5

FACTORISATION

5.1 FACTORS

*When an algebraic expression can be written as the product of two or more expressions, then each of these expressions is called a **factor** of the given expression.*

For example, $x^2 - 3x + 2 = (x - 1)(x - 2)$, therefore, $(x - 1)$ and $(x - 2)$ are the factors of $x^2 - 3x + 2$.

We already know the process of finding the product of two or more algebraic expressions, and now we shall learn the *reverse process*.

*The process of finding two or more expressions (factors) whose product is the given expression is called **factorisation**.*

However, in this book, we shall deal only with some special types of expressions.

5.2 FACTORISING BY TAKING OUT COMMON FACTORS

If the different terms of a given polynomial have common factors, then the given polynomial can be factorised by the following procedure :

(*i*) Find the H.C.F. of all the terms of the given polynomial.

(*ii*) Divide each term of the given polynomial by H.C.F. Enclose the quotient within the brackets and keep the common factor outside the bracket.

ILLUSTRATIVE EXAMPLES

Example 1. *Factorise the following :*

(*i*) $24x^3 - 32x^2$ (*ii*) $15ab^2 - 21a^2b$

(*iii*) $6xy^2 + 9x^2y - 21xy$ (*iv*) $14x^2y^2 - 10x^2y + 8xy^2$.

Solution. (*i*) H.C.F. of $24x^3$ and $32x^2$ is $8x^2$.

Divide each term by $8x^2$ and keep $8x^2$ outside the bracket.

$\therefore \quad 24x^3 - 32x^2 = 8x^2(3x - 4)$.

(*ii*) H.C.F. of $15ab^2$ and $21a^2b$ is $3ab$.

$\therefore \quad 15ab^2 - 21a^2b = 3ab(5b - 7a)$.

(*iii*) H.C.F. of $6xy^2$, $9x^2y$ and $21xy$ is $3xy$.

$\therefore \quad 6xy^2 + 9x^2y - 21xy = 3xy\,(2y + 3x - 7)$.

(*iv*) H.C.F. of $14x^2y^2$, $10x^2y$ and $8xy^2$ is $2xy$.

$\therefore \quad 14x^2y^2 - 10x^2y + 8xy^2 = 2xy\,(7xy - 5x + 4y)$.

Example 2. *Factorise the following :*

(*i*) $3x\,(y + 2z) + 5a\,(y + 2z)$

(*ii*) $10\,(p - 2q)^3 + 6\,(p - 2q)^2 - 20\,(p - 2q)$.

Solution. (*i*) H.C.F. of the expressions $3x\,(y + 2z)$ and $5a\,(y + 2z)$ is $y + 2z$.

$\therefore \quad 3x\,(y + 2z) + 5a\,(y + 2z) = (y + 2z)\,(3x + 5a)$.

Divide each term by $y + 2z$ and keep $y + 2z$ outside the bracket.

(*ii*) H.C.F. of the expressions

$10\,(p - 2q)^3$, $6\,(p - 2q)^2$ and $20\,(p - 2q)$ is $2(p - 2q)$

$\therefore \quad 10\,(p - 2q)^3 + 6\,(p - 2q)^2 - 20\,(p - 2q)$

$= 2\,(p - 2q)\,[5(p - 2q)^2 + 3(p - 2q) - 10]$.

Exercise 5.1

Factorise the following (1 to 9) :

1. (*i*) $8xy^3 + 12x^2y^2$ (*ii*) $15ax^3 - 9ax^2$.
2. (*i*) $21py^2 - 56py$ (*ii*) $4x^3 - 6x^2$.
3. (*i*) $2\pi r^2 - 4\pi r$ (*ii*) $18m + 16n$.
4. (*i*) $25abc^2 - 15a^2b^2c$ (*ii*) $28p^2q^2r - 42pq^2r^2$.
5. (*i*) $8x^3 - 6x^2 + 10x$ (*ii*) $14mn + 22m - 62p$.
6. (*i*) $18p^2q^2 - 24pq^2 + 30p^2q$ (*ii*) $27a^3b^3 - 18a^2b^3 + 75a^3b^2$.
7. (*i*) $15a\,(2p - 3q) - 10b\,(2p - 3q)$ (*ii*) $3a\,(x^2 + y^2) + 6b\,(x^2 + y^2)$.
8. (*i*) $6\,(x + 2y)^3 + 8\,(x + 2y)^2$ (*ii*) $14\,(a - 3b)^3 - 21p\,(a - 3b)$.
9. (*i*) $10a\,(2p + q)^3 - 15b\,(2p + q)^2 + 35\,(2p + q)$
   (*ii*) $x\,(x^2 + y^2 - z^2) + y\,(-x^2 - y^2 + z^2) - z\,(x^2 + y^2 - z^2)$.

5.3 FACTORISING BY GROUPING OF TERMS

When the grouping of terms of the given polynomial gives rise to common factor, then the given polynomial can be factorised by the following procedure :

(*i*) Arrange the terms of the given polynomial in groups in such a way that each group has a common factor.

(*ii*) Factorise each group.

(*iii*) Take out the factor which is common to each group.

ILLUSTRATIVE EXAMPLES

Example 1. *Factorise the following :*

(*i*) $ax - ay + bx - by$ (*ii*) $4x^2 - 10xy - 6xz + 15yz$.

Solution. (*i*) $ax - ay + bx - by = (ax - ay) + (bx - by)$
$= a(x - y) + b(x - y)$
$= (x - y)(a + b).$

(*ii*) $4x^2 - 10xy - 6xz + 15yz = (4x^2 - 10xy) - (6xz - 15yz)$
$= 2x(2x - 5y) - 3z(2x - 5y)$
$= (2x - 5y)(2x - 3z).$

Example 2. *Factorise the following :*

(*i*) $x^3 + 2x^2 + x + 2$ (*ii*) $1 + p + pq + p^2q.$

Solution. (*i*) $x^3 + 2x^2 + x + 2 = (x^3 + 2x^2) + (x + 2)$
$= x^2(x + 2) + 1(x + 2)$
$= (x + 2)(x^2 + 1).$

(*ii*) $1 + p + pq + p^2q = (1 + p) + (pq + p^2q)$
$= 1(1 + p) + pq(1 + p)$
$= (1 + p)(1 + pq).$

Example 3. *Factorise the following :*

(*i*) $xy - pq + qy - px$ (*ii*) $ab(x^2 + y^2) + xy(a^2 + b^2).$

Solution. (*i*) Since xy and pq have nothing in common, we do not group the terms in pairs in order in which the given expression is written. Hence we interchange $-pq$ and $-px$

$\therefore$ $xy - pq + qy - px = (xy - px) + (qy - pq)$
$= x(y - p) + q(y - p)$
$= (y - p)(x + q)$

(*ii*) $ab(x^2 + y^2) + xy(a^2 + b^2) = abx^2 + aby^2 + a^2xy + b^2xy$
$= (abx^2 + a^2xy) + (aby^2 + b^2xy)$
$= ax(bx + ay) + by(ay + bx)$
$= (bx + ay)(ax + by)$

Example 4. *Factorise the following :*

(*i*) $a(a + b - c) - bc$ (*ii*) $a^2x^2 + (ax^2 + 1)x + a.$

Solution. (*i*) $a(a + b - c) - bc = a^2 + ab - ac - bc$
$= (a^2 + ab) - (ac + bc)$
$= a(a + b) - c(a + b)$
$= (a + b)(a - c).$

(*ii*) $a^2x^2 + (ax^2 + 1)x + a = a^2x^2 + ax^3 + x + a$
$= (ax^3 + a^2x^2) + (x + a)$
$= ax^2(x + a) + 1(x + a)$
$= (x + a)(ax^2 + 1).$

Example 5. *Factorise the following :*

(*i*) $ax + by + bx + az + ay + bz$ (*ii*) $x^3 - x^2 + ax + x - a - 1.$

Solution. (*i*) $ax + by + bx + az + ay + bz$
$= (ax + ay + az) + (bx + by + bz)$

$$= a(x + y + z) + b(x + y + z)$$
$$= (x + y + z)(a + b).$$

(ii) $x^3 - x^2 + ax + x - a - 1 = (x^3 - x^2) + (ax - a) + (x - 1)$
$$= x^2(x - 1) + a(x - 1) + 1(x - 1)$$
$$= (x - 1)(x^2 + a + 1).$$

Example 6. *Factorise the following :*

(i) $p(x - y)^2 - qy + qx + 3x - 3y$ (ii) $ax - (ax + by)^2 + a^2x + aby + by.$

Solution. (i) $p(x - y)^2 - qy + qx + 3x - 3y$
$$= p(x - y)^2 + (qx - qy) + (3x - 3y)$$
$$= p(x - y)^2 + q(x - y) + 3(x - y)$$
$$= (x - y)[p(x - y) + q + 3].$$

(ii) $ax - (ax + by)^2 + a^2x + aby + by$
$$= (ax + a^2x) + (aby + by) - (ax + by)^2$$
$$= ax(1 + a) + by(a + 1) - (ax + by)^2$$
$$= (1 + a)(ax + by) - (ax + by)^2$$
$$= (ax + by)[1 + a - (ax + by)]$$
$$= (ax + by)(1 + a - ax - by).$$

Example 7. *Factorise the following :*

(i) $a^3x + a^2(x - y) - a(y + z) - z$ (ii) $(x^2 - 2x)^2 - 5(x^2 - 2x) - y(x^2 - 2x) + 5y.$

Solution. (i) $a^3x + a^2(x - y) - a(y + z) - z$
$$= a^3x + a^2x - a^2y - ay - az - z$$
$$= (a^3x + a^2x) - (a^2y + ay) - (az + z)$$
$$= a^2x(a + 1) - ay(a + 1) - z(a + 1)$$
$$= (a + 1)(a^2x - ay - z).$$

(ii) $(x^2 - 2x)^2 - 5(x^2 - 2x) - y(x^2 - 2x) + 5y$
$$= ((x^2 - 2x)^2 - 5(x^2 - 2x)) - (y(x^2 - 2x) - 5y)$$
$$= (x^2 - 2x)(x^2 - 2x - 5) - y(x^2 - 2x - 5)$$
$$= (x^2 - 2x - 5)(x^2 - 2x - y).$$

Example 8. *Factorise :* $a^2 + b^2 - 2(ab - ac + bc).$

Solution. $a^2 + b^2 - 2(ab - ac + bc) = (a^2 + b^2 - 2ab) - 2(-ac + bc)$
$$= (a - b)^2 - 2(-c)(a - b)$$
$$= (a - b)[(a - b) + 2c]$$
$$= (a - b)(a - b + 2c).$$

Exercise 5.2

Factorise the following (1 to 13) :

1. (i) $x^2 + xy - x - y$ (ii) $y^2 - yz - 5y + 5z.$
2. (i) $5xy + 7y - 5y^2 - 7x$ (ii) $5p^2 - 8pq - 10p + 16q.$
3. (i) $a^2b - ab^2 + 3a - 3b$ (ii) $x^3 - 3x^2 + x - 3.$

4. (i) $6xy^2 - 3xy - 10y + 5$ (ii) $3ax - 6ay - 8by + 4bx$.
5. (i) $1 - a - b + ab$ (ii) $a(a - 2b - c) + 2bc$.
6. (i) $x^2 + xy(1 + y) + y^3$ (ii) $y^2 - xy(1 - x) - x^3$.
7. (i) $ab^2 + (a - 1)b - 1$ (ii) $2a - 4b - xa + 2bx$.
8. (i) $5ph - 10qk + 2rph - 4qrk$ (ii) $x^2 - x(a + 2b) + 2ab$.
9. (i) $ab(x^2 + y^2) - xy(a^2 + b^2)$ (ii) $(ax + by)^2 + (bx - ay)^2$.
10. (i) $a^3 + ab(1 - 2a) - 2b^2$ (ii) $3x^2y - 3xy + 12x - 12$.

Hint

(ii) 3 is a common factor. First take 3 outside.

11. $a^2b + ab^2 - abc - b^2c + axy + bxy$.
12. $ax^2 - bx^2 + ay^2 - by^2 + az^2 - bz^2$.
13. $x - 1 - (x - 1)^2 + ax - a$.

5.4 DIFFERENCE OF TWO SQUARES

We shall use the identity $a^2 - b^2 = (a + b)(a - b)$.

ILLUSTRATIVE EXAMPLES

Example 1. *Factorise the following :*

(i) $4x^2 - 169y^2$ (ii) $1 - (b - c)^2$

(iii) $x^2 - 2y + xy - 4$ (iv) $a(a - 3) - b(b - 3)$.

Solution. (i)
$$4x^2 - 169y^2 = (2x)^2 - (13y)^2 = (2x + 13y)(2x - 13y).$$

(ii)
$$1 - (b - c)^2 = (1)^2 - (b - c)^2 = (1 + \overline{b-c})(1 - \overline{b-c}) = (1 + b - c)(1 - b + c).$$

(iii)
$$x^2 - 2y + xy - 4 = (x^2 - 4) + (xy - 2y) = (x + 2)(x - 2) + y(x - 2) = (x - 2)(\overline{x + 2} + y) = (x - 2)(x + y + 2).$$

(iv)
$$a(a - 3) - b(b - 3) = a^2 - 3a - b^2 + 3b = (a^2 - b^2) - 3a + 3b = (a - b)(a + b) - 3(a - b) = (a - b)(a + b - 3).$$

Example 2. *Factorise the following :*

(i) $16y^3 - 4y$ (ii) $9x^2 - 4a^2 + 4ay - y^2$ (iii) $x^3 - 3x^2 - x + 3$.

Solution. (i)
$$16y^3 - 4y = 4y(4y^2 - 1) = 4y[(2y)^2 - (1)^2] = 4y(2y + 1)(2y - 1).$$

(*ii*) $9x^2 - 4a^2 + 4ay - y^2 = 9x^2 - (4a^2 - 4ay + y^2)$

$= (3x)^2 - (2a - y)^2$

$= (3x + \overline{2a - y})(3x - \overline{2a - y})$

$= (3x - y + 2a)(3x + y - 2a)$.

(*iii*) $x^3 - 3x^2 - x + 3 = (x^3 - 3x^2) + (-x + 3)$

$= x^2(x - 3) - 1(x - 3)$

$= (x - 3)(x^2 - 1)$

$= (x - 3)(x^2 - 1^2)$

$= (x - 3)(x + 1)(x - 1)$.

Example 3. *Factorise the following :*

(*i*) $3x^5 - 48x$ (*ii*) $2(ab + cd) - a^2 - b^2 + c^2 + d^2$

(*iii*) $(1 - x^2)(1 - y^2) + 4xy$ (*iv*) $x^4 + y^4 - 11x^2 y^2$.

Solution. (*i*) $3x^5 - 48x = 3x(x^4 - 16) = 3x[(x^2)^2 - (4)^2]$

$= 3x(x^2 + 4)(x^2 - 4)$

$= 3x(x^2 + 4)(x + 2)(x - 2)$.

(*ii*) $2(ab + cd) - a^2 - b^2 + c^2 + d^2 = 2ab + 2cd - a^2 - b^2 + c^2 + d^2$

$= (c^2 + 2cd + d^2) - (a^2 - 2ab + b^2)$

$= (c + d)^2 - (a - b)^2$

$= (\overline{c + d} + \overline{a - b})(\overline{c + d} - \overline{a - b})$

$= (c + d + a - b)(c + d - a + b)$.

(*iii*) $(1 - x^2)(1 - y^2) + 4xy = 1 - x^2 - y^2 + x^2y^2 + 4xy$

$= x^2y^2 + 1 + 2xy - x^2 - y^2 + 2xy$

$= (x^2y^2 + 2xy + 1) - (x^2 - 2xy + y^2)$

$= (xy + 1)^2 - (x - y)^2$

$= (\overline{xy + 1} + \overline{x - y})(\overline{xy + 1} - \overline{x - y})$

$= (xy + x - y + 1)(xy - x + y + 1)$.

(*iv*) $x^4 + y^4 - 11x^2y^2 = (x^4 + y^4 - 2x^2y^2) - 9x^2y^2$

$= (x^2 - y^2)^2 - (3xy)^2$

$= (x^2 - y^2 + 3xy)(x^2 - y^2 - 3xy)$.

Example 4. *Factorise the following :*

(*i*) $x^4 + 4$ (*ii*) $x^4 + x^2 + 1$ (*iii*) $x^4 + x^2y^2 + y^4$.

Solution. (*i*) $x^4 + 4 = x^4 + 4x^2 + 4 - 4x^2$ (Adding and subtracting $4x^2$)

$= (x^2 + 2)^2 - (2x)^2$

$= (x^2 + 2 + 2x)(x^2 + 2 - 2x)$

$= (x^2 + 2x + 2)(x^2 - 2x + 2)$.

(*ii*) $x^4 + x^2 + 1 = x^4 + 2x^2 + 1 - x^2$ (Adding and subtracting x^2)

$= (x^2 + 1)^2 - x^2$

$= (x^2 + 1 + x)(x^2 + 1 - x)$

$= (x^2 + x + 1)(x^2 - x + 1).$

(*iii*) $x^4 + x^2y^2 + y^4 = x^4 + 2x^2y^2 + y^4 - x^2y^2$ (Adding and subtracting x^2y^2)

$= (x^2 + y^2)^2 - (xy)^2$

$= (x^2 + y^2 + xy)(x^2 + y^2 - xy).$

Example 5. *Factorise completely* $(x^2 + y^2 - z^2)^2 - 4x^2y^2$.

Solution. $(x^2 + y^2 - z^2)^2 - 4x^2y^2 = (x^2 + y^2 - z^2)^2 - (2xy)^2$

$= (x^2 + y^2 - z^2 + 2xy)(x^2 + y^2 - z^2 - 2xy)$

$= \left(\overline{x^2 + 2xy + y^2} - z^2\right)\left(\overline{x^2 - 2xy + y^2} - z^2\right)$

$= [(x + y)^2 - z^2][(x - y)^2 - z^2]$

$= (x + y + z)(x + y - z)(x - y + z)(x - y - z).$

Example 6. *Express* $(x^2 - 4x + 9)(x^2 + 4x - 9)$ *as a difference of two squares.*

Solution. $(x^2 - 4x + 9)(x^2 + 4x - 9) = \left(x^2 - \overline{4x - 9}\right)\left(x^2 + \overline{4x - 9}\right)$

[Expressing as $(a - b)(a + b)$]

$= (x^2)^2 - (4x - 9)^2.$

Example 7. *If* $x > 1$ *and* $x + \frac{1}{x} = 2\frac{1}{12}$, *find the values of*

(*i*) $x^2 + \frac{1}{x^2}$ (*ii*) $x - \frac{1}{x}$ (*iii*) $x^3 - \frac{1}{x^3}$ (*iv*) $x^4 - \frac{1}{x^4}$.

Solution. (*i*) Given $x + \frac{1}{x} = 2\frac{1}{12} = \frac{25}{12}$

$\Rightarrow \left(x + \frac{1}{x}\right)^2 = \left(\frac{25}{12}\right)^2$

$\Rightarrow x^2 + \frac{1}{x^2} + 2 = \frac{625}{144}$

$\Rightarrow x^2 + \frac{1}{x^2} = \frac{625}{144} - 2 = \frac{337}{144}.$

(*ii*) $\left(x - \frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} - 2 = \frac{337}{144} - 2 = \frac{49}{144}$

$\Rightarrow x - \frac{1}{x} = \pm\frac{7}{12}.$

But $x > 1 \Rightarrow x - \frac{1}{x} > 0,$

$\therefore x - \frac{1}{x} = \frac{7}{12}.$

(iii) $\left(x-\frac{1}{x}\right)^3 = x^3 - \frac{1}{x^3} - 3 \cdot x \cdot \frac{1}{x}\left(x-\frac{1}{x}\right)$

$\Rightarrow \left(\frac{7}{12}\right)^3 = x^3 - \frac{1}{x^3} - 3 \cdot \frac{7}{12}$

$\Rightarrow x^3 - \frac{1}{x^3} = \frac{343}{1728} + \frac{7}{4} = \frac{3367}{1728}.$

(iv) $x^4 - \frac{1}{x^4} = \left(x^2 - \frac{1}{x^2}\right)\left(x^2 + \frac{1}{x^2}\right)$

$= \left(x-\frac{1}{x}\right)\left(x+\frac{1}{x}\right)\left(x^2+\frac{1}{x^2}\right)$

$= \frac{7}{12} \times \frac{25}{12} \times \frac{337}{144} = \frac{58975}{20736}.$

Exercise 5.3

Factorise the following (1 to 17) :

1. (i) $4x^2 - 25y^2$ (ii) $9x^2 - 1$.
2. (i) $150 - 6a^2$ (ii) $32x^2 - 18y^2$.
3. (i) $(x - y)^2 - 9$ (ii) $9(x + y)^2 - x^2$.
4. (i) $20x^2 - 45y^2$ (ii) $9x^2 - 4(y + 2x)^2$.
5. (i) $2(x - 2y)^2 - 50y^2$ (ii) $32 - 2(x - 4)^2$.
6. (i) $108a^2 - 3(b - c)^2$ (ii) $\pi a^5 - \pi^3 ab^2$.
7. (i) $50x^2 - 2(x - 2)^2$ (ii) $(x - 2)(x + 2) + 3$.

Hint

(ii) $(x - 2)(x + 2) + 3 = (x^2 - 4) + 3 = x^2 - 1$.

8. (i) $x - 2y - x^2 + 4y^2$ (ii) $4a^2 - b^2 + 2a + b$.
9. (i) $a(a - 2) - b(b - 2)$ (ii) $a(a - 1) - b(b - 1)$.
10. (i) $9 - x^2 + 2xy - y^2$ (ii) $9x^4 - (x^2 + 2x + 1)$.
11. (i) $9x^4 - x^2 - 12x - 36$ (ii) $x^3 - 5x^2 - x + 5$.
12. (i) $a^4 - b^4 + 2b^2 - 1$ (ii) $x^3 - 25x$.
13. (i) $2x^4 - 32$ (ii) $a^2(b + c) - (b + c)^3$.
14. (i) $(a + b)^3 - a - b$ (ii) $x^2 - 2xy + y^2 - a^2 - 2ab - b^2$.
15. (i) $(a^2 - b^2)(c^2 - d^2) - 4abcd$ (ii) $4x^2 - y^2 - 3xy + 2x - 2y$.

Hint

(i) Given expression $= (ac - bd)^2 - (bc + ad)^2$.

(ii) Given expression $= (x^2 - y^2) + (3x^2 - 3xy) + (2x - 2y)$.

16. (i) $x^2 + \frac{1}{x^2} - 11$ (ii) $x^4 + 5x^2 + 9$.
17. (i) $a^4 + b^4 - 7a^2b^2$ (ii) $x^4 - 14x^2 + 1$.

18. Express each of the following as the difference of two squares :

(*i*) $(x^2 - 5x + 7)(x^2 + 5x + 7)$ (*ii*) $(x^2 - 5x + 7)(x^2 - 5x - 7)$

(*iii*) $(x^2 + 5x - 7)(x^2 - 5x + 7)$.

19. Evaluate the following by using factors :

(*i*) $(979)^2 - (21)^2$ (*ii*) $(99{\cdot}9)^2 - (0{\cdot}1)^2$.

20. If $x > 0$ and $x - \frac{1}{x} = \frac{5}{6}$, find the values of

(*i*) $x^2 + \frac{1}{x^2}$ (*ii*) $x + \frac{1}{x}$

(*iii*) $x^3 + \frac{1}{x^3}$ (*iv*) $x^4 - \frac{1}{x^4}$.

21. If $x + \frac{1}{x} = \sqrt{3}$, find the values of

(*i*) $x^3 + \frac{1}{x^3}$ (*ii*) $x^6 - \frac{1}{x^6}$.

5.5 TRINOMIALS

When a trinomial is of the form $ax^2 + bx + c$, split b (the coefficient of x) into two parts such that the *algebraic sum* of these two parts is b and their product is ac, then factorise by grouping method.

Remark

It is not always possible to factorise a trinomial $ax^2 + bx + c$ (*i.e.* a quadratic expression); the following rule can save lot of time :

For the expression $ax^2 + bx + c$, work out $b^2 - 4ac$. If it is a perfect square, then the given expression will factorise; otherwise, not.

ILLUSTRATIVE EXAMPLES

Example 1. *Factorise the following :*

(*i*) $x^2 + 7x + 12$ (*ii*) $x^2 - 5x - 6$

(*iii*) $3x^2 + 14x + 8$ (*iv*) $1 - 18y - 63y^2$.

Solution. (*i*) Since $3 + 4 = 7$ and $3.4 = 12$,

$$\therefore \quad x^2 + 7x + 12 = x^2 + 3x + 4x + 12 = x(x + 3) + 4(x + 3) = (x + 3)(x + 4).$$

(*ii*) Since $-6 + 1 = -5$ and $(-6).1 = -6$,

$$\therefore \quad x^2 - 5x - 6 = x^2 - 6x + x - 6 = x(x - 6) + 1(x - 6) = (x - 6)(x + 1).$$

(*iii*) Since $12 + 2 = 14$ and $12.2 = 24$,

$$\therefore \quad 3x^2 + 14x + 8 = 3x^2 + 12x + 2x + 8 = 3x(x + 4) + 2(x + 4) = (x + 4)(3x + 2).$$

(*iv*) Since $-21 + 3 = -18$ and $(-21).3 = -63$,

$\therefore \quad 1 - 18y - 63y^2 = 1 - 21y + 3y - 63y^2$

$= 1(1 - 21y) + 3y(1 - 21y)$

$= (1 - 21y)(1 + 3y).$

Example 2. *Factorise the following :*

(*i*) $7\sqrt{2}x^2 - 10x - 4\sqrt{2}$ (*ii*) $x^2 + \frac{1}{4}x - \frac{1}{8}$.

Solution. (*i*) Since $-14 + 4 = 10$ and $(-14).4 = 7\sqrt{2}.(-4\sqrt{2})$,

$\therefore \quad 7\sqrt{2}x^2 - 10x - 4\sqrt{2} = 7\sqrt{2}x^2 - 14x + 4x - 4\sqrt{2}$

$= 7\sqrt{2}x(x - \sqrt{2}) + 4(x - \sqrt{2})$

$= (x - \sqrt{2})(7\sqrt{2}x + 4).$

(*ii*) Since $\frac{1}{2} - \frac{1}{4} = \frac{1}{4}$ and $\frac{1}{2}\left(-\frac{1}{4}\right) = -\frac{1}{8}$,

$\therefore \quad x^2 + \frac{1}{4}x - \frac{1}{8} = x^2 + \frac{1}{2}x - \frac{1}{4}x - \frac{1}{8}$

$= x\left(x + \frac{1}{2}\right) - \frac{1}{4}\left(x + \frac{1}{2}\right)$

$= \left(x + \frac{1}{2}\right)\left(x - \frac{1}{4}\right).$

Example 3. *Factorise the following :*

(*i*) $3x^2 - 5xy - 12y^2$ (*ii*) $2x^3 + 5x^2y - 12xy^2$

(*iii*) $8(a - 2b)^2 - 2a + 4b - 1$ (*iv*) $9x^2 - (x^2 - 4)^2$.

Solution. (*i*) Since $-9 + 4 = -5$ and $(-9).4 = -36$,

$\therefore \quad 3x^2 - 5xy - 12y^2 = 3x^2 - 9xy + 4xy - 12y^2$

$= 3x(x - 3y) + 4y(x - 3y)$

$= (x - 3y)(3x + 4y).$

(*ii*) $2x^3 + 5x^2y - 12xy^2 = x(2x^2 + 5xy - 12y^2)$

$= x(2x^2 + 8xy - 3xy - 12y^2)$

[$\because 8 + (-3) = 5$ and $8.(-3) = -24$]

$= x[2x(x + 4y) - 3y(x + 4y)]$

$= x(x + 4y)(2x - 3y).$

(*iii*) $8(a - 2b)^2 - 2a + 4b - 1 = 8(a - 2b)^2 - 2(a - 2b) - 1$

$= 8x^2 - 2x - 1$ where $x = a - 2b$

$= 8x^2 - 4x + 2x - 1$

[$\because -4 + 2 = -2$ and $(-4).2 = -8$]

$= 4x(2x - 1) + 1(2x - 1)$

$= (2x - 1)(4x + 1)$

$= (2.\overline{a - 2b} - 1)(4.\overline{a - 2b} + 1)$

[replacing back the value of x]

$= (2a - 4b - 1)(4a - 8b + 1).$

(iv) $9x^2 - (x^2 - 4)^2 = (3x)^2 - (x^2 - 4)^2$

$= (3x + \overline{x^2 - 4}\,)\,(3x - \overline{x^2 - 4}\,)$

$= (x^2 + 3x - 4)\,(4 + 3x - x^2)$

$= (x^2 + 4x - x - 4)\,(4 + 4x - x - x^2)$

$= [x\,(x + 4) - 1\,(x + 4)]\,[4(1 + x) - x\,(1 + x)]$

$= (x + 4)\,(x - 1)\,(1 + x)\,(4 - x).$

Example 4. *Factorise :* $(x^2 - 3x)^2 - 8(x^2 - 3x) - 20.$

Solution. Let $x^2 - 3x = y$, then

$(x^2 - 3x)^2 - 8(x^2 - 3x) - 20 = y^2 - 8y - 20$

$= y^2 - 10y + 2y - 20$

$= y(y - 10) + 2(y - 10)$

$= (y - 10)\,(y + 2)$

$= (x^2 - 3x - 10)\,(x^2 - 3x + 2)$

$= (x^2 - 5x + 2x - 10)\,(x^2 - 2x - x + 2)$

$= [x(x - 5) + 2(x - 5)]\,[x(x - 2) - 1(x - 2)]$

$= (x - 5)\,(x + 2)\,(x - 2)\,(x - 1).$

Example 5. *Factorise :* $5 - (3x^2 - 2x)\,(6 - 3x^2 + 2x).$

Solution. $5 - (3x^2 - 2x)\,(6 - 3x^2 + 2x) = 5 - (3x^2 - 2x)\,(6 - \overline{3x^2 - 2x}\,)$

$= 5 - y\,(6 - y)$ where $y = 3x^2 - 2x$

$= 5 - 6y + y^2 = 5 - 5y - y + y^2$

$= 5\,(1 - y) - y\,(1 - y) = (1 - y)\,(5 - y)$

$= (1 - 3x^2 + 2x)\,(5 - 3x^2 + 2x)$

$= (1 + 3x - x - 3x^2)\,(5 + 5x - 3x - 3x^2)$

$= [1\,(1 + 3x) - x\,(1 + 3x)]\,[5\,(1 + x) - 3x\,(1 + x)]$

$= (1 + 3x)\,(1 - x)\,(1 + x)\,(5 - 3x).$

Example 6. *Factorise the following :*

(i) $x^4 - 14x^2y^2 - 51y^4$ *(ii)* $(x^2 + x)^2 + 4(x^2 + x) - 12.$

Solution. (i) Since $-17 + 3 = -14$ and $(-17)\,.\,3 = -51$,

$\therefore$ $x^4 - 14x^2y^2 - 51y^2 = x^4 - 17x^2y^2 + 3x^2y^2 - 51y^4$

$= x^2(x^2 - 17y^2) + 3y^2(x^2 - 17y^2)$

$= (x^2 - 17y^2)\,(x^2 + 3y^2)$

$= (x^2 - (\sqrt{17}\,y)^2)\,(x^2 + 3y^2)$

$= (x - \sqrt{17}\,y)\,(x + \sqrt{17}\,y)\,(x^2 + 3y^2).$

(ii) $(x^2 + x)^2 + 4(x^2 + x) - 12 = y^2 + 4y - 12$ where $y = x^2 + x$

$= y^2 + 6y - 2y - 12$

$= y(y + 6) - 2(y + 6)$

$= (y + 6)\,(y - 2)$

$= (x^2 + x + 6)\,(x^2 + x - 2)$

$= (x^2 + x + 6)(x^2 + 2x - x - 2)$

$= (x^2 + x + 6)[x(x + 2) - 1(x + 2)]$

$= (x^2 + x + 6)(x + 2)(x - 1).$

Now compare $x^2 + x + 6$ with $ax^2 + bx + c$.

Here $a = 1$, $b = 1$ and $c = 6$.

$\therefore$ $b^2 - 4ac = 1^2 - 4 . 1 . 6 = -23$, which is not a perfect square.

Therefore, $x^2 + x + 6$ cannot be factorised. (See remark)

Hence $(x^2 + x)^2 + 4(x^2 + x) - 12 = (x^2 + x + 6)(x + 2)(x - 1)$.

Exercise 5.4

Factorise the following (1 to 18) :

1. (*i*) $x^2 + 5x + 6$ (*ii*) $x^2 - 8x + 7$.
2. (*i*) $x^2 + 6x - 7$ (*ii*) $y^2 + 7y - 18$.
3. (*i*) $y^2 - 7y - 18$ (*ii*) $a^2 - 3a - 54$.
4. (*i*) $2x^2 - 7x + 6$ (*ii*) $6x^2 + 13x - 5$.
5. (*i*) $6x^2 + 11x - 10$ (*ii*) $6x^2 - 7x - 3$.
6. (*i*) $2x^2 - x - 6$ (*ii*) $3x^2 - 4x - 7$.
7. (*i*) $2y^2 + y - 45$ (*ii*) $5 - 4x - 12x^2$.
8. (*i*) $x(12x + 7) - 10$ (*ii*) $(4 - x)^2 - 2x$.

Hint

(*ii*) $(4 - x)^2 - 2x = x^2 - 10x + 16$.

9. (*i*) $60x^2 - 70x - 30$ (*ii*) $x^2 - 6xy - 7y^2$.
10. (*i*) $2x^2 + 13xy - 24y^2$ (*ii*) $6x^2 - 5xy - 6y^2$.
11. (*i*) $5x^2 + 17xy - 12y^2$ (*ii*) $x^2y^2 - 8xy - 48$.
12. (*i*) $2a^2b^2 - 7ab - 30$ (*ii*) $a(2a - b) - b^2$.
13. (*i*) $(x - y)^2 - 6(x - y) + 5$ (*ii*) $(2x - y)^2 - 11(2x - y) + 28$.
14. (*i*) $4(a - 1)^2 - 4(a - 1) - 3$ (*ii*) $1 - 2a - 2b - 3(a + b)^2$.
15. (*i*) $3 - 5a - 5b - 12(a + b)^2$ (*ii*) $a^4 - 11a^2 + 10$.
16. (*i*) $(x + 4)^2 - 5xy - 20y - 6y^2$ (*ii*) $(x^2 - 2x)^2 - 23(x^2 - 2x) + 120$.

Hint

(*i*) $(x + 4)^2 - 5xy - 20y - 6y^2 = (x + 4)^2 - 5y(x + 4) - 6y^2$

$= z^2 - 5yz - 6y^2$ where $z = x + 4$

$= (z + y)(z - 6y)$.

17. $4(2a - 3)^2 - 3(2a - 3)(a - 1) - 7(a - 1)^2$.

Hint

Let $2a - 3 = x$ and $a - 1 = y$,

then given expression $= 4x^2 - 3xy - 7y^2 = (x + y)(4x - 7y)$.

18. $(2x^2 + 5x)(2x^2 + 5x - 19) + 84$

Hint

Let $2x^2 + 5x = y$, then

$(2x^2 + 5x)(2x^2 + 5x - 19) + 84 = y(y - 19) + 84$

$= y^2 - 19y + 84 = (y - 7)(y - 12).$

5.6 SUM OR DIFFERENCE OF TWO CUBES

We shall use the following identities :

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2),$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2).$$

ILLUSTRATIVE EXAMPLES

Example 1. *Resolve the following into factors :*

(i) $8x^3 + 125y^3$ *(ii)* $27x^3 - \frac{343}{x^3}$ *(iii)* $27x^4 - 8x.$

Solution. (*i*) $8x^3 + 125y^3 = (2x)^3 + (5y)^3$

$= (2x + 5y)[(2x)^2 - 2x.5y + (5y)^2]$

$= (2x + 5y)(4x^2 - 10xy + 25y^2).$

(*ii*) $27x^3 - \frac{343}{x^3} = (3x)^3 - \left(\frac{7}{x}\right)^3$

$= \left(3x - \frac{7}{x}\right)\left[(3x)^2 + 3x.\frac{7}{x} + \left(\frac{7}{x}\right)^2\right]$

$= \left(3x - \frac{7}{x}\right)\left(9x^2 + \frac{49}{x^2} + 21\right).$

(*iii*) $27x^4 - 8x = x(27x^3 - 8) = x[(3x)^3 - (2)^3]$

$= x(3x - 2)[(3x)^2 + 3x.2 + (2)^2]$

$= x(3x - 2)(9x^2 + 6x + 4).$

Example 2. *Factorise the following :*

(i) $64 - a^3b^3 + 8 - 2ab$ *(ii)* $64a^6 - b^6.$

Solution. (*i*) $64 - a^3b^3 + 8 - 2ab = [(4)^3 - (ab)^3] + 2(4 - ab)$

$= (4 - ab)(16 + 4.ab + a^2b^2) + 2(4 - ab)$

$= (4 - ab)(16 + 4ab + a^2b^2 + 2)$

$= (4 - ab)(18 + 4ab + a^2b^2).$

(*ii*) $64a^6 - b^6 = (8a^3)^2 - (b^3)^2$

$= (8a^3 + b^3)(8a^3 - b^3)$

$= [(2a)^3 + b^3][(2a)^3 - b^3]$

$= (2a + b)(4a^2 - 2a.b + b^2)(2a - b)(4a^2 + 2a.b + b^2)$

$= (2a + b)(2a - b)(4a^2 - 2ab + b^2)(4a^2 + 2ab + b^2).$

Example 3. *Factorise the following :*

(i) $a^7 - ab^6$ (ii) $27(x + y)^3 - 8(x - y)^3$.

Solution. (i) $a^7 - ab^6 = a(a^6 - b^6) = a((a^3)^2 - (b^3)^2)$

$= a(a^3 + b^3)(a^3 - b^3)$

$= a(a + b)(a^2 - ab + b^2)(a - b)(a^2 + ab + b^2)$.

(ii) $27(x + y)^3 - 8(x - y)^3 = (3(x + y))^3 - (2(x - y))^3$

$= (3(x + y) - 2(x - y))[(3(x + y))^2 + 3(x + y).2(x - y) + (2(x - y))^2]$

$= (x + 5y)[9(x^2 + 2xy + y^2) + 6(x^2 - y^2) + 4(x^2 - 2xy + y^2)]$

$= (x + 5y)(19x^2 + 10xy + 7y^2)$

Example 4. *Factorise the following :*

(i) $x^3p^2 - 8y^3p^2 - 4x^3q^2 + 32y^3q^2$ (ii) $x^3 + 3x^2y + 3xy^2 + 2y^3$.

Solution. (i) $x^3p^2 - 8y^3p^2 - 4x^3q^2 + 32y^3q^2 = p^2(x^3 - 8y^3) - 4q^2(x^3 - 8y^3)$

$= (x^3 - 8y^3)(p^2 - 4q^2)$

$= [x^3 - (2y)^3][p^2 - (2q)^2]$

$= (x - 2y)(x^2 + 2xy + 4y^2)(p + 2q)(p - 2q)$.

(ii) $x^3 + 3x^2y + 3xy^2 + 2y^3 = (x^3 + 3x^2y + 3xy^2 + y^3) + y^3$ (Note this step)

$= (x + y)^3 + y^3$

$= p^3 + y^3$, where $p = x + y$

$= (p + y)(p^2 - py + y^2)$

$= (\overline{x + y} + y)[(x + y)^2 - (x + y)y + y^2]$

$= (x + 2y)(x^2 + 2xy + y^2 - xy - y^2 + y^2)$

$= (x + 2y)(x^2 + xy + y^2)$.

Example 5. *Factorise the following :*

(i) $x^3 + 3x^2 + 3x - 7$ (ii) $x^3 - 3x^2 + 3x + 7$.

Solution. (i) $x^3 + 3x^2 + 3x - 7 = (x^3 + 3x^2 + 3x + 1) - 8$ (Note this step)

$= (x + 1)^3 - (2)^3$

$= \{(x + 1) - 2\}\{(x + 1)^2 + 2(x + 1) + 2^2\}$

$= (x - 1)(x^2 + 2x + 1 + 2x + 2 + 4)$

$= (x - 1)(x^2 + 4x + 7)$.

(ii) $x^3 - 3x^2 + 3x + 7 = (x^3 - 3x^2 + 3x - 1) + 8$ (Note this step)

$= (x^3 - 3x^2 + 3x - 1) + 8$

$= (x - 1)^3 + (2)^3$

$= \{(x - 1) + 2\}\{(x - 1)^2 - 2(x - 1) + 2^2\}$

$= (x + 1)(x^2 - 2x + 1 - 2x + 2 + 4)$

$= (x + 1)(x^2 - 4x + 7)$.

Example 6. *Factorise :* $x^6 - 26x^3 - 27$.

Solution. $x^6 - 26x^3 - 27 = y^2 - 26y - 27$ where $y = x^3$

$= y^2 - 27y + y - 27$

$= y(y - 27) + 1(y - 27)$
$= (y - 27)(y + 1)$
$= (x^3 - 27)(x^3 + 1)$
$= [x^3 - (3)^3][x^3 + 1^3]$
$= (x - 3)(x^2 + 3x + 9)(x + 1)(x^2 - x + 1).$

Exercise 5.5

Factorise the following (1 to 13) :

1. (i) $8x^3 + y^3$ (ii) $64x^3 - 125y^3$.
2. (i) $64x^3 + 1$ (ii) $7a^3 + 56b^3$.
3. (i) $\frac{x^6}{343} + \frac{343}{x^6}$ (ii) $8x^3 - \frac{1}{27y^3}$.
4. (i) $x^2 + x^5$ (ii) $32x^4 - 500x$.
5. (i) $27x^3y^3 - 8$ (ii) $27(x + y)^3 + 8(2x - y)^3$.

Hint

(ii) Given expression $= [3(x + y) + 2(2x - y)][9(x + y)^2 - 3(x + y).2(2x - y) + 4(2x - y)^2)$.

6. (i) $a^3 + b^3 + a + b$ (ii) $a^3 - b^3 - a + b$.
7. (i) $x^3 + x + 2$ (ii) $a^3 - a - 120$.

Hint

(i) $x^3 + x + 2 = (x^3 + 1) + (x + 1)$
(ii) $a^3 - a - 120 = (a^3 - 125) - (a - 5)$.

8. (i) $x^3 + 6x^2 + 12x + 16$ (ii) $a^3 - 3a^2b + 3ab^2 - 2b^3$.
9. (i) $2a^3 + 16b^3 - 5a - 10b$ (ii) $a^3 - \frac{1}{a^3} - 2a + \frac{2}{a}$.
10. (i) $a^6 - b^6$ (ii) $x^6 - 1$.
11. (i) $64x^6 - 729y^6$ (ii) $x^2 - \frac{8}{x}$.
12. (i) $250(a - b)^3 + 2$ (ii) $32a^2x^3 - 8b^2x^3 - 4a^2y^3 + b^2y^3$.
13. (i) $x^9 + y^9$ (ii) $x^6 - 7x^3 - 8$.

CHAPTER TEST

Factorize the following (1 to 12) :

1. (*i*) $15(2x-3)^3 - 10(2x-3)$ (*ii*) $a(b-c)(b+c) - d(c-b)$.
2. (*i*) $2a^2x - bx + 2a^2 - b$ (*ii*) $p^2 - (a+2b)p + 2ab$.
3. (*i*) $(x^2-y^2)z + (y^2-z^2)x$ (*ii*) $5a^4 - 5a^3 + 30a^2 - 30a$.
4. (*i*) $b(c-d)^2 + a(d-c) + 3c - 3d$ (*ii*) $x^3 - x^2 - xy + x + y - 1$.
5. (*i*) $x(x+z) - y(y+z)$ (*ii*) $a^{12}x^4 - a^4x^{12}$.
6. (*i*) $9x^2 + 12x + 4 - 16y^2$ (*ii*) $x^4 + 3x^2 + 4$.

Hint

(*ii*) $x^4 + 3x^2 + 4 = (x^2+2)^2 - x^2$.

7. (*i*) $21x^2 - 59xy + 40y^2$ (*ii*) $4x^3y - 44x^2y + 112xy$.
8. (*i*) $x^2y^2 - xy - 72$ (*ii*) $9x^3y + 41x^2y + 20xy^3$.
9. (*i*) $(3a-2b)^2 + 3(3a-2b) - 10$ (*ii*) $(x^2-3x)(x^2-3x+7) + 10$.

Hint

(*ii*) Given expression $= y(y+7) + 10$ where $y = x^2 - 3x$
$= y^2 + 7y + 10 = (y+5)(y+2)$.

10. (*i*) $(x^2-x)(4x^2-4x-5) - 6$ (*ii*) $x^4 + 9x^2y^2 + 81y^4$.

Hint

(*ii*) $x^4 + 9x^2y^2 + 81y^4 = (x^4 + 18x^2y^2 + 81y^4) - 9x^2y^2$
$= (x^2+9y^2)^2 - (3xy)^2$.

11. (*i*) $\frac{8}{27}x^3 - \frac{1}{8}y^3$ (*ii*) $x^6 + 63x^3 - 64$.
12. (*i*) $x^3 + x^2 - \frac{1}{x^2} + \frac{1}{x^3}$ (*ii*) $(x+1)^6 - (x-1)^6$.

Hint

(*ii*) Let $x + 1 = a$, $x - 1 = b$.
Given expression $= a^6 - b^6 = (a^3-b^3)(a^3+b^3)$
$= (a-b)(a^2+ab+b^2)(a+b)(a^2-ab+b^2)$.

13. Show that $97^3 + 14^3$ is divisible by 111.
14. If $a + b = 8$ and $ab = 15$, find the value of $a^4 + a^2b^2 + b^4$.

Hint

$a^4 + a^2b^2 + b^4 = (a^2+b^2)^2 - a^2b^2$.