10

INDICES

10.1 INDICES

Repeated multiplication by the same number is very common. For example, 2.2.2.2.2. It is usually written as 2^5.

Thus $2.2.2.2.2 = 2^5$, where 2 is called the **base**, 5 is called the **exponent** or **index** and 2^5 is the **exponential expression** (read as 2 to the 5th power or 2 to the power 5 or simply 2 power 5). This leads to :

Definition. *If a is any real number and n is a positive integer, then*

(i) $a^0 = 1$

(ii) $a^n = a.a.a....$ *up to n factors*

(iii) $a^{-n} = \left(\frac{1}{a}\right)^n = \frac{1}{a^n},\ a \neq 0,$

where a is called the **base**, *0, n or − n (as the case may be) is called the* **exponent** or **index**.

a^n is read as a to the nth power or a to the power n.

For example,

(i) $3^0 = 1,\quad 3^5 = 3.3.3.3.3 = 243,\quad 3^{-5} = \frac{1}{3^5} = \frac{1}{243}.$

(ii) $\left(\frac{2}{5}\right)^0 = 1, \left(\frac{2}{5}\right)^3 = \frac{2}{5}\cdot\frac{2}{5}\cdot\frac{2}{5} = \frac{8}{125}, \left(\frac{2}{5}\right)^{-3} = \frac{1}{\left(\frac{2}{5}\right)^3} = \frac{1}{\frac{8}{125}} = \frac{125}{8}.$

(iii) $(-2)^0 = 1, (-2)^3 = (-2)(-2)(-2) = -8, (-2)^{-3} = \frac{1}{(-2)^3} = \frac{1}{-8} = -\frac{1}{8}.$

10.1.1 Fractional indices (or surds)

Since $9 = 3^2$, 3 is a square root of 9, we write it as $\sqrt{9}$. Similarly, as $27 = 3^3$, 3 is a cube root of 27, we write it as $(27)^{1/3}$ or $\sqrt[3]{27}$. This leads to :

Definition. *If a is any real number and n is a positive integer, then x is called the* ***nth root*** *of a if and only if $x^n = a$. We write it as $a^{\frac{1}{n}}$ or $\sqrt[n]{a}$.*

When a is positive (unless mentioned otherwise), $a^{\frac{1}{n}}$ or $\sqrt[n]{a}$ will mean the positive real nth root of a.

In particular, when n is a rational number, then $n = \frac{p}{q}$, where p and q are integers having no common factor (except 1) and $q > 0$. If a is any real number, then a^n is defined as

$$a^n = a^{\frac{p}{q}} = \left(a^{\frac{1}{q}}\right)^p = (a^p)^{\frac{1}{q}}.$$

Note that $a \neq 0$ if n is negative.

Remark

$(a^p)^{\frac{1}{q}}$ is sometimes written as $\sqrt[q]{a^p}$.

10.2 RULES FOR INDICES

If a and b are any two real numbers, and m and n are any two rational numbers, then the following results hold :

- ☆ $a^m . a^n = a^{m+n}$.
- ☆ $\frac{a^m}{a^n} = a^{m-n}$, $a \neq 0$.
- ☆ $(a^m)^n = a^{mn}$.
- ☆ $\frac{1}{a^{-n}} = a^n$, $a \neq 0$.
- ☆ $(ab)^n = a^n b^n$.
- ☆ $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$, $b \neq 0$.
- ☆ $a^n = b^n$, $n \neq 0 \Rightarrow a = b$ provided a and b are positive.
- ☆ $a^n = a^m \Rightarrow n = m$ provided $a > 0$ and $a \neq 1$.

Note

If p and q are different positive prime integers, then $p^m q^n = p^l q^k \Rightarrow m = l$ and $n = k$.

ILLUSTRATIVE EXAMPLES

Example 1. *Simplify the following :*

(i) $(3x^4y^3)(18x^3y^5)$ *(ii)* $\frac{3x^4y^3}{18x^3y^5}$ *(iii)* $\left(-\frac{2x^2}{y^3}\right)^3$ *(iv)* $\sqrt[3]{27^{-2}}$.

Solution. *(i)* $(3x^4y^3)(18x^3y^5) = 3.18.x^4.x^3.y^3.y^5 = 54\,x^7y^8$.

(ii) $\frac{3x^4y^3}{18x^3y^5} = \frac{3}{18}.\frac{x^4}{x^3}.\frac{y^3}{y^5} = \frac{1}{6}.x^{4-3}.\frac{1}{y^{5-3}} = \frac{x}{6y^2}$.

(iii) $\left(-\frac{2x^2}{y^3}\right)^3 = (-2)^3.\frac{(x^2)^3}{(y^3)^3} = -8.\frac{x^6}{y^9}$.

(iv) $\sqrt[3]{27^{-2}} = (27^{-2})^{1/3} = (27)^{-2/3} = (3^3)^{-2/3} = 3^{3\times(-2/3)} = 3^{-2} = \frac{1}{3^2} = \frac{1}{9}$.

Example 2. *Simplify :* $\left(\frac{81}{16}\right)^{-\frac{3}{4}} \times \left[\left(\frac{25}{9}\right)^{-\frac{3}{2}} \div \left(\frac{5}{2}\right)^{-3}\right]$.

Solution. $\left(\frac{81}{16}\right)^{-\frac{3}{4}} \times \left[\left(\frac{25}{9}\right)^{-\frac{3}{2}} \div \left(\frac{5}{2}\right)^{-3}\right]$

$$= \left(\left(\frac{3}{2}\right)^{4}\right)^{-\frac{3}{4}} \times \left[\left(\left(\frac{5}{3}\right)^{2}\right)^{-\frac{3}{2}} \div \left(\frac{5}{2}\right)^{-3}\right]$$

$$= \left(\frac{3}{2}\right)^{4 \times \left(-\frac{3}{4}\right)} \times \left[\left(\frac{5}{3}\right)^{2 \times \left(-\frac{3}{2}\right)} \div \left(\frac{5}{2}\right)^{-3}\right]$$

$$= \left(\frac{3}{2}\right)^{-3} \times \left[\left(\frac{5}{3}\right)^{-3} \div \left(\frac{5}{2}\right)^{-3}\right]$$

$$= \left(\frac{2}{3}\right)^{3} \times \left[\left(\frac{3}{5}\right)^{3} \div \left(\frac{2}{5}\right)^{3}\right] \qquad \left[\because a^{-n} = \left(\frac{1}{a}\right)^{n}\right]$$

$$= \frac{2^3}{3^3} \times \left[\frac{3^3}{5^3} \div \frac{2^3}{5^3}\right] = \frac{2^3}{3^3} \times \frac{3^3}{5^3} \times \frac{5^3}{2^3} = 1.$$

Example 3. *Evaluate* $\sqrt{\frac{1}{4}} + (\cdot 01)^{-1/2} - (27)^{2/3}$. *Leave your answer as a fraction.*

Solution. $\sqrt{\frac{1}{4}} + (\cdot 01)^{-1/2} - (27)^{2/3} = \frac{1}{2} + \left(\frac{1}{100}\right)^{-1/2} - (3^3)^{2/3}$

$$= \frac{1}{2} + (100)^{1/2} - 3^2 \qquad \left[\because a^{-n} = \left(\frac{1}{a}\right)^{n}\right]$$

$$= \frac{1}{2} + \left((10)^2\right)^{1/2} - 9$$

$$= \frac{1}{2} + 10 - 9 = \frac{1}{2} + 1 = 1\frac{1}{2}.$$

Example 4. *Simplify* $\left(\frac{1}{4}\right)^{-2} - 3(8)^{2/3}(4)^0 + \left(\frac{9}{16}\right)^{-1/2}$.

Solution. $\left(\frac{1}{4}\right)^{-2} - 3(8)^{2/3}(4)^0 + \left(\frac{9}{16}\right)^{-1/2} = (4)^2 - 3 \cdot (2^3)^{2/3} \cdot 1 + \left(\frac{16}{9}\right)^{1/2}$

$$\left[\because a^{-n} = \left(\frac{1}{a}\right)^{n} \text{ and } a^0 = 1\right]$$

$$= 16 - 3 \times 2^2 \times 1 + \left(\left(\frac{4}{3}\right)^2\right)^{1/2}$$

$$= 16 - 3 \times 4 + \frac{4}{3} = 16 - 12 + \frac{4}{3}$$

$$= 4 + \frac{4}{3} = \frac{16}{3} = 5\frac{1}{3}.$$

Example 5. *Simplify the following :*

(i) $\frac{5^{n+2} - 6.5^{n+1}}{13.5^n - 2.5^{n+1}}$ (ii) $\left(\frac{x^m}{x^n}\right)^{m+n} \left(\frac{x^n}{x^l}\right)^{n+l} \left(\frac{x^l}{x^m}\right)^{l+m}$

(iii) $\left(x^{\frac{1}{3}} + x^{-\frac{1}{3}}\right)\left(x^{\frac{2}{3}} - 1 + x^{-\frac{2}{3}}\right)$.

Solution. (*i*) $\dfrac{5^{n+2} - 6.5^{n+1}}{13.5^n - 2.5^{n+1}} = \dfrac{5^n.5^2 - 6.5^n.5^1}{13.5^n - 2.5^1.5^n}$

$$= \frac{5^n(5^2 - 6.5)}{5^n(13 - 2.5)} = \frac{25 - 30}{13 - 10} = \frac{-5}{3} = -\frac{5}{3}.$$

(*ii*) $\left(\dfrac{x^m}{x^n}\right)^{m+n}\left(\dfrac{x^n}{x^l}\right)^{n+l}\left(\dfrac{x^l}{x^m}\right)^{l+m}$

$$= (x^{m-n})^{m+n} . (x^{n-l})^{n+l} . (x^{l-m})^{l+m}$$

$$= x^{m^2-n^2} . x^{n^2-l^2} . x^{l^2-m^2}$$

$$= x^{m^2-n^2+n^2-l^2+l^2-m^2} = x^0 = 1.$$

(*iii*) $\left(x^{\frac{1}{3}} + x^{-\frac{1}{3}}\right)\left(x^{\frac{2}{3}} - 1 + x^{-\frac{2}{3}}\right) = x^{\frac{1}{3}} . x^{\frac{2}{3}} - x^{\frac{1}{3}} + x^{\frac{1}{3}} . x^{-\frac{2}{3}} + x^{-\frac{1}{3}} . x^{\frac{2}{3}} - x^{-\frac{1}{3}} + x^{-\frac{1}{3}} . x^{-\frac{2}{3}}$

$$= x^1 - x^{\frac{1}{3}} + x^{-\frac{1}{3}} + x^{\frac{1}{3}} - x^{-\frac{1}{3}} + x^{-1}$$

$$= x + x^{-1} = x + \frac{1}{x}.$$

Example 6. *If $a = b^{2x}$, $b = c^{2y}$ and $c = a^{2z}$, prove that $xyz = \frac{1}{8}$.*

Solution. Given $a = b^{2x}$ ...(*i*) $b = c^{2y}$ ...(*ii*) $c = a^{2z}$ ...(*iii*)

Substituting the value of b from (*ii*) in (*i*), we get

$$a = (c^{2y})^{2x} = c^{4xy} \quad ...(iv)$$

Substituting the value of c from (*iii*) in (*iv*), we get

$$a = (a^{2z})^{4xy} = a^{8xyz}$$

$\Rightarrow$ $a^1 = a^{8xyz} \Rightarrow 1 = 8xyz$ (Assume $a > 0$, $a \neq 1$)

$\Rightarrow$ $xyz = \frac{1}{8}$.

Example 7. *If $a^x = b^y = c^z$ and $b^2 = ac$, prove that $y = \frac{2xz}{z + x}$.*

Solution. Let $a^x = b^y = c^z = k$ (say), then

$$a = k^{\frac{1}{x}},\ b = k^{\frac{1}{y}} \text{ and } c = k^{\frac{1}{z}}.$$

Given $b^2 = ac \Rightarrow \left(k^{\frac{1}{y}}\right)^2 = k^{\frac{1}{x}} . k^{\frac{1}{z}}$

$\Rightarrow$ $k^{\frac{2}{y}} = k^{\frac{1}{x}+\frac{1}{z}} \Rightarrow \frac{2}{y} = \frac{1}{x} + \frac{1}{z}$

$\Rightarrow$ $\frac{2}{y} = \frac{z + x}{xz} \Rightarrow y = \frac{2xz}{z + x}$.

Example 8. *If $abc = 1$, show that*

$$\frac{1}{1 + a + b^{-1}} + \frac{1}{1 + b + c^{-1}} + \frac{1}{1 + c + a^{-1}} = 1.$$

Solution. $\dfrac{1}{1 + a + b^{-1}} + \dfrac{1}{1 + b + c^{-1}} + \dfrac{1}{1 + c + a^{-1}}$

$$= \frac{1}{1 + a + \frac{1}{b}} + \frac{1}{1 + b + \frac{1}{c}} + \frac{1}{1 + c + \frac{1}{a}}$$

$$= \frac{b}{b+ab+1} + \frac{1}{1+b+ab} + \frac{1}{1+\frac{1}{ab}+\frac{1}{a}} \qquad \left(\because abc = 1 \Rightarrow \frac{1}{c} = ab \text{ and } c = \frac{1}{ab}\right)$$

$$= \frac{b}{1+b+ab} + \frac{1}{1+b+ab} + \frac{ab}{ab+1+b}$$

$$= \frac{b+1+ab}{1+b+ab} = 1.$$

Example 9. *If $x = \sqrt[3]{28}$ and $y = \sqrt[3]{27}$, find the value of $x + y - \frac{1}{x^2+xy+y^2}$.*

Solution. $x + y - \frac{1}{x^2+xy+y^2} = x + y - \frac{x-y}{(x-y)(x^2+xy+y^2)}$ (Note this step)

$$= x + y - \frac{x-y}{x^3-y^3}$$

$$= x + y - \frac{x-y}{((28)^{1/3})^3 - ((27)^{1/3})^3}$$

$$= x + y - \frac{x-y}{28-27} = x + y - \frac{x-y}{1}$$

$$= x + y - (x - y) = 2y$$

$$= 2 \times \sqrt[3]{27} = 2 \times (27)^{1/3}$$

$$= 2 \times (3^3)^{\frac{1}{3}} = 2 \times 3^1 = 6.$$

Example 10. *Given $1176 = 2^p . 3^q . 7^r$, find*

(i) the numerical values of p, q and r *(ii) the value of $2^p . 3^q . 7^{-r}$ as a fraction.*

Solution. (*i*) Given $1176 = 2^p.3^q.7^r$

$\Rightarrow 2 \times 2 \times 2 \times 3 \times 7 \times 7 = 2^p.3^q.7^r$

$\Rightarrow 2^3.3^1.7^2 = 2^p.3^q.7^r$

$\Rightarrow p = 3, q = 1$ and $r = 2$. (See note page 150)

(*ii*) $2^p.3^q.7^{-r} = 2^3.3^1.7^{-2} = \frac{8 \times 3}{7^2} = \frac{24}{49}$.

Example 11. *Solve the following equations for x :*

(i) $4^{2x} = \frac{1}{32}$ *(ii)* $\sqrt{\left(\frac{3}{5}\right)^{1-2x}} = 4\frac{17}{27}$.

Solution. (*i*) Given $4^{2x} = \frac{1}{32} \Rightarrow (2^2)^{2x} = \left(\frac{1}{2}\right)^5$ $\left[\because \left(\frac{1}{a}\right)^n = a^{-n}\right]$

$\Rightarrow 2^{4x} = 2^{-5}$

$\Rightarrow 4x = -5 \Rightarrow x = -\frac{5}{4}$.

(*ii*) Given $\sqrt{\left(\frac{3}{5}\right)^{1-2x}} = 4\frac{17}{27} \Rightarrow \left(\left(\frac{3}{5}\right)^{1-2x}\right)^{1/2} = \frac{125}{27}$

$\Rightarrow \left(\frac{3}{5}\right)^{\frac{1-2x}{2}} = \left(\frac{5}{3}\right)^3 \Rightarrow \left(\frac{3}{5}\right)^{\frac{1-2x}{2}} = \left(\frac{3}{5}\right)^{-3}$ $\left[\because \left(\frac{1}{a}\right)^n = a^{-n}\right]$

$\Rightarrow \dfrac{1-2x}{2} = -3 \Rightarrow 1 - 2x = -6$

$\Rightarrow -2x = -6 - 1 \Rightarrow -2x = -7 \Rightarrow x = \dfrac{7}{2}.$

Example 12. *Solve the following equations for x :*

(i) $\sqrt{\left(8^0 + \dfrac{2}{3}\right)} = (0{\cdot}6)^{2-3x}$ (ii) $2^3(5^0 + 3^{2x}) = 8\dfrac{8}{27}.$

Solution. (i) Given $\sqrt{\left(8^0 + \dfrac{2}{3}\right)} = (0{\cdot}6)^{2-3x}$

$\Rightarrow \sqrt{\left(1 + \dfrac{2}{3}\right)} = \left(\dfrac{3}{5}\right)^{2-3x} \Rightarrow \left(\dfrac{5}{3}\right)^{\frac{1}{2}} = \left(\dfrac{3}{5}\right)^{2-3x}$

$\Rightarrow \left(\dfrac{3}{5}\right)^{-\frac{1}{2}} = \left(\dfrac{3}{5}\right)^{2-3x}$ $\left[\because \left(\dfrac{1}{a}\right)^n = a^{-n}\right]$

$\Rightarrow -\dfrac{1}{2} = 2 - 3x$

$\Rightarrow 3x = 2 + \dfrac{1}{2} \Rightarrow 3x = \dfrac{5}{2} \Rightarrow x = \dfrac{5}{6}.$

(ii) Given $2^3(5^0 + 3^{2x}) = 8\dfrac{8}{27}$

$\Rightarrow 8(1 + 3^{2x}) = 8 + \dfrac{8}{27} \Rightarrow 8 + 8 \times 3^{2x} = 8 + \dfrac{8}{27}$

$\Rightarrow 8 \times 3^{2x} = \dfrac{8}{27} \Rightarrow 3^{2x} = \dfrac{1}{27}$

$\Rightarrow 3^{2x} = \dfrac{1}{3^3} \Rightarrow 3^{2x} = 3^{-3}$

$\Rightarrow 2x = -3 \Rightarrow x = -\dfrac{3}{2}.$

Example 13. *If* $\left(\dfrac{p^{-1}q^2}{p^3q^{-2}}\right)^{1/3} \div \left(\dfrac{p^6q^{-3}}{p^{-2}q^3}\right)^{1/2} = p^aq^b$, *prove that* $a + b + 1 = 0$, *where p and q are different positive primes.*

Solution. $\left(\dfrac{p^{-1}q^2}{p^3q^{-2}}\right)^{1/3} = (p^{-1-3}q^{2-(-2)})^{1/3} = (p^{-4}q^4)^{1/3} = p^{-\frac{4}{3}}q^{\frac{4}{3}},$

$\left(\dfrac{p^6q^{-3}}{p^{-2}q^3}\right)^{1/2} = (p^{6-(-2)}q^{-3-3})^{1/2} = (p^8q^{-6})^{1/2} = p^4q^{-3}.$

$\therefore \left(\dfrac{p^{-1}q^2}{p^3q^{-2}}\right)^{1/3} \div \left(\dfrac{p^6q^{-3}}{p^{-2}q^3}\right) = p^aq^b$

$\Rightarrow \dfrac{p^{-\frac{4}{3}}q^{\frac{4}{3}}}{p^4q^{-3}} \Rightarrow p^aq^b \Rightarrow p^{-\frac{4}{3}-4}q^{\frac{4}{3}-(-3)} = p^aq^b$

$\Rightarrow p^{-\frac{16}{3}}q^{\frac{13}{3}} = p^aq^b$

$\Rightarrow -\dfrac{16}{3} = a$ and $\dfrac{13}{3} = b$ ($\because$ p, q are different positive primes)

$\therefore a + b + 1 = -\dfrac{16}{3} + \dfrac{13}{3} + 1 = \dfrac{-16+13+3}{3} = 0.$

Example 14. *If $5^{2x-1} = 25^{x-1} + 100$, find the value of 3^{1+x}.*

Solution. Given $5^{2x-1} = 25^{x-1} + 100$

$\Rightarrow \quad 5^{2x-1} = (5^2)^{x-1} + 100 \Rightarrow 5^{2x-1} - 5^{2x-2} = 100$

$\Rightarrow \quad 5^{2x-2} . 5^1 - 5^{2x-2} = 100 \Rightarrow 5^{2x-2}(5-1) = 100$

$\Rightarrow \quad 5^{2x-2} \times 4 = 100 \Rightarrow 5^{2x-2} = 25$

$\Rightarrow \quad 5^{2x-2} = 5^2 \Rightarrow 2x - 2 = 2$

$\Rightarrow \quad 2x = 4 \Rightarrow x = 2.$

$\therefore \quad 3^{1+x} = 3^{1+2} = 3^3 = 27.$

Example 15. *Determine $(8x)^x$ if $9^{x+2} = 240 + 9^x$.*

Solution. Given $9^{x+2} = 240 + 9^x$

$\Rightarrow \quad 9^x . 9^2 - 9^x = 240 \quad \Rightarrow \quad (9^2 - 1)\, 9^x = 240$

$\Rightarrow \quad (81 - 1)\, 9^x = 240 \quad \Rightarrow \quad 80 \times (3^2)^x = 240$

$\Rightarrow \quad 3^{2x} = \frac{240}{80} = 3 = 3^1 \quad \Rightarrow \quad 2x = 1 \quad \Rightarrow \quad x = \frac{1}{2}.$

$\therefore \quad (8x)^x = \left(8 \times \frac{1}{2}\right)^{\frac{1}{2}} = 4^{\frac{1}{2}} = (2^2)^{\frac{1}{2}} = 2^{2 \times \frac{1}{2}} = 2^1 = 2.$

Example 16. *Solve for x and y :*

$$\left(\sqrt{32}\right)^x \div 2^{y+1} = 1, \quad 16^{4-\frac{x}{2}} - 8^y = 0.$$

Solution. Given $\left(\sqrt{32}\right)^x \div 2^{y+1} = 1$

$$\Rightarrow \quad \left(\left(2^5\right)^{\frac{1}{2}}\right)^x \div 2^{y+1} = 1$$

$$\Rightarrow \quad \frac{\left(2^{\frac{5}{2}}\right)^x}{2^{y+1}} = 1 \Rightarrow 2^{\frac{5x}{2} - y - 1} = 2^0$$

$$\Rightarrow \quad \frac{5x}{2} - y - 1 = 0 \Rightarrow 5x - 2y - 2 = 0 \qquad \ldots(i)$$

Also $\quad 16^{4-\frac{x}{2}} - 8^y = 0 \Rightarrow \left(2^4\right)^{4-\frac{x}{2}} = (2^3)^y$

$\Rightarrow \quad 2^{16-2x} = 2^{3y} \Rightarrow 16 - 2x = 3y$

$\Rightarrow \quad 2x + 3y - 16 = 0 \qquad \ldots(ii)$

Multiplying (*i*) by 3 and (*ii*) by 2, we get

$15x - 6y - 6 = 0 \qquad \ldots(iii)$

and $\quad 4x + 6y - 32 = 0 \qquad \ldots(iv)$

Adding (*iii*) and (*iv*), we get

$19x - 38 = 0 \Rightarrow x = 2.$

Substituting $x = 2$ in (*ii*), we get

$2.2 + 3y - 16 = 0 \Rightarrow 3y = 12 \Rightarrow y = 4.$

Hence the solution is $x = 2, y = 4$.

Exercise 10

Simplify the following (1 to 20) :

1. (*i*) $(ab^2)^5$ (*ii*) $\left(\frac{2x^2y}{8x^3y^2}\right)^2$.

2. (*i*) $(2a^{-3}b^2)^3$ (*ii*) $\frac{a^{-1}+b^{-1}}{(ab)^{-1}}$.

3. (*i*) $\frac{x^{-1}y^{-1}}{x^{-1}+y^{-1}}$ (*ii*) $\frac{(4\times 10^7)(6\times 10^{-5})}{8\times 10^{10}}$.

4. (*i*) $\frac{3a}{b^{-1}}+\frac{2b}{a^{-1}}$ (*ii*) $5^0\times 4^{-1}+8^{1/3}$.

5. (*i*) $\left(\frac{8}{125}\right)^{-\frac{1}{3}}$ (*ii*) $(0{\cdot}027)^{\frac{1}{3}}$.

6. (*i*) $\left(-\frac{1}{27}\right)^{-\frac{2}{3}}$ (*ii*) $(64)^{-\frac{2}{3}}\div 9^{-\frac{3}{2}}$.

7. (*i*) $\frac{(27)^{\frac{2n}{3}}\times(8)^{-\frac{n}{6}}}{(18)^{-\frac{n}{2}}}$ (*ii*) $\frac{5.(25)^{n+1}-25.(5)^{2n}}{5.(5)^{2n+3}-(25)^{n+1}}$.

8. (*i*) $\left[8^{-\frac{4}{3}}\div 2^{-2}\right]^{1/2}$ (*ii*) $\left(\frac{27}{8}\right)^{2/3}-\left(\frac{1}{4}\right)^{-2}+5^0$.

9. (*i*) $(3x^2)^{-3}\times(x^9)^{2/3}$ (*ii*) $(8x^4)^{1/3}\div x^{1/3}$.

10. (*i*) $(3^2)^0+3^{-4}\times 3^6+\left(\frac{1}{3}\right)^{-2}$ (*ii*) $9^{5/2}-3.(5)^0-\left(\frac{1}{81}\right)^{-1/2}$.

11. (*i*) $16^{3/4}+2\left(\frac{1}{2}\right)^{-1}(3)^0$ (*ii*) $(81)^{3/4}-\left(\frac{1}{32}\right)^{-2/5}+(8)^{1/3}\left(\frac{1}{2}\right)^{-1}(2)^0$.

12. (*i*) $\left(\frac{64}{125}\right)^{-\frac{2}{3}}\div\frac{1}{\left(\frac{256}{625}\right)^{\frac{1}{4}}}+\left(\frac{\sqrt{25}}{\sqrt[3]{64}}\right)^0$ (*ii*) $\frac{5^{n+3}-6\times 5^{n+1}}{9\times 5^n-2^2\times 5^n}$.

13. (*i*) $\left[(64)^{\frac{2}{3}}\,2^{-2}\div 8^{\circ}\right]^{-1/2}$ (*ii*) $3^n\times 9^{n+1}\div(3^{n-1}\times 9^{n-1})$.

14. (*i*) $\frac{\sqrt{2^2}\times\sqrt[4]{256}}{\sqrt[3]{64}}-\left(\frac{1}{2}\right)^{-2}$ (*ii*) $\frac{3^{-\frac{6}{7}}\times 4^{-\frac{3}{7}}\times 9^{\frac{3}{7}}\times 2^{\frac{6}{7}}}{2^2+2^0+2^{-2}}$.

15. (*i*) $\frac{(32)^{\frac{2}{5}}\times(4)^{-\frac{1}{2}}\times(8)^{\frac{1}{3}}}{2^{-2}\div(64)^{-1/3}}$ (*ii*) $\frac{5^{2(x+6)}\times(25)^{-7+2x}}{(125)^{2x}}$.

16. (i) $\dfrac{7^{2n+3}-(49)^{n+2}}{\left((343)^{n+1}\right)^{2/3}}$ (ii) $(27)^{4/3}+(32)^{0\cdot 8}+(0\cdot 8)^{-1}$.

17. (i) $\left(\sqrt{32}-\sqrt{5}\right)^{\frac{1}{3}}\left(\sqrt{32}+\sqrt{5}\right)^{\frac{1}{3}}$ (ii) $\left(x^{\frac{1}{3}}-x^{-\frac{1}{3}}\right)\left(x^{\frac{2}{3}}+1+x^{-\frac{2}{3}}\right)$.

18. (i) $\left(\dfrac{x^m}{x^n}\right)^l\cdot\left(\dfrac{x^n}{x^l}\right)^m\cdot\left(\dfrac{x^l}{x^m}\right)^n$ (ii) $\left(\dfrac{x^{a+b}}{x^c}\right)^{a-b}\cdot\left(\dfrac{x^{b+c}}{x^a}\right)^{b-c}\cdot\left(\dfrac{x^{c+a}}{x^b}\right)^{c-a}$.

19. (i) $\sqrt[lm]{\dfrac{x^l}{x^m}}\cdot\sqrt[mn]{\dfrac{x^m}{x^n}}\cdot\sqrt[nl]{\dfrac{x^n}{x^l}}$ (ii) $\left(\dfrac{x^a}{x^b}\right)^{a^2+ab+b^2}\cdot\left(\dfrac{x^b}{x^c}\right)^{b^2+bc+c^2}\cdot\left(\dfrac{x^c}{x^a}\right)^{c^2+ac+a^2}$

(iii) $\left(\dfrac{x^a}{x^{-b}}\right)^{a^2-ab+b^2}\cdot\left(\dfrac{x^b}{x^{-c}}\right)^{b^2-bc+c^2}\cdot\left(\dfrac{x^c}{x^{-a}}\right)^{c^2-ca+a^2}$.

20. (i) $(a^{-1}+b^{-1})\div(a^{-2}-b^{-2})$ (ii) $\dfrac{1}{1+a^{m-n}}+\dfrac{1}{1+a^{n-m}}$.

21. Prove the following :

(i) $(a+b)^{-1}\ (a^{-1}+b^{-1})=\dfrac{1}{ab}$ (ii) $\dfrac{x+y+z}{x^{-1}y^{-1}+y^{-1}z^{-1}+z^{-1}x^{-1}}=xyz$.

22. If $a=c^z$, $b=a^x$ and $c=b^y$, prove that $xyz=1$.

23. If $a=xy^{p-1}$, $b=xy^{q-1}$ and $c=xy^{r-1}$, prove that
$a^{q-r}.b^{r-p}.c^{p-q}=1$.

24. If $2^x=3^y=6^{-z}$, prove that $\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=0$.

25. If $2^x=3^y=12^z$, prove that $x=\dfrac{2yz}{y-z}$.

Hint

Let $2^x=3^y=12^z=k$, then $2=k^{\frac{1}{x}}$, $3=k^{\frac{1}{y}}$ and $12=k^{\frac{1}{z}}$.

Now $12=2^2\times 3 \Rightarrow k^{\frac{1}{z}}=\left(k^{\frac{1}{x}}\right)^2\times k^{\frac{1}{y}}$

$\Rightarrow \dfrac{1}{z}=\dfrac{2}{x}+\dfrac{1}{y}\Rightarrow\dfrac{1}{z}-\dfrac{1}{y}=\dfrac{2}{x}$.

26. Simplify and express with positive exponents :

$(3x^2)^0$, $(xy)^{-2}$, $(-27\,a^9)^{2/3}$.

27. If $a=3$ and $b=-2$, find the values of :

(i) a^a+b^b (ii) a^b+b^a.

28. If $x=10^3\times 0\cdot 0099$, $y=10^{-2}\times 110$, find the value of $\sqrt{\dfrac{x}{y}}$.

29. Evaluate $x^{1/2} . y^{-1} . z^{2/3}$ when $x = 9$, $y = 2$ and $z = 8$.

30. If $x^4 y^2 z^3 = 49392$, find the values of x, y and z, where x, y and z are different positive primes.

31. If $\sqrt[3]{a^6 b^{-4}} = a^x . b^{2y}$, find x and y, where a, b are different positive primes.

32. If $(p + q)^{-1} (p^{-1} + q^{-1}) = p^a q^b$, prove that $a + b + 2 = 0$, where p and q are different positive primes.

33. If $\left(\frac{p^{-1} q^2}{p^2 q^{-4}}\right)^7 \div \left(\frac{p^3 q^{-5}}{p^{-2} q^3}\right)^{-5} = p^x q^y$, find $x + y$, where p and q are different positive primes.

34. Solve the following equations for x :

(i) $5^{2x+3} = 1$ (ii) $(13)^{\sqrt{x}} = 4^4 - 3^4 - 6$

(iii) $\left(\sqrt{\frac{3}{5}}\right)^{x+1} = \frac{125}{27}$ (iv) $\left(\sqrt[3]{4}\right)^{2x+\frac{1}{2}} = \frac{1}{32}$.

35. Solve the following equations for x :

(i) $\sqrt{\frac{p}{q}} = \left(\frac{q}{p}\right)^{1-2x}$ (ii) $4^{x-1} \times (0{\cdot}5)^{3-2x} = \left(\frac{1}{8}\right)^x$.

36. If $5^{3x} = 125$ and $(10)^y = 0{\cdot}001$, find x and y.

37. If $\frac{9^n \cdot 3^2 \cdot 3^n - (27)^n}{3^{3m} \cdot 2^3} = \frac{1}{27}$, prove that $m = 1 + n$.

38. If $3^{4x} = (81)^{-1}$ and $(10)^{1/y} = 0{\cdot}0001$, find the value of $2^{-x} . (16)^y$.

39. If $3^{x+1} = 9^{x-2}$, find the value of 2^{1+x}.

40. Solve the following equations :

(i) $3(2^x + 1) - 2^{x+2} + 5 = 0$ (ii) $3^x = 9 . 3^y$, $8 . 2^y = 4^x$.

CHAPTER TEST

1. If $2^x.3^y.5^z = 2160$ find, the values of x, y and z. Hence compute the value of $3^x.2^{-y}.5^{-z}$.
2. If $x = 2$ and $y = -3$, find the values of (*i*) $x^x + y^y$ (*ii*) $x^y + y^x$.
3. If $p = x^{m+n}.y^l$, $q = x^{n+l}.y^m$ and $r = x^{l+m}.y^n$, prove that $p^{m-n}.q^{n-l}.r^{l-m} = 1$.
4. If $x = a^{m+n}$, $y = a^{n+l}$ and $z = a^{l+m}$, prove that $x^m y^n z^l = x^n y^l z^m$.
5. Show that $\dfrac{\left(p+\frac{1}{q}\right)^m \times \left(p-\frac{1}{q}\right)^n}{\left(q+\frac{1}{p}\right)^m \times \left(q-\frac{1}{p}\right)^n} = \left(\dfrac{p}{q}\right)^{m+n}$.
6. Show that $\dfrac{a^{-1}}{a^{-1}+b^{-1}} + \dfrac{a^{-1}}{a^{-1}-b^{-1}} = \dfrac{2b^2}{b^2-a^2}$.
7. Show that $\dfrac{1}{1+a^{y-x}+a^{z-x}} + \dfrac{1}{1+a^{z-y}+a^{x-y}} + \dfrac{1}{1+a^{x-z}+a^{y-z}} = 1$.

Hint

$$\frac{1}{1+a^{y-x}+a^{z-x}} = \frac{1}{1+\frac{a^y}{a^x}+\frac{a^z}{a^x}} = \frac{a^x}{a^x+a^y+a^z} \text{ etc.}$$

8. If $3^x = 5^y = (75)^z$, show that $z = \dfrac{xy}{2x+y}$.
9. Solve the following equations :
   (*i*) $3^{x+1} = 27.3^4$
   (*ii*) $4^{2x} = \left(\sqrt[3]{16}\right)^{-\frac{6}{y}} = \left(\sqrt{8}\right)^2$
   (*iii*) $3^{x-1} \times 5^{2y-3} = 225$
   (*iv*) $8^{x+1} = 16^{y+2}$, $\left(\frac{1}{2}\right)^{3+x} = \left(\frac{1}{4}\right)^{3y}$.