4 EXPANSIONS

4.1 SPECIAL PRODUCTS

The following products occur frequently in algebra, and the students are advised to memorize them (without resorting to actual multiplication) until they can recognise each, both the product from the factors and the factors from the product.

1. $(a + b)^2 = a^2 + 2ab + b^2$.
2. $(a - b)^2 = a^2 - 2ab + b^2$.
3. $(a + b)(a - b) = a^2 - b^2$.
4. $\left(a + \frac{1}{a}\right)^2 = a^2 + \frac{1}{a^2} + 2$.
5. $\left(a - \frac{1}{a}\right)^2 = a^2 + \frac{1}{a^2} - 2$.
6. $\left(a + \frac{1}{a}\right)\left(a - \frac{1}{a}\right) = a^2 - \frac{1}{a^2}$.
7. (i) $(x + a)(x + b) = x^2 + (a + b)x + ab$.
   (ii) $(x + a)(x - b) = x^2 + (a - b)x - ab$.
   (iii) $(x - a)(x + b) = x^2 - (a - b)x - ab$.
   (iv) $(x - a)(x - b) = x^2 - (a + b)x + ab$.
8. $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$.
9. $(a + b)^3 = a^3 + b^3 + 3ab(a + b) = a^3 + 3a^2b + 3ab^2 + b^3$.
10. $(a - b)^3 = a^3 - b^3 - 3ab(a - b) = a^3 - 3a^2b + 3ab^2 - b^3$.
11. $(a + b)(a^2 - ab + b^2) = a^3 + b^3$.
12. $(a - b)(a^2 + ab + b^2) = a^3 - b^3$.
13. $(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) = a^3 + b^3 + c^3 - 3abc$.
14. $(x + a)(x + b)(x + c) = x^3 + (a + b + c)x^2 + (ab + bc + ca)x + abc$.

Remark

- The expression $a^2 + 2ab + b^2$ is called the **expansion** of $(a + b)^2$, $a^2 - 2ab + b^2$ is the expansion of $(a - b)^2$ and $a^3 + b^3 + 3ab(a + b)$ is the expansion of $(a + b)^3$ and so on.
- The above results 1 to 3 and 7 to 14 are true for all values of the variables involved, and the results 4 to 6 are true for all values of a except 0. *An equation which is true for all values of the variables involved is called an* ***identity.*** Thus, the above results 1 to 14 are all identities; of course, $a \neq 0$ in the results 4 to 6.

4.2 APPLICATIONS OF THE SPECIAL PRODUCTS

We learn the applications of the special products with the help of the following examples:

ILLUSTRATIVE EXAMPLES

Example 1. *Find the expansions of the following :*

(i) $(2a + 3b)^2$ (ii) $(3x - 4y)^2$ (iii) $(2a + 3b - 4c)^2$.

Solution. (i) $(2a + 3b)^2 = (2a)^2 + 2(2a)(3b) + (3b)^2$

$= 4a^2 + 12ab + 9b^2$.

(ii) $(3x - 4y)^2 = (3x)^2 - 2(3x)(4y) + (4y)^2$

$= 9x^2 - 24xy + 16y^2$.

(iii) $(2a + 3b - 4c)^2 = [2a + 3b + (-4c)]^2$

$= (2a)^2 + (3b)^2 + (-4c)^2 + 2[(2a)(3b) + (3b)(-4c) + (-4c)(2a)]$

$= 4a^2 + 9b^2 + 16c^2 + 2(6ab - 12bc - 8ca)$

$= 4a^2 + 9b^2 + 16c^2 + 12ab - 24bc - 16ca$.

Example 2. *Find the expansions of the following :*

(i) $\left(\frac{2}{3}x + \frac{5}{7}y\right)^2$ (ii) $\left(\frac{3a}{2} - \frac{2}{3b}\right)^2$ (iii) $\left(\frac{1}{2}x - \frac{2}{3}y - \frac{4}{5}z\right)^2$.

Solution. (i) $\left(\frac{2}{3}x + \frac{5}{7}y\right)^2 = \left(\frac{2}{3}x\right)^2 + 2\left(\frac{2}{3}x\right)\left(\frac{5}{7}y\right) + \left(\frac{5}{7}y\right)^2$

$= \frac{4}{9}x^2 + \frac{20}{21}xy + \frac{25}{49}y^2$.

(ii) $\left(\frac{3a}{2} - \frac{2}{3b}\right)^2 = \left(\frac{3a}{2}\right)^2 - 2\left(\frac{3a}{2}\right)\left(\frac{2}{3b}\right) + \left(\frac{2}{3b}\right)^2$

$= \frac{9}{4}a^2 - 2\frac{a}{b} + \frac{4}{9b^2}$.

(iii) $\left(\frac{1}{2}x - \frac{2}{3}y - \frac{4}{5}z\right)^2 = \left(\frac{1}{2}x + \left(-\frac{2}{3}y\right) + \left(-\frac{4}{5}z\right)\right)^2$

$$= \left(\frac{1}{2}x\right)^2 + \left(-\frac{2}{3}y\right)^2 + \left(-\frac{4}{5}z\right)^2$$

$$+ 2\left[\left(\frac{1}{2}x\right)\left(-\frac{2}{3}y\right) + \left(-\frac{2}{3}y\right)\left(-\frac{4}{5}z\right) + \left(-\frac{4}{5}z\right)\left(\frac{1}{2}x\right)\right]$$

$$= \frac{1}{4}x^2 + \frac{4}{9}y^2 + \frac{16}{25}z^2 + 2\left[-\frac{1}{3}xy + \frac{8}{15}yz - \frac{2}{5}zx\right]$$

$$= \frac{1}{4}x^2 + \frac{4}{9}y^2 + \frac{16}{25}z^2 - \frac{2}{3}xy + \frac{16}{15}yz - \frac{4}{5}zx.$$

Example 3. *Simplify the following :*

(i) $(2x - 3y + 5)(2x - 3y - 5)$ *(ii)* $\left(2x - \frac{3}{x} + 1\right)\left(2x + \frac{3}{x} + 1\right)$.

Solution. (*i*) Given expression $= (2x - 3y + 5)(2x - 3y - 5)$.

Let $2x - 3y = a$, then

given expression $= (a + 5)(a - 5) = a^2 - 5^2 = a^2 - 25$

$$= (2x - 3y)^2 - 25$$

$$= (2x)^2 - 2(2x)(3y) + (3y)^2 - 25$$

$$= 4x^2 - 12xy + 9y^2 - 25.$$

(*ii*) Given expression $= \left\{(2x+1) - \frac{3}{x}\right\}\left\{(2x+1) + \frac{3}{x}\right\}$.

Let $2x + 1 = a$, then

given expression $= \left(a - \frac{3}{x}\right)\left(a + \frac{3}{x}\right) = a^2 - \left(\frac{3}{x}\right)^2$

$$= (2x + 1)^2 - \frac{9}{x^2}$$

$$= (2x)^2 + 2(2x)(1) + (1)^2 - \frac{9}{x^2}$$

$$= 4x^2 + 4x + 1 - \frac{9}{x^2}.$$

Example 4. *Simplify the following :*

(i) $\left(2p - \frac{q}{5} - 3\right)\left(2p + \frac{q}{5} + 3\right)$ *(ii)* $(3x - 2y)(3x + 2y)(9x^2 + 4y^2)$.

Solution. (*i*) Given expression $= \left(2p - \frac{q}{5} - 3\right)\left(2p + \frac{q}{5} + 3\right)$

$$= \left\{2p - \left(\frac{q}{5} + 3\right)\right\}\left\{2p + \left(\frac{q}{5} + 3\right)\right\}$$

$$= (2p)^2 - \left(\frac{q}{5} + 3\right)^2$$

$$= 4p^2 - \left\{\left(\frac{q}{5}\right)^2 + 2\left(\frac{q}{5}\right)(3) + (3)^2\right\}$$

$$= 4p^2 - \frac{q^2}{25} - \frac{6q}{5} - 9.$$

(*ii*) Given expression $= (3x - 2y)(3x + 2y)(9x^2 + 4y^2)$

$= ((3x - 2y)(3x + 2y))(9x^2 + 4y^2)$

$= ((3x)^2 - (2y)^2)(9x^2 + 4y^2)$

$= (9x^2 - 4y^2)(9x^2 + 4y^2)$

$= (9x^2)^2 - (4y^2)^2$

$= 81x^4 - 16y^4.$

Example 5. *Find the expansions of the following :*

(i) $(2a + 3b)^3$ *(ii)* $(3x - 4y)^3$ *(iii)* $(x + y - 1)^3$.

Solution. (*i*) $(2a + 3b)^3 = (2a)^3 + (3b)^3 + 3(2a)(3b)(2a + 3b)$

$= 8a^3 + 27b^3 + 18ab(2a + 3b)$

$= 8a^3 + 27b^3 + 36a^2b + 54ab^2.$

(*ii*) $(3x - 4y)^3 = (3x)^3 - (4y)^3 - 3(3x)(4y)(3x - 4y)$

$= 27x^3 - 64y^3 - 36xy(3x - 4y)$

$= 27x^3 - 64y^3 - 108x^2y + 144xy^2.$

(*iii*) $(x + y - 1)^3 = (\overline{x+y} - 1)^3$ [consider $x + y$ as one term]

$= (x + y)^3 - (1)^3 - 3(x + y)(1)(\overline{x+y} - 1)$

$= (x + y)^3 - 1 - 3(x + y)(\overline{x+y} - 1)$

$= [x^3 + y^3 + 3xy(x + y)] - 1 - 3(x + y)^2 + 3(x + y)$

$= x^3 + y^3 + 3x^2y + 3xy^2 - 1 - 3(x^2 + 2xy + y^2) + 3x + 3y$

$= x^3 + y^3 + 3x^2y + 3xy^2 - 3x^2 - 6xy - 3y^2 + 3x + 3y - 1.$

Example 6. *Find the product of the following (using standard results) :*

(i) $(2x + 5y + 3)(2x + 5y + 4)$ *(ii)* $(3x - y - 4)(3x - y + 5)$.

Solution. (*i*) Given expression $= (2x + 5y + 3)(2x + 5y + 4)$.

Let $2x + 5y = a$, then

given expression $= (a + 3)(a + 4)$

$= a^2 + (3 + 4)a + 3 \times 4$

[$\because (x + a)(x + b) = x^2 + (a + b)x + ab$]

$= a^2 + 7a + 12$

$= (2x + 5y)^2 + 7(2x + 5y) + 12$

$= (2x)^2 + 2(2x)(5y) + (5y)^2 + 14x + 35y + 12$

$= 4x^2 + 20xy + 25y^2 + 14x + 35y + 12$

(*ii*) Given expression $= (3x - y - 4)(3x - y + 5)$

Let $3x - y = a$, then

given expression $= (a - 4)(a + 5)$

$= a^2 - (4 - 5)a - 4 \times 5$

[$\because (x - a)(x + b) = x^2 - (a - b)x - ab$]

$= a^2 + a - 20$
$= (3x - y)^2 + (3x - y) - 20$
$= (3x)^2 - 2(3x)(y) + (y)^2 + 3x - y - 20$
$= 9x^2 - 6xy + y^2 + 3x - y - 20.$

Example 7. *Simplify the following :*

(i) $(3x + 5y)(9x^2 - 15xy + 25y^2)$ *(ii)* $\left(x - \frac{2}{x}\right)\left(x^2 + 2 + \frac{4}{x^2}\right)$

Solution. (*i*) We know that $(a + b)(a^2 - ab + b^2) = a^3 + b^3$,

$\therefore$ Given expression $= (3x + 5y)[(3x)^2 - (3x)(5y) + (5y)^2]$
$= (3x)^3 + (5y)^3$
$= 27x^3 + 125y^3.$

(*ii*) We know that $(a - b)(a^2 + ab + b^2) = a^3 - b^3$,

$\therefore$ given expression $= \left(x - \frac{2}{x}\right)\left[x^2 + x \cdot \frac{2}{x} + \left(\frac{2}{x}\right)^2\right]$
$= (x)^3 - \left(\frac{2}{x}\right)^3$
$= x^3 - \frac{8}{x^3}.$

Example 8. *Simplify :* $(x + 3y + 5z)(x^2 + 9y^2 + 25z^2 - 3xy - 15yz - 5zx).$

Solution. We know that

$(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) = a^3 + b^3 + c^3 - 3abc.$

$\therefore$ Given expression $= (x + 3y + 5z)[(x)^2 + (3y)^2 + (5z)^2 - (x)(3y) - (3y)(5z) - (5z)(x)]$
$= (x)^3 + (3y)^3 + (5z)^3 - 3(x)(3y)(5z)$
$= x^3 + 27y^3 + 125z^3 - 45xyz.$

Example 9. *Multiply* $(9x^2 + 25y^2 + 15xy + 12x - 20y + 16)$ *by* $(3x - 5y - 4)$*, using a suitable identity.*

Solution. We know that

$(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) = a^3 + b^3 + c^3 - 3abc.$

$\therefore$ $(3x - 5y - 4)(9x^2 + 25y^2 + 15xy + 12x - 20y + 16)$
$= (3x - 5y - 4)[(3x)^2 + (-5y)^2 + (-4)^2 - (3x)(-5y) - (-5y)(-4) - (-4)(3x)]$
$= (3x)^3 + (-5y)^3 + (-4)^3 - 3(3x)(-5y)(-4)$
$= 27x^3 - 125y^3 - 64 - 180xy$
$= 27x^3 - 125y^3 - 180xy - 64.$

Example 10. *Find the products of :*

(i) $(x + 2)(x + 3)(x + 5)$ *(ii)* $(x + 1)(x - 3)(x - 4).$

Solution. We know that

$(x + a)(x + b)(x + c) = x^3 + (a + b + c)x^2 + (ab + bc + ca)x + abc.$

(*i*) Compare $(x + 2)(x + 3)(x + 5)$ with $(x + a)(x + b)(x + c)$.

Here $a = 2, b = 3, c = 5.$

$\therefore$ Given product $= x^3 + (2 + 3 + 5)\,x^2 + (2.3 + 3.5 + 5.2)\,x + 2.3.5$
$= x^3 + 10x^2 + 31x + 30.$

(*ii*) Compare $(x + 1)\,(x - 3)\,(x - 4)$ with $(x + a)\,(x + b)\,(x + c)$.
Here $a = 1$, $b = -3$ and $c = -4$.

$\therefore$ Given product $= x^3 + (1 + (-3) + (-4))\,x^2 + [1.(-3) + (-3)\,(-4) + (-4).1]\,x + 1.(-3)\,(-4)$
$= x^3 - 6x^2 + 5x + 12.$

Example 11. *Find the coefficient of x^2 and x in the product of $(x - 5)\,(x + 3)\,(x + 7)$.*

Solution. We know that
$(x + a)\,(x + b)\,(x + c) = x^3 + (a + b + c)\,x^2 + (ab + bc + ca)\,x + abc.$
Compare $(x - 5)\,(x + 3)\,(x + 7)$ with $(x + a)\,(x + b)\,(x + c)$.
Here $a = -5$, $b = 3$, and $c = 7$.
$\therefore$ Coefficient of $x^2 = a + b + c = (-5) + 3 + 7 = 5$ and
coefficient of $x = ab + bc + ca = (-5).3 + 3.7 + 7.(-5) = -29.$

Example 12. *Find the coefficient of x^2 in the expansion of $(x^2 + 2x + 3)^2 + (x^2 - 2x + 3)^2$.*

Solution. Given expression $= (x^2 + 2x + 3)^2 + (x^2 - 2x + 3)^2$
$= ((x^2 + 3) + (2x))^2 + ((x^2 + 3) - (2x))^2.$

Let $x^2 + 3 = a$ and $2x = b$, then

given expression $= (a + b)^2 + (a - b)^2$
$= (a^2 + 2ab + b^2) + (a^2 - 2ab + b^2)$
$= 2(a^2 + b^2)$
$= 2[(x^2 + 3)^2 + (2x)^2]$
$= 2[(x^2)^2 + 2(x^2)\,(3) + (3)^2 + 4x^2]$
$= 2(x^4 + 6x^2 + 9 + 4x^2)$
$= 2x^4 + 20x^2 + 18.$

$\therefore$ Coefficent of $x^2 = 20$.

Example 13. *By using $(a + b)^2 = a^2 + 2ab + b^2$, find the value of $(10{\cdot}3)^2$.*

Solution. $(10{\cdot}3)^2 = (10 + {\cdot}3)^2$
$= (10)^2 + 2 \times 10 \times {\cdot}3 + ({\cdot}3)^2$
$= 100 + 6 + {\cdot}09 = 106{\cdot}09.$

Example 14. *Using a suitable identity, find the value of $(98)^3$.*

Solution. We know that $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$.

Now $(98)^3 = (100 - 2)^3$
$= (100)^3 - 3 \times (100)^2 \times 2 + 3 \times 100 \times (2)^2 - (2)^3$
(take $a = 100$ and $b = 2$)
$= 1000000 - 60000 + 1200 - 8$
$= 1001200 - 60008 = 941192.$

Example 15. *If $a + b + c = 0$, prove that $a^3 + b^3 + c^3 = 3abc$.*

Solution. Given $a + b + c = 0 \Rightarrow a + b = -c$.

On cubing both sides, we get $(a + b)^3 = (-c)^3$

$\Rightarrow a^3 + b^3 + 3ab\,(a + b) = -c^3$ but $a + b = -c$

$\Rightarrow \quad a^3 + b^3 + 3ab(-c) = -c^3$

$\Rightarrow \quad a^3 + b^3 + c^3 - 3abc = 0$

$\Rightarrow \quad a^3 + b^3 + c^3 = 3abc.$

Example 16. *Without actually calculating the cubes, find the values of :*

(i) $(-23)^3 + 15^3 + 8^3$ (ii) $\left(\frac{1}{4}\right)^3 + \left(\frac{1}{3}\right)^3 - \left(\frac{7}{12}\right)^3$.

Solution. (i) Let $a = -23$, $b = 15$ and $c = 8$, then

$a + b + c = -23 + 15 + 8 = 0$

$\therefore \quad a^3 + b^3 + c^3 = 3abc$ (See example 15)

$\Rightarrow \quad (-23)^3 + 15^3 + 8^3 = 3(-23) \times 15 \times 8 = -8280.$

(ii) Let $a = \frac{1}{4}$, $b = \frac{1}{3}$ and $c = -\frac{7}{12}$, then

$$a + b + c = \frac{1}{4} + \frac{1}{3} - \frac{7}{12} = \frac{3+4-7}{12} = \frac{0}{12} = 0.$$

$\therefore \quad a^3 + b^3 + c^3 = 3abc$ (See example 15)

$$\Rightarrow \left(\frac{1}{4}\right)^3 + \left(\frac{1}{3}\right)^3 - \left(\frac{7}{12}\right)^3 = \left(\frac{1}{4}\right)^3 + \left(\frac{1}{3}\right)^3 + \left(-\frac{7}{12}\right)^3$$

$$= 3\left(\frac{1}{4}\right)\left(\frac{1}{3}\right)\left(-\frac{7}{12}\right) = -\frac{7}{48}.$$

Example 17. *If $x = 2y + 6$, then find the value of $x^3 - 8y^3 - 36xy - 216$.*

Solution. Given $x = 2y + 6 \Rightarrow x - 2y - 6 = 0$

$\Rightarrow \quad x + (-2y) + (-6) = 0$

$\Rightarrow \quad (x)^3 + (-2y)^3 + (-6)^3 = 3(x)(-2y)(-6)$

($\because$ if $a + b + c = 0$, then $a^3 + b^3 + c^3 = 3abc$)

$\Rightarrow \quad x^3 - 8y^3 - 216 = 36xy$

$\Rightarrow \quad x^3 - 8y^3 - 36xy - 216 = 0.$

Example 18. *If $a = 1$, $b = -2$ and $c = -3$, find the value of*

$$\frac{a^3 + b^3 + c^3 - 3abc}{ab + bc + ca - (a^2 + b^2 + c^2)}.$$

Solution. We know that $(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) = a^3 + b^3 + c^3 - 3abc$,

$$\therefore \quad \frac{a^3 + b^3 + c^3 - 3abc}{ab + bc + ca - (a^2 + b^2 + c^2)} = \frac{(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)}{-(a^2 + b^2 + c^2 - ab - bc - ca)}$$

$$= -(a + b + c)$$

$$= -(1 - 2 - 3) = -(-4) = 4.$$

Example 19. *Using suitable identity, find the value of :*

$$\frac{0{\cdot}75 \times 0{\cdot}75 \times 0{\cdot}75 + 0{\cdot}25 \times 0{\cdot}25 \times 0{\cdot}25}{0{\cdot}75 \times 0{\cdot}75 - 0{\cdot}75 \times 0{\cdot}25 + 0{\cdot}25 \times 0{\cdot}25}.$$

Solution. We know that $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

$$\Rightarrow \quad \frac{a^3 + b^3}{a^2 - ab + b^2} = a + b.$$

Putting $a = 0{\cdot}75$ and $b = 0{\cdot}25$ in this result, we get

$$\frac{0{\cdot}75 \times 0{\cdot}75 \times 0{\cdot}75 + 0{\cdot}25 \times 0{\cdot}25 \times 0{\cdot}25}{0{\cdot}75 \times 0{\cdot}75 - 0{\cdot}75 \times 0{\cdot}25 + 0{\cdot}25 \times 0{\cdot}25} = \frac{a^3 + b^3}{a^2 - ab + b^2}$$

$$= a + b = 0{\cdot}75 + 0{\cdot}25 = 1.$$

Example 20. *If $a + b + c = 0$, find the value of $\frac{(b+c)^2}{bc} + \frac{(c+a)^2}{ca} + \frac{(a+b)^2}{ab}$.*

Solution. Given $a + b + c = 0 \Rightarrow a^3 + b^3 + c^3 = 3abc$ ...(*i*)

(See example 15)

Also $a + b + c = 0 \Rightarrow b + c = -a,\ c + a = -b,\ a + b = -c.$

$$\therefore \quad \frac{(b+c)^2}{bc} + \frac{(c+a)^2}{ca} + \frac{(a+b)^2}{ab} = \frac{(-a)^2}{bc} + \frac{(-b)^2}{ca} + \frac{(-c)^2}{ab}$$

$$= \frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab}$$

$$= \frac{a^3 + b^3 + c^3}{abc} = \frac{3abc}{abc} \quad \text{[Using } (i)]$$

$$= 3.$$

Example 21. *Simplify : $\frac{(a^2 - b^2)^3 + (b^2 - c^2)^3 + (c^2 - a^2)^3}{(a \pm b)^3 + (b - c)^3 + (c - a)^3}$.*

Solution. We know that if $x + y + z = 0$, then $x^3 + y^3 + z^3 = 3xyz$.

Here, $(a^2 - b^2) + (b^2 - c^2) + (c^2 - a^2) = 0$,

$\therefore \quad (a^2 - b^2)^3 + (b^2 - c^2)^3 + (c^2 - a^2)^3 = 3(a^2 - b^2)(b^2 - c^2)(c^2 - a^2)$.

Also, $(a - b) + (b - c) + (c - a) = 0$,

$\therefore \quad (a - b)^3 + (b - c)^3 + (c - a)^3 = 3(a - b)(b - c)(c - a)$.

$$\therefore \quad \frac{(a^2 - b^2)^3 + (b^2 - c^2)^3 + (c^2 - a^2)^3}{(a - b)^3 + (b - c)^3 + (c - a)^3} = \frac{3(a^2 - b^2)(b^2 - c^2)(c^2 - a^2)}{3(a - b)(b - c)(c - a)}$$

$$= \frac{(a - b)(a + b)(b - c)(b + c)(c - a)(c + a)}{(a - b)(b - c)(c - a)}$$

$$= (a + b)(b + c)(c + a).$$

Exercise 4.1

By using standard formulae, expand the following (1 to 9) :

1. (*i*) $(2x + 7y)^2$ (*ii*) $\left(\frac{1}{2}x + \frac{2}{3}y\right)^2$.

2. (*i*) $\left(3x + \frac{1}{2x}\right)^2$ (*ii*) $(3x^2y + 5z)^2$.

3. (*i*) $\left(3x - \frac{1}{2x}\right)^2$ (*ii*) $\left(\frac{1}{2}x - \frac{3}{2}y\right)^2$.

4. (*i*) $(x + 3)(x + 5)$ (*ii*) $(x + 3)(x - 5)$

(*iii*) $(x - 7)(x + 9)$ (*iv*) $(x - 2y)(x - 3y)$.

5. (i) $(x - 2y - z)^2$ (ii) $(2x - 3y + 4z)^2$.

6. (i) $\left(2x + \frac{3}{x} - 1\right)^2$ (ii) $\left(\frac{2}{3}x - \frac{3}{2x} - 1\right)^2$.

7. (i) $(x + 2)^3$ (ii) $(2a + b)^3$.

8. (i) $\left(3x + \frac{1}{x}\right)^3$ (ii) $(2x - 1)^3$.

9. (i) $(5x - 3y)^3$ (ii) $\left(2x - \frac{1}{3y}\right)^3$.

Simplify the following (10 to 19) :

10. (i) $(a + b)^2 + (a - b)^2$ (ii) $(a + b)^2 - (a - b)^2$.

11. (i) $\left(a + \frac{1}{a}\right)^2 + \left(a - \frac{1}{a}\right)^2$ (ii) $\left(a + \frac{1}{a}\right)^2 - \left(a - \frac{1}{a}\right)^2$.

12. (i) $(3x - 1)^2 - (3x - 2)(3x + 1)$. (ii) $(4x + 3y)^2 - (4x - 3y)^2 - 48xy$.

13. (i) $(7p + 9q)(7p - 9q)$ (ii) $\left(2x - \frac{3}{x}\right)\left(2x + \frac{3}{x}\right)$.

14. (i) $(2x - y + 3)(2x - y - 3)$ (ii) $(3x + y - 5)(3x - y - 5)$.

15. (i) $\left(x + \frac{2}{x} - 3\right)\left(x - \frac{2}{x} - 3\right)$ (ii) $(5 - 2x)(5 + 2x)(25 + 4x^2)$.

16. (i) $(x + 2y + 3)(x + 2y + 7)$ (ii) $(2x + y + 5)(2x + y - 9)$
(iii) $(x - 2y - 5)(x - 2y + 3)$ (iv) $(3x - 4y - 2)(3x - 4y - 6)$.

17. (i) $(2p + 3q)(4p^2 - 6pq + 9q^2)$ (ii) $\left(x + \frac{1}{x}\right)\left(x^2 - 1 + \frac{1}{x^2}\right)$.

18. (i) $(3p - 4q)(9p^2 + 12pq + 16q^2)$ (ii) $\left(x - \frac{3}{x}\right)\left(x^2 + 3 + \frac{9}{x^2}\right)$.

19. $(2x + 3y + 4z)(4x^2 + 9y^2 + 16z^2 - 6xy - 12yz - 8zx)$.

20. Find the product of the following :
(i) $(x + 1)(x + 2)(x + 3)$ (ii) $(x - 2)(x - 3)(x + 4)$.

21. Find the coefficient of x^2 and x in the product of $(x - 3)(x + 7)(x - 4)$.

22. If $a^2 + 4a + x = (a + 2)^2$, find the value of x.

23. Use $(a + b)^2 = a^2 + 2ab + b^2$ to evaluate the following :
(i) $(101)^2$ (ii) $(1003)^2$ (iii) $(10{\cdot}2)^2$.

24. Use $(a - b)^2 = a^2 - 2ab + b^2$ to evaluate the following :
(i) $(99)^2$ (ii) $(997)^2$ (iii) $(9{\cdot}8)^2$.

25. By using suitable identities, evaluate the following :
(i) $(103)^2$ (ii) $(99)^3$ (iii) $(10{\cdot}1)^3$.

26. If $2a - b + c = 0$, prove that $4a^2 - b^2 + c^2 + 4ac = 0$.

Hint

$2a - b + c = 0 \Rightarrow 2a + c = b \Rightarrow (2a + c)^2 = b^2$.

27. If $a + b + 2c = 0$, prove that $a^3 + b^3 + 8c^3 = 6abc$.

28. If $x + y = 4$, then find the value of $x^3 + y^3 + 12xy - 64$.

29. Without actually calculating the cubes, find the values of :

(i) $(27)^3 + (-17)^3 + (-10)^3$ (ii) $(-28)^3 + (15)^3 + (13)^3$.

30. Using suitable identity, find the value of :

$$\frac{86 \times 86 \times 86 - 14 \times 14 \times 14}{86 \times 86 - 86 \times 14 + 14 \times 14}.$$

4.3 MORE APPLICATIONS OF SPECIAL PRODUCTS

ILLUSTRATIVE EXAMPLES

Example 1. *If $a + b = 3$ and $ab = 2$, find the values of :*

(i) $a^2 + b^2$ *(ii)* $a - b$ *(iii)* $a^2 - b^2$ *(iv)* $a^3 + b^3$.

Solution. (i) We know that $(a + b)^2 = a^2 + b^2 + 2ab$

$\Rightarrow$ $a^2 + b^2 = (a + b)^2 - 2ab$ but $a + b = 3$ and $ab = 2$

$\Rightarrow$ $a^2 + b^2 = (3)^2 - 2.2 = 9 - 4 = 5$.

(ii) $(a - b)^2 = a^2 + b^2 - 2ab = 5 - 2.2 = 5 - 4 = 1$

$\Rightarrow$ $a - b = \pm\sqrt{1} = \pm 1$.

(iii) $a^2 - b^2 = (a + b)(a - b) = 3 \times (\pm 1) = \pm 3$.

(iv) We know that $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$

$\Rightarrow$ $a^3 + b^3 = (a + b)^3 - 3ab(a + b)$

$\Rightarrow$ $a^3 + b^3 = (3)^3 - 3.2.3 = 27 - 18 = 9$.

Example 2. *If $x - \frac{1}{x} = 5$, find the values of :*

(i) $x^2 + \frac{1}{x^2}$ *(ii)* $x^4 + \frac{1}{x^4}$ *(iii)* $x^3 - \frac{1}{x^3}$.

Solution. (i) We know that $\left(x - \frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} - 2$

$\Rightarrow$ $x^2 + \frac{1}{x^2} = \left(x - \frac{1}{x}\right)^2 + 2$

$\Rightarrow$ $x^2 + \frac{1}{x^2} = (5)^2 + 2 = 25 + 2 = 27$.

(ii) $\left(x^2 + \frac{1}{x^2}\right)^2 = x^4 + \frac{1}{x^4} + 2$

$\Rightarrow$ $x^4 + \frac{1}{x^4} = \left(x^2 + \frac{1}{x^2}\right)^2 - 2$

$\Rightarrow$ $x^4 + \frac{1}{x^4} = (27)^2 - 2 = 729 - 2 = 727$.

(*iii*) We know that $\left(x-\frac{1}{x}\right)^3 = x^3 - \frac{1}{x^3} - 3x.\frac{1}{x}\left(x-\frac{1}{x}\right)$

$\Rightarrow \quad x^3 - \frac{1}{x^3} = \left(x-\frac{1}{x}\right)^3 + 3\left(x-\frac{1}{x}\right)$

$\Rightarrow \quad x^3 - \frac{1}{x^3} = (5)^3 + 3 \times 5 = 125 + 15 = 140.$

Example 3. *If* $x^2 + \frac{1}{x^2} = 7$, *find the values of :*

(*i*) $x + \frac{1}{x}$ (*ii*) $x - \frac{1}{x}$ (*iii*) $2x^2 - \frac{2}{x^2}$.

Solution. (*i*) $\left(x+\frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} + 2 = 7 + 2 = 9$

$\Rightarrow \quad x + \frac{1}{x} = \pm\sqrt{9} = \pm 3.$

(*ii*) $\left(x-\frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} - 2 = 7 - 2 = 5$

$\Rightarrow \quad x - \frac{1}{x} = \pm\sqrt{5}.$

(*iii*) $2x^2 - \frac{2}{x^2} = 2\left(x^2 - \frac{1}{x^2}\right) = 2\left(x+\frac{1}{x}\right)\left(x-\frac{1}{x}\right)$

$= 2 \times (\pm 3)(\pm\sqrt{5}) = \pm 6\sqrt{5}.$

Example 4. *If* $a^2 - 3a - 1 = 0$, *find the value of* $a^2 + \frac{1}{a^2}$.

Solution. Given $a^2 - 3a - 1 = 0$, dividing each term by a, we get

$a - 3 - \frac{1}{a} = 0 \Rightarrow a - \frac{1}{a} = 3.$

Now $\left(a-\frac{1}{a}\right)^2 = a^2 + \frac{1}{a^2} - 2$

$\Rightarrow \quad a^2 + \frac{1}{a^2} = \left(a-\frac{1}{a}\right)^2 + 2$

$\Rightarrow \quad a^2 + \frac{1}{a^2} = (3)^2 + 2 = 9 + 2 = 11.$

Example 5. *If* $\frac{x^2+1}{x} = 2\frac{1}{2}$, *find the values of :*

(*i*) $x - \frac{1}{x}$ (*ii*) $x^3 - \frac{1}{x^3}$.

Solution. Given $\frac{x^2+1}{x} = 2\frac{1}{2} \Rightarrow x + \frac{1}{x} = \frac{5}{2}.$

(*i*) Now $\left(x-\frac{1}{x}\right)^2 = \left(x^2+\frac{1}{x^2}\right) - 2$

$= \left(\left(x+\frac{1}{x}\right)^2 - 2\right) - 2 = \left(x+\frac{1}{x}\right)^2 - 4$

$= \left(\frac{5}{2}\right)^2 - 4 = \frac{25}{4} - 4 = \frac{9}{4}$

$\Rightarrow \quad x - \frac{1}{x} = \pm\sqrt{\frac{9}{4}} = \pm\frac{3}{2}.$

(ii) $\left(x-\frac{1}{x}\right)^3 = x^3 - \frac{1}{x^3} - 3 \cdot x \cdot \frac{1}{x}\left(x-\frac{1}{x}\right) = x^3 - \frac{1}{x^3} - 3\left(x-\frac{1}{x}\right)$

$\Rightarrow \quad x^3 - \frac{1}{x^3} = \left(x-\frac{1}{x}\right)^3 + 3\left(x-\frac{1}{x}\right).$

Two cases arise :

Case I. When $x - \frac{1}{x} = \frac{3}{2}$,

$$x^3 - \frac{1}{x^3} = \left(\frac{3}{2}\right)^3 + 3 \cdot \frac{3}{2} = \frac{27}{8} + \frac{9}{2} = \frac{63}{8} = 7\frac{7}{8}.$$

Case II. When $x - \frac{1}{x} = -\frac{3}{2}$,

$$x^3 - \frac{1}{x^3} = \left(-\frac{3}{2}\right)^3 + 3\left(-\frac{3}{2}\right) = -\frac{27}{8} - \frac{9}{2} = -\frac{63}{8} = -7\frac{7}{8}.$$

Example 6. *If $x^2 + \frac{1}{25x^2} = 8\frac{3}{5}$, find the value of $x^3 + \frac{1}{125x^3}$.*

Solution. Given $x^2 + \frac{1}{25x^2} = 8\frac{3}{5} = \frac{43}{5}$.

We know that $\left(x+\frac{1}{5x}\right)^2 = x^2 + \frac{1}{25x^2} + 2 \cdot x \cdot \frac{1}{5x}$

$$= \frac{43}{5} + \frac{2}{5} = \frac{45}{5} = 9$$

$\Rightarrow \quad x + \frac{1}{5x} = \pm\sqrt{9} = \pm 3.$

Now $\left(x+\frac{1}{5x}\right)^3 = x^3 + \frac{1}{125x^3} + 3 \cdot x \cdot \frac{1}{5x}\left(x+\frac{1}{5x}\right)$

$\Rightarrow \quad x^3 + \frac{1}{125x^3} = \left(x+\frac{1}{5x}\right)^3 - \frac{3}{5}\left(x+\frac{1}{5x}\right).$

Two cases arise :

Case I. When $x + \frac{1}{5x} = 3$,

$$x^3 + \frac{1}{125\,x^3} = (3)^3 - \frac{3}{5} \cdot 3 = 27 - \frac{9}{5} = \frac{126}{5} = 25\frac{1}{5}.$$

Case II. When $x + \frac{1}{5x} = -3$,

$$x^3 + \frac{1}{125\,x^3} = (-3)^3 - \frac{3}{5}(-3) = -27 + \frac{9}{5} = -\frac{126}{5} = -25\frac{1}{5}.$$

Example 7. *If $x^4 + \frac{1}{x^4} = 194$, find the values of :*

(i) $x^2 + \frac{1}{x^2}$ *(ii)* $x + \frac{1}{x}$ *(iii)* $x^3 + \frac{1}{x^3}$.

Solution. (i) $\left(x^2+\frac{1}{x^2}\right)^2 = x^4 + \frac{1}{x^4} + 2 = 194 + 2 = 196$

$\Rightarrow \quad x^2 + \frac{1}{x^2} = \pm 14$ but $x^2 + \frac{1}{x^2}$ is always positive,

$\therefore \quad x^2 + \frac{1}{x^2} = 14.$

(ii) $\left(x+\frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} + 2 = 14 + 2$ (Using part (i))

$\Rightarrow \left(x+\frac{1}{x}\right)^2 = 16$

$\Rightarrow x + \frac{1}{x} = \pm 4.$

(iii) $\left(x+\frac{1}{x}\right)^3 = x^3 + \frac{1}{x^3} + 3 \,.\, x \,.\, \frac{1}{x}\left(x+\frac{1}{x}\right)$

$= x^3 + \frac{1}{x^3} + 3\left(x+\frac{1}{x}\right)$

$\Rightarrow x^3 + \frac{1}{x^3} = \left(x+\frac{1}{x}\right)^3 - 3\left(x+\frac{1}{x}\right).$

Two cases arise :

Case I. When $x + \frac{1}{x} = 4$,

$x^3 + \frac{1}{x^3} = (4)^3 - 3 \,.\, 4 = 64 - 12 = 52.$

Case II. When $x + \frac{1}{x} = -4$,

$x^3 + \frac{1}{x^3} = (-4)^3 - 3 \,.\, (-4) = -64 + 12 = -52.$

Example 8. *If $x = 7 - 4\sqrt{3}$, find the value of $\sqrt{x} + \frac{1}{\sqrt{x}}$.*

Solution. Given $x = 7 - 4\sqrt{3}$ ...(i)

$\therefore \quad \frac{1}{x} = \frac{1}{7-4\sqrt{3}} = \frac{1}{7-4\sqrt{3}} \times \frac{7+4\sqrt{3}}{7+4\sqrt{3}} = \frac{7+4\sqrt{3}}{7^2-(4\sqrt{3})^2}$

$= \frac{7+4\sqrt{3}}{49-48} = \frac{7+4\sqrt{3}}{1}$

$\Rightarrow \quad \frac{1}{x} = 7 + 4\sqrt{3}$ ...(ii)

On adding (i) and (ii), we get

$x + \frac{1}{x} = 14$ ...(iii)

Now $\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2 = x + \frac{1}{x} + 2.\sqrt{x}.\frac{1}{\sqrt{x}} = x + \frac{1}{x} + 2$

$= 14 + 2$ (using (iii))

$\Rightarrow \left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2 = 16 \Rightarrow \sqrt{x} + \frac{1}{\sqrt{x}} = \pm\sqrt{16} = \pm 4.$

As $\sqrt{x} > 0$, $\sqrt{x} + \frac{1}{\sqrt{x}} > 0$. So, we reject -4.

$\therefore \quad \sqrt{x} + \frac{1}{\sqrt{x}} = 4.$

Example 9. *If $x = 3 + 2\sqrt{2}$, find the value of $x^3 - \frac{1}{x^3}$.*

Solution. Given $x = 3 + 2\sqrt{2}$ ...(*i*)

$$\therefore \quad \frac{1}{x} = \frac{1}{3+2\sqrt{2}} = \frac{1}{3+2\sqrt{2}} \times \frac{3-2\sqrt{2}}{3-2\sqrt{2}} = \frac{3-2\sqrt{2}}{(3)^2-(2\sqrt{2})^2}$$

$$= \frac{3-2\sqrt{2}}{9-8} = \frac{3-2\sqrt{2}}{1}$$

$$\Rightarrow \quad \frac{1}{x} = 3 - 2\sqrt{2} \qquad \text{...(ii)}$$

On subtracting (*ii*) from (*i*), we get

$$x - \frac{1}{x} = 4\sqrt{2} \qquad \text{...(iii)}$$

$$\text{Now } \left(x - \frac{1}{x}\right)^3 = x^3 - \frac{1}{x^3} - 3.x.\frac{1}{x}\left(x - \frac{1}{x}\right)$$

$$= x^3 - \frac{1}{x^3} - 3\left(x - \frac{1}{x}\right)$$

$$\Rightarrow \quad x^3 - \frac{1}{x^3} = \left(x - \frac{1}{x}\right)^3 + 3\left(x - \frac{1}{x}\right)$$

$$\Rightarrow \quad x^3 - \frac{1}{x^3} = (4\sqrt{2})^3 + 3 \times 4\sqrt{2} \qquad \text{(using (iii))}$$

$$= 128\sqrt{2} + 12\sqrt{2}$$

$$\Rightarrow \quad x^3 - \frac{1}{x^3} = 140\sqrt{2}.$$

Example 10. *If $a^2 + b^2 + c^2 = 50$ and $ab + bc + ca = 47$, find $a + b + c$.*

Solution. $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$

$$= 50 + 2 \times 47 = 50 + 94 = 144$$

$$\Rightarrow \quad a + b + c = \pm\sqrt{144} = \pm 12.$$

Example 11. *If $x + y - z = 4$ and $x^2 + y^2 + z^2 = 38$, then find the value of $xy - yz - zx$.*

Solution. We know that

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca).$$

$$\therefore \quad (x + y - z)^2 = x^2 + y^2 + (-z)^2 + 2[xy + y(-z) + (-z)x]$$

$$= x^2 + y^2 + z^2 + 2(xy - yz - zx)$$

$$\Rightarrow \quad 4^2 = 38 + 2(xy - yz - zx)$$

$$\Rightarrow \quad 16 = 38 + 2(xy - yz - zx)$$

$$\Rightarrow \quad 2(xy - yz - zx) = 16 - 38 = -22$$

$$\Rightarrow \quad xy - yz - zx = -11.$$

Example 12. *If $a + b + c = 2$, $ab + bc + ca = -1$ and $abc = -2$, find the value of $a^3 + b^3 + c^3$.*

Solution. We know that

$$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

$$= (a + b + c)(a^2 + b^2 + c^2 + 2ab + 2bc + 2ca - 3ab - 3bc - 3ca)$$

$$= (a + b + c)[(a + b + c)^2 - 3(ab + bc + ca)]$$

$$= 2[(2)^2 - 3(-1)] = 2(4 + 3) = 14$$

$\Rightarrow \quad a^3 + b^3 + c^3 - 3(-2) = 14$

$\Rightarrow \quad a^3 + b^3 + c^3 + 6 = 14$

$\Rightarrow \quad a^3 + b^3 + c^3 = 8.$

Example 13. *If $a + b = 10$ and $a^2 + b^2 = 58$, find the value of $a^3 + b^3$.*

Solution. Given $a + b = 10$ ...(*i*) and $a^2 + b^2 = 58$ ...(*ii*)

Now $(a + b)^2 = a^2 + b^2 + 2ab$

$\Rightarrow \quad 10^2 = 58 + 2ab$ (using (*i*) and (*ii*))

$\Rightarrow \quad 100 - 58 = 2ab \Rightarrow 42 = 2ab$

$\Rightarrow \quad ab = 21$ ...(*iii*)

We know that $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$

$\Rightarrow \quad 10^3 = a^3 + b^3 + 3 \times 21 \times 10$ (using (*i*), (*ii*) and (*iii*))

$\Rightarrow \quad 1000 = a^3 + b^3 + 630$

$\Rightarrow \quad a^3 + b^3 = 370.$

Example 14. *If $\frac{a}{b} = \frac{b}{c}$, prove that $(a + b + c)(a - b + c) = a^2 + b^2 + c^2$.*

Solution. Let $\frac{a}{b} = \frac{b}{c} = k$ (say)

$\Rightarrow \quad a = bk$ and $b = ck$

$\Rightarrow \quad a = ck.k$ and $b = ck$

$\Rightarrow \quad a = ck^2$ ...(*i*)

and $\quad b = ck$ ...(*ii*)

Now, L.H.S. of the given result $= (a + b + c)(a - b + c)$

$= (ck^2 + ck + c)(ck^2 - ck + c)$ [Using (*i*) and (*ii*)]

$= c(k^2 + k + 1)\, c(k^2 - k + 1)$

$= c^2 \left(\overline{k^2 + 1} + k\right)\left(\overline{k^2 + 1} - k\right)$

$= c^2 [(k^2 + 1)^2 - k^2]$

$= c^2 (k^4 + 2k^2 + 1 - k^2)$

$= c^2 (k^4 + k^2 + 1)$ ...(*iii*)

And R.H.S. of the given result $= a^2 + b^2 + c^2$

$= (ck^2)^2 + (ck)^2 + c^2$

$= c^2k^4 + c^2k^2 + c^2$

$= c^2 (k^4 + k^2 + 1)$ ...(*iv*)

From (*iii*) and (*iv*), it follows that $(a + b + c)(a - b + c) = a^2 + b^2 + c^2$.

Example 15. *If the number p is 7 more than the number q and the sum of the squares of p and q is 85, find the product of p and q.*

Solution. Given $p = q + 7$ and $p^2 + q^2 = 85$.

Now $\quad p = q + 7 \Rightarrow p - q = 7$

$\Rightarrow \quad (p - q)^2 = 7^2 \Rightarrow p^2 + q^2 - 2pq = 49$

$\Rightarrow \quad 85 - 2pq = 49$ $(\because p^2 + q^2 = 85)$

$\Rightarrow \quad 85 - 49 = 2pq$

$\Rightarrow \quad 2pq = 36 \Rightarrow pq = 18.$

Example 16. *If the sum of two numbers is 7 and the sum of their cubes is 133, find the sum of their squares.*

Solution. Let the two numbers be a and b, then

$a + b = 7$ and $a^3 + b^3 = 133$.

Now $a + b = 7 \Rightarrow (a + b)^3 = 7^3$

$\Rightarrow \quad a^3 + b^3 + 3ab\,(a + b) = 343$

$\Rightarrow \quad 133 + 3ab \times 7 = 343$

$\Rightarrow \quad 21ab = 343 - 133 = 210$

$\Rightarrow \quad ab = 10.$

We know that $(a + b)^2 = a^2 + b^2 + 2ab$

$\Rightarrow \quad 7^2 = a^2 + b^2 + 2 \times 10$

$\Rightarrow \quad 49 = a^2 + b^2 + 20$

$\Rightarrow \quad a^2 + b^2 = 49 - 20 = 29.$

Hence the sum of the squares of the numbers is 29.

Exercise 4.2

1. If $x - y = 8$ and $xy = 5$, find $x^2 + y^2$.

2. If $x + y = 10$ and $xy = 21$, find $2(x^2 + y^2)$.

3. If $2a + 3b = 7$ and $ab = 2$, find $4a^2 + 9b^2$.

4. If $3x - 4y = 16$ and $xy = 4$ find the value of $9x^2 + 16y^2$.

5. If $x + y = 8$ and $x - y = 2$, find the value of $2x^2 + 2y^2$.

Hint

$2(x^2 + y^2) = (x + y)^2 + (x - y)^2$.

6. If $a^2 + b^2 = 13$ and $ab = 6$, find (*i*) $a + b$ (*ii*) $a - b$.

7. If $a + b = 4$ and $ab = -12$, find (*i*) $a - b$ (*ii*) $a^2 - b^2$.

8. If $p - q = 9$ and $pq = 36$, evaluate

(*i*) $p + q$ (*ii*) $p^2 - q^2$.

9. If $x + y = 6$ and $x - y = 4$, find (*i*) $x^2 + y^2$ (*ii*) xy.

Hint

(*ii*) $4xy = (x + y)^2 - (x - y)^2$.

10. If $x - 3 = \frac{1}{x}$, find the value of $x^2 + \frac{1}{x^2}$.

Hint

$x - 3 = \frac{1}{x} \Rightarrow x - \frac{1}{x} = 3.$

11. If $x + y = 8$ and $xy = 3\frac{3}{4}$, find the values of

(*i*) $x - y$ (*ii*) $3(x^2 + y^2)$ (*iii*) $5(x^2 + y^2) + 4(x - y)$.

12. If $x^2 + y^2 = 34$ and $xy = 10\frac{1}{2}$, find the value of $2(x + y)^2 + (x - y)^2$.

13. If $a - b = 3$ and $ab = 4$, find $a^3 - b^3$.

14. If $2a - 3b = 3$ and $ab = 2$, find the value of $8a^3 - 27b^3$.

15. If $x + \frac{1}{x} = 4$, find the values of

(i) $x^2 + \frac{1}{x^2}$ (ii) $x^4 + \frac{1}{x^4}$ (iii) $x^3 + \frac{1}{x^3}$ (iv) $x - \frac{1}{x}$.

16. If $x - \frac{1}{x} = 5$, find the value of $x^4 + \frac{1}{x^4}$.

17. If $x - \frac{1}{x} = \sqrt{5}$, find the values of

(i) $x^2 + \frac{1}{x^2}$ (ii) $x + \frac{1}{x}$ (iii) $x^3 + \frac{1}{x^3}$.

18. If $x + \frac{1}{x} = 6$, find (i) $x - \frac{1}{x}$ (ii) $x^2 - \frac{1}{x^2}$.

19. If $x + \frac{1}{x} = 2$, prove that $x^2 + \frac{1}{x^2} = x^3 + \frac{1}{x^3} = x^4 + \frac{1}{x^4}$.

20. If $x - \frac{2}{x} = 3$, find the value of $x^3 - \frac{8}{x^3}$.

21. If $a + 2b = 5$, prove that $a^3 + 8b^3 + 30ab = 125$.

22. If $a + \frac{1}{a} = p$, prove that $a^3 + \frac{1}{a^3} = p(p^2 - 3)$.

23. If $x^2 + \frac{1}{x^2} = 27$, find the value of $x - \frac{1}{x}$.

24. If $x^2 + \frac{1}{x^2} = 27$, find the value of $3x^3 + 5x - \frac{3}{x^3} - \frac{5}{x}$.

25. If $x^2 + \frac{1}{25x^2} = 8\frac{3}{5}$, find $x + \frac{1}{5x}$.

26. If $x^2 + \frac{1}{4x^2} = 8$, find $x^3 + \frac{1}{8x^3}$.

27. If $a^2 - 3a + 1 = 0$, find (i) $a^2 + \frac{1}{a^2}$ (ii) $a^3 + \frac{1}{a^3}$.

28. If $a = \frac{1}{a-5}$, find (i) $a - \frac{1}{a}$ (ii) $a + \frac{1}{a}$ (iii) $a^2 - \frac{1}{a^2}$.

Hint

$a = \frac{1}{a-5} \Rightarrow a^2 - 5a - 1 = 0.$

29. If $\left(x + \frac{1}{x}\right)^2 = 3$, find $x^3 + \frac{1}{x^3}$.

30. If $x = 5 - 2\sqrt{6}$, find the value of $\sqrt{x} + \frac{1}{\sqrt{x}}$.

31. If $a + b + c = 12$ and $ab + bc + ca = 22$, find $a^2 + b^2 + c^2$.

32. If $a + b + c = 12$ and $a^2 + b^2 + c^2 = 100$, find $ab + bc + ca$.

33. If $a^2 + b^2 + c^2 = 125$ and $ab + bc + ca = 50$, find $a + b + c$.

34. If $a + b - c = 5$ and $a^2 + b^2 + c^2 = 29$, find the value of $ab - bc - ca$.

35. If $a - b = 7$ and $a^2 + b^2 = 85$, then find the value of $a^3 - b^3$.

36. If the number x is 3 less than the number y and the sum of the squares of x and y is 29, find the product of x and y.

37. If the sum and the product of two numbers are 8 and 15 respectively, find the sum of their cubes.

CHAPTER TEST

1. Find the expansions of the following :

   (i) $(2x + 3y + 5)(2x + 3y - 5)$ (ii) $(6 - 4a - 7b)^2$

   (iii) $(7 - 3xy)^3$ (iv) $(x + y + 2)^3$.

2. Simplify $(x - 2)(x + 2)(x^2 + 4)(x^4 + 16)$.
3. Evaluate 1002×998 by using a special product.
4. If $a + 2b + 3c = 0$, prove that $a^3 + 8b^3 + 27c^3 = 18abc$.
5. If $2x = 3y - 5$, then find the value of $8x^3 - 27y^3 - 90xy + 125$.
6. If $a^2 - \frac{1}{a^2} = 5$, evaluate $a^4 + \frac{1}{a^4}$.
7. If $a + \frac{1}{a} = p$ and $a - \frac{1}{a} = q$, find the relation between p and q.
8. If $\frac{a^2 + 1}{a} = 4$, find the value of $2a^3 + \frac{2}{a^3}$.
9. If $x = \frac{1}{4 - x}$, find the values of

   (i) $x + \frac{1}{x}$ (ii) $x^3 + \frac{1}{x^3}$ (iii) $x^6 + \frac{1}{x^6}$.

10. If $x - \frac{1}{x} = 3 + 2\sqrt{2}$, find the value of $\frac{1}{4}\left(x^3 - \frac{1}{x^3}\right)$
11. If $x + \frac{1}{x} = 3\frac{1}{3}$, find the value of $x^3 - \frac{1}{x^3}$.
12. If $x = 2 - \sqrt{3}$, then find the value of $x^3 - \frac{1}{x^3}$.
13. If the sum of two numbers is 11 and sum of their cubes is 737, find the sum of their squares.
14. If $a - b = 7$ and $a^3 - b^3 = 133$, find (i) ab (ii) $a^2 + b^2$.
15. Find the coefficient of x^2 in the expansion of $(x^2 + x + 1)^2 + (x^2 - x + 1)^2$.

Hint

Given expression $= \{(x^2 + 1) + x\}^2 + \{(x^2 + 1) - x\}^2$
$= 2\{(x^2 + 1)^2 + x^2\}$.