$ is rejected.

$\therefore$ The solution of the given equation is $x = 13$.

Example 13. *Find the value of x if* $\log_{10} x - \log_{10}(2x - 1) = 1.$

Solution. Given $\log_{10} x - \log_{10}(2x - 1) = 1$

$\Rightarrow$ $\log_{10} \frac{x}{2x-1} = 1 \Rightarrow \frac{x}{2x-1} = 10^1$

$\Rightarrow$ $\frac{x}{2x-1} = 10 \Rightarrow 20x - 10 = x$

$\Rightarrow$ $19x = 10 \Rightarrow x = \frac{10}{19}.$

Example 14. *Solve the following equations for x :*

(*i*) $\log_x 25 - \log_x 5 + \log_x \frac{1}{125} = 2$ (*ii*) $\log_x(8x - 3) - \log_x 4 = 2$

(*iii*) $3^{\log x} - 2^{\log x} = 2^{\log x + 1} - 3^{\log x - 1}.$

Solution. (*i*) Given $\log_x 25 - \log_x 5 + \log_x \frac{1}{125} = 2$

$\Rightarrow$ $\log_x \frac{25 \times \frac{1}{125}}{5} = 2 \Rightarrow \log_x \frac{1}{25} = 2$

$\Rightarrow$ $\log_x \left(\frac{1}{5}\right)^2 = 2 \Rightarrow 2\log_x \frac{1}{5} = 2$

$\Rightarrow$ $\log_x \frac{1}{5} = 1 \Rightarrow x^1 = \frac{1}{5}$

$\Rightarrow$ $x = \frac{1}{5}.$

(*ii*) Given $\log_x(8x - 3) - \log_x 4 = 2$

$\Rightarrow$ $\log_x \frac{8x-3}{4} = 2 \Rightarrow x^2 = \frac{8x-3}{4} \Rightarrow 4x^2 = 8x - 3$

$\Rightarrow \quad 4x^2 - 8x + 3 = 0 \Rightarrow 4x^2 - 6x - 2x + 3 = 0$

$\Rightarrow \quad 2x(2x - 3) - 1(2x - 3) = 0 \Rightarrow (2x - 3)(2x - 1) = 0$

$\Rightarrow \quad 2x - 3 = 0, 2x - 1 = 0$

$\Rightarrow \quad x = \frac{3}{2}, \frac{1}{2}.$

(*iii*) Given $3^{\log x} - 2^{\log x} = 2^{\log x + 1} - 3^{\log x - 1}$

$\Rightarrow \quad 3^{\log x} + 3^{\log x - 1} = 2^{\log x + 1} - 2^{\log x}$

$\Rightarrow \quad 3^{\log x} + 3^{\log x} \times 3^{-1} = 2^{\log x} \times 2^1 - 2^{\log x}$

$\Rightarrow \quad \left(1 + \frac{1}{3}\right) 3^{\log x} = (2 + 1)\, 2^{\log x}$

$\Rightarrow \quad \frac{4}{3} \times 3^{\log x} = 3 \times 2^{\log x}$

$\Rightarrow \quad \frac{3^{\log x}}{2^{\log x}} = \frac{9}{4} \Rightarrow \left(\frac{3}{2}\right)^{\log x} = \left(\frac{3}{2}\right)^2$

$\Rightarrow \quad \log x = 2 \Rightarrow x = 10^2$

$\Rightarrow \quad x = 100.$

Example 15. *Solve for x :* $\log_2 x + \log_4 x + \log_{16} x = \frac{21}{4}$.

Solution. Given $\log_2 x + \log_4 x + \log_{16} x = \frac{21}{4}$

$\Rightarrow \quad \frac{1}{\log_x 2} + \frac{1}{\log_x 4} + \frac{1}{\log_x 16} = \frac{21}{4}$

$\Rightarrow \quad \frac{1}{\log_x 2} + \frac{1}{\log_x 2^2} + \frac{1}{\log_x 2^4} = \frac{21}{4} \Rightarrow \frac{1}{\log_x 2} + \frac{1}{2\log_x 2} + \frac{1}{4\log_x 2} = \frac{21}{4}$

$\Rightarrow \quad \frac{1}{\log_x 2}\left(1 + \frac{1}{2} + \frac{1}{4}\right) = \frac{21}{4} \Rightarrow \frac{7}{4} \cdot \frac{1}{\log_x 2} = \frac{21}{4}$

$\Rightarrow \quad \log_x 2 = \frac{7}{4} \cdot \frac{4}{21} = \frac{1}{3}$

$\Rightarrow \quad x^{1/3} = 2 \Rightarrow x = 2^3 = 8.$

Example 16. *If* $a = 1 + \log_{10} 2 - \log_{10} 5$, $b = 2 \log_{10} 3$ *and* $c = \log_{10} m - \log_{10} 5$, *find the value of m given that* $a + b = 2c$.

Solution. Given $a + b = 2c$

$\Rightarrow \quad 1 + \log_{10} 2 - \log_{10} 5 + 2\log_{10} 3 = 2(\log_{10} m - \log_{10} 5)$

$\Rightarrow \quad \log_{10} 10 + \log_{10} 2 - \log_{10} 5 + 2\log_{10} 3 + 2\log_{10} 5 = 2\log_{10} m$ $[\because \log_{10} 10 = 1]$

$\Rightarrow \quad \log_{10} 10 + \log_{10} 2 + \log_{10} 5 + \log_{10} 3^2 = 2\log_{10} m$

$\Rightarrow \quad \log_{10}(10 \times 2 \times 5 \times 3^2) = 2\log_{10} m$

$\Rightarrow \quad \log_{10} 900 = 2\log_{10} m \Rightarrow \log_{10}(30)^2 = 2\log_{10} m$

$\Rightarrow \quad 2\log_{10} 30 = 2\log_{10} m \Rightarrow \log_{10} 30 = \log_{10} m$

$\Rightarrow \quad 30 = m$ *i.e.* $m = 30.$

Example 17. *If $a^2 + b^2 = 7ab$, prove that $2 \log (a + b) = \log 9 + \log a + \log b$.*

Solution. Given $a^2 + b^2 = 7ab$.

Adding $2ab$ to both sides, we get

$a^2 + 2ab + b^2 = 9ab$

$\Rightarrow$ $(a + b)^2 = 9ab$, taking logs of both sides, we get

$\log (a + b)^2 = \log 9ab$

$\Rightarrow$ $2 \log (a + b) = \log 9 + \log a + \log b$.

Example 18. *If $\frac{\log (x + y)}{\log 2} = \frac{\log (x - y)}{\log 3} = \frac{\log 64}{\log 0{\cdot}125}$, find the values of x and y.*

Solution. $\frac{\log 64}{\log 0{\cdot}125} = \frac{\log 64}{\log \frac{1}{8}} = \frac{\log 8^2}{\log 8^{-1}} = \frac{2 \log 8}{(-1) \log 8} = -2.$

$\therefore$ $\frac{\log (x + y)}{\log 2} = -2 \Rightarrow \log (x + y) = -2 \log 2 = \log 2^{-2} = \log \frac{1}{4}$

$\Rightarrow$ $x + y = \frac{1}{4}$ ...(*i*)

Also, $\frac{\log (x - y)}{\log 3} = -2 \Rightarrow \log (x - y) = -2 \log 3 = \log 3^{-2} = \log \frac{1}{9}$

$\Rightarrow$ $x - y = \frac{1}{9}$ ...(*ii*)

On adding (*i*) and (*ii*), we get

$$2x = \frac{1}{4} + \frac{1}{9} = \frac{9 + 4}{36} = \frac{13}{36} \Rightarrow x = \frac{13}{72}.$$

Subtracting (*ii*) from (*i*), we get

$$2y = \frac{1}{4} - \frac{1}{9} = \frac{9 - 4}{36} = \frac{5}{36} \Rightarrow y = \frac{5}{72}.$$

Hence $x = \frac{13}{72}$ and $y = \frac{5}{72}$.

Example 19. *Solve the following equations for x and y :*

$$\log_{10}(xy) = 2, \log_{10}\left(\frac{x}{y}\right) + 2 \log_{10} 2 = 2.$$

Solution. $\log_{10} (xy) = 2 \Rightarrow xy = 10^2 \Rightarrow xy = 100$ ...(*i*)

$\log_{10} \left(\frac{x}{y}\right) + 2 \log_{10} 2 = 2$

$\Rightarrow$ $\log_{10} \left(\frac{x}{y}\right) = 2 - 2 \log_{10} 2 = 2 (1 - \log_{10} 2)$

$= 2 (\log_{10} 10 - \log_{10} 2) = 2 \log_{10} \frac{10}{2} = 2 \log_{10} 5$

$= \log_{10} 5^2 = \log_{10} 25$

$\Rightarrow$ $\frac{x}{y} = 25$ ...(*ii*)

Multiplying (*i*) and (*ii*), we get

$$(xy) \times \left(\frac{x}{y}\right) = 100 \times 25 \Rightarrow x^2 = 2500 \Rightarrow x = \pm 50.$$

When $x = 50$, from (i), $y = \frac{100}{x} = \frac{100}{50} = 2$;

when $x = -50$, from (i), $y = \frac{100}{x} = \frac{100}{-50} = -2$.

Hence the solutions are $x = 50, y = 2; x = -50, y = -2$.

Note that both these solutions satisfy the given equations.

Exercise 11.2

1. Simplify the following :

(i) $\log a^3 - \log a^2$ (ii) $\log a^3 \div \log a^2$ (iii) $\frac{\log 4}{\log 2}$

(iv) $\frac{\log 8 \log 9}{\log 27}$ (v) $\frac{\log 27}{\log \sqrt{3}}$ (vi) $\frac{\log 9 - \log 3}{\log 27}$.

2. Evaluate the following :

(i) $\log \left(10 \div \sqrt[3]{10}\right)$ (ii) $2 + \frac{1}{2} \log (10^{-3})$

(iii) $2 \log 5 + \log 8 - \frac{1}{2} \log 4$

(iv) $2 \log 10^3 + 3 \log 10^{-2} - \frac{1}{3} \log 5^{-3} + \frac{1}{2} \log 4$

(v) $2 \log 2 + \log 5 - \frac{1}{2} \log 36 - \log \frac{1}{30}$

(vi) $2 \log 5 + \log 3 + 3 \log 2 - \frac{1}{2} \log 36 - 2 \log 10$

(vii) $\log 2 + 16 \log \frac{16}{15} + 12 \log \frac{25}{24} + 7 \log \frac{81}{80}$

$(viii)$ $2 \log_{10} 5 + \log_{10} 8 - \frac{1}{2} \log_{10} 4$.

3. Express each of the following as a single logarithm :

(i) $2 \log 3 - \frac{1}{2} \log 16 + \log 12$ (ii) $2 \log_{10} 5 - \log_{10} 2 + 3 \log_{10} 4 + 1$

(iii) $\frac{1}{2} \log 36 + 2 \log 8 - \log 1{\cdot}5$ (iv) $\frac{1}{2} \log 25 - 2 \log 3 + 1$

(v) $7 \log \frac{16}{15} + 5 \log \frac{25}{24} + 3 \log \frac{81}{80}$.

4. Prove the following :

(i) $\log_{10} 4 \div \log_{10} 2 = \log_3 9$

(ii) $\log_{10} 25 + \log_{10} 4 = \log_5 25$.

5. If $x = (100)^a$, $y = (10000)^b$ and $z = (10)^c$, express $\log \frac{10\sqrt{y}}{x^2 z^3}$ in terms of a, b, c.

6. If $a = \log_{10} x$, find the following in terms of a :

(i) x (ii) $\log_{10} \sqrt[5]{x^2}$.

7. If $a = \log \frac{2}{3}$, $b = \log \frac{3}{5}$ and $c = 2 \log \sqrt{\frac{5}{2}}$, find the value of

(i) $a + b + c$ (ii) 5^{a+b+c}.

8. If $x = \log \frac{3}{5}$, $y = \log \frac{5}{4}$ and $z = 2 \log \frac{\sqrt{3}}{2}$, find the values of

(i) $x + y - z$ (ii) 3^{x+y-z}.

9. If $x = \log_{10} 12$, $y = \log_4 2 \times \log_{10} 9$ and $z = \log_{10} 0{\cdot}4$, find the values of

(i) $x - y - z$ (ii) 7^{x-y-z}.

Hint

(i) $x - y - z = \log_{10} 12 - \log_4 2 \times \log_{10} 9 - \log_{10} 0{\cdot}4$

$= \log_{10} (4 \times 3) - \log_4 4^{1/2} \times \log_{10} 3^2 - \log_{10} \frac{4}{10}$

$= \log_{10} 4 + \log_{10} 3 - \frac{1}{2} \log_4 4 \times 2 \log_{10} 3 - (\log_{10} 4 - \log_{10} 10)$

$= \log_{10} 4 + \log_{10} 3 - \frac{1}{2} \times 1 \times 2 \log_{10} 3 - \log_{10} 4 + 1 = 1.$

10. If $\log V + \log 3 = \log \pi + \log 4 + 3 \log r$, find V in terms of other quantities.

11. Given $3 (\log 5 - \log 3) - (\log 5 - 2 \log 6) = 2 - \log n$, find n.

12. Given that $\log_{10} y + 2 \log_{10} x = 2$, express y in terms of x.

13. Express $\log_{10} 2 + 1$ in the form $\log_{10} x$.

14. If $a^2 = \log_{10} x$, $b^3 = \log_{10} y$ and $\frac{a^2}{2} - \frac{b^3}{3} = \log_{10} z$, express z in terms of x and y.

15. Given that $\log m = x + y$ and $\log n = x - y$, express the value of $\log m^2 n$ in terms of x and y.

16. Given that $\log x = m + n$ and $\log y = m - n$, express the value of $\log \left(\frac{10x}{y^2}\right)$ in terms of m and n.

17. If $\frac{\log x}{2} = \frac{\log y}{3}$, find the value of $\frac{y^4}{x^6}$.

18. Solve for x :

(i) $\log x + \log 5 = 2 \log 3$ (ii) $\log_3 x - \log_3 2 = 1$

(iii) $x = \frac{\log 125}{\log 25}$ (iv) $\frac{\log 8}{\log 2} \times \frac{\log 3}{\log \sqrt{3}} = 2 \log x$.

19. Given $2 \log_{10} x + 1 = \log_{10} 250$, find (i) x (ii) $\log_{10} 2x$.

20. If $\frac{\log x}{\log 5} = \frac{\log y^2}{\log 2} = \frac{\log 9}{\log \frac{1}{3}}$, find x and y.

21. Prove the following :

(i) $3^{\log 4} = 4^{\log 3}$ (ii) $27^{\log 2} = 8^{\log 3}$.

22. Solve the following equations :

(i) $\log (2x + 3) = \log 7$

(ii) $\log (x + 1) + \log (x - 1) = \log 24$

(iii) $\log (10x + 5) - \log (x - 4) = 2$

(iv) $\log_{10} 5 + \log_{10} (5x + 1) = \log_{10} (x + 5) + 1$

(v) $\log (4y - 3) = \log (2y + 1) - \log 3$

(vi) $\log_{10} (x + 2) + \log_{10} (x - 2) = \log_{10} 3 + 3 \log_{10} 4$

(vii) $\log(3x + 2) + \log(3x - 2) = 5 \log 2$.

23. Solve for x : $\log_3 (x + 1) - 1 = 3 + \log_3 (x - 1)$.

24. Solve for x : $5^{\log x} + 3^{\log x} = 3^{\log x + 1} - 5^{\log x - 1}$.

25. If $\log \frac{x - y}{2} = \frac{1}{2} (\log x + \log y)$, prove that $x^2 + y^2 = 6xy$.

26. If $x^2 + y^2 = 23xy$, prove that $\log \frac{x + y}{5} = \frac{1}{2} (\log x + \log y)$.

27. If $p = \log_{10} 20$ and $q = \log_{10} 25$, find the value of x if $2 \log_{10}(x + 1) = 2p - q$.

28. Show that :

(i) $\frac{1}{\log_2 42} + \frac{1}{\log_3 42} + \frac{1}{\log_7 42} = 1$

(ii) $\frac{1}{\log_8 36} + \frac{1}{\log_9 36} + \frac{1}{\log_{18} 36} = 2$.

29. Prove the following identities :

(i) $\frac{1}{\log_a abc} + \frac{1}{\log_b abc} + \frac{1}{\log_c abc} = 1$

(ii) $\log_b a \cdot \log_c b \cdot \log_d c = \log_d a$.

30. Given that $\log_a x = \frac{1}{\alpha}$, $\log_b x = \frac{1}{\beta}$, $\log_c x = \frac{1}{\gamma}$, find $\log_{abc} x$.

Hint

$\frac{1}{\alpha} = \log_a x = \frac{\log x}{\log a} \Rightarrow \log a = \alpha \log x$ etc. and

$\log_{abc} x = \frac{\log x}{\log abc} = \frac{\log x}{\log a + \log b + \log c}$.

31. Solve for x :

(i) $\log_3 x + \log_9 x + \log_{81} x = \frac{7}{4}$

(ii) $\log_2 x + \log_8 x + \log_{32} x = \frac{23}{15}$.

CHAPTER TEST

1. Expand $\log_a \sqrt[3]{x^7 y^8 + \sqrt[4]{z}}$.

2. Find the value of $\log_{\sqrt{3}} 3\sqrt{3} - \log_5 (0\cdot04)$.

3. Prove the following :

   (*i*) $(\log x)^2 - (\log y)^2 = \log \frac{x}{y} . \log xy$

   (*ii*) $2 \log \frac{11}{13} + \log \frac{130}{77} - \log \frac{55}{91} = \log 2$.

4. If $\log (m + n) = \log m + \log n$, show that $n = \frac{m}{m-1}$.

5. If $\log \frac{x+y}{2} = \frac{1}{2}(\log x + \log y)$, prove that $x = y$.

6. If a, b are positive real numbers, $a > b$ and $a^2 + b^2 = 27ab$, prove that $\log \left(\frac{a-b}{5}\right) = \frac{1}{2}(\log a + \log b)$.

7. Solve the following equations for x :

   (*i*) $\log_x \frac{1}{49} = -2$ (*ii*) $\log_x \frac{1}{4\sqrt{2}} = -5$

   (*iii*) $\log_x \frac{1}{243} = 10$ (*iv*) $\log_4 32 = x - 4$

   (*v*) $\log_7 (2x^2 - 1) = 2$ (*vi*) $\log (x^2 - 21) = 2$

   (*vii*) $\log_6 (x - 2)(x + 3) = 1$ (*viii*) $\log_6 (x - 2) + \log_6 (x + 3) = 1$

   (*ix*) $\log (x + 1) + \log (x - 1) = \log 11 + 2 \log 3$.

8. Solve for x and y :

   $\frac{\log x}{3} = \frac{\log y}{2}$ and $\log (xy) = 5$.

Hint

$\frac{\log x}{3} = \frac{\log y}{2} \Rightarrow 2 \log x - 3 \log y = 0$ ...(*i*)

$\log (xy) = 5 \Rightarrow \log x + \log y = 5$ ...(*ii*)

Multiplying (*ii*) by 3 and adding to (*i*), we get

$5 \log x = 15 \Rightarrow \log x = 3 \Rightarrow x = 10^3 = 1000$.

9. If $a = 1 + \log_x yz$, $b = 1 + \log_y zx$ and $c = 1 + \log_z xy$, then show that

   $ab + bc + ca = abc$.

Hint

$a = 1 \log_x yz = \log_x x + \log_x yz = \log_x xyz \Rightarrow \frac{1}{a} = \log_{xyz} x$ etc.

$\Rightarrow \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \log_{xyz} x + \log_{xyz} y + \log_{xyz} z = 1$.