UNIT 3 : Algebra

4 Expansions

4.1 INTRODUCTION

Expansion is the process in which the contents of brackets are evaluated.

Recall of concepts of expansions learned in earlier classes :

1. Since, $(a+b)^2 = (a+b)(a+b)$
$= a(a+b) + b(a+b)$
$= a^2 + ab + ab + b^2$
$= a^2 + 2ab + b^2$ I

$a^2 + 2ab + b^2$ is the expansion of $(a+b)^2$

Similarly,

2. $(a-b)^2 = a^2 - 2ab + b^2$ II
3. $(a+b)^2 + (a-b)^2 = 2(a^2+b^2)$ [On adding I and II]
4. $(a+b)^2 - (a-b)^2 = 4ab$ [On subtracting II from I]

If a ≠ 0, then :

5. $(a + \frac{1}{a})^2 = a^2 + \frac{1}{a^2} + 2 \Rightarrow a^2 + \frac{1}{a^2} = (a + \frac{1}{a})^2 - 2$
6. $(a - \frac{1}{a})^2 = a^2 + \frac{1}{a^2} - 2 \Rightarrow a^2 + \frac{1}{a^2} = (a - \frac{1}{a})^2 + 2$
7. $(a + \frac{1}{a})^2 + (a - \frac{1}{a})^2 = 2(a^2 + \frac{1}{a^2})$
8. $(a + \frac{1}{a})^2 - (a - \frac{1}{a})^2 = 4$

4.2 IDENTITIES

Consider the expansion : $(a+b)^2 = a^2 + 2ab + b^2$

1. If $a = 5$ and $b = 3$
$(a+b)^2 = (5+3)^2 = 8^2 = 64$ and
$a^2 + 2ab + b^2 = 5^2 + 2 \times 5 \times 3 + 3^2 = 25 + 30 + 9 = 64.$
i.e. $(a+b)^2 = a^2 + 2ab + b^2$

2. If $a = -8$ and $b = 5$
$(a+b)^2 = (-8+5)^2 = (-3)^2 = 9$ and
$a^2 + 2ab + b^2 = (-8)^2 + 2 \times -8 \times 5 + 5^2 = 64 - 80 + 25 = 9$
i.e. $(a+b)^2 = a^2 + 2ab + b^2$

In the same way, if we give any number of values to a and b; every time $(a+b)^2$ and $a^2 + 2ab + b^2$ will come same (equal).

> An equation, which is true for all values of its variables, is called an **identity**.
> Each equation (expansion) given above in article 4.1 is an identity.

1 Evaluate : (i) $(a + 2b)^2$ (ii) $(2a - 3b)^2$.

Solution :

(i) $(a + 2b)^2 = (a)^2 + 2 \times a \times 2b + (2b)^2$

$= a^2 + 4ab + 4b^2$ **Ans.**

(ii) $(2a - 3b)^2 = (2a)^2 - 2 \times 2a \times 3b + (3b)^2$

$= 4a^2 - 12ab + 9b^2$ **Ans.**

2 If $a + b = 9$ and $ab = -22$, find : (i) $a - b$ (ii) $a^2 - b^2$.

Solution :

(i) $(a + b)^2 - (a - b)^2 = 4ab$

$\Rightarrow (a - b)^2 = (a + b)^2 - 4ab$

$= (9)^2 - 4 \times -22$

$= 81 + 88 = 169$

$\therefore a - b = \pm\sqrt{169} = \pm 13$ **Ans.**

OR, $(a + b)^2 = 9^2$

$\Rightarrow a^2 + b^2 + 2ab = 81$

$\Rightarrow a^2 + b^2 + 2 \times -22 = 81$

$\Rightarrow a^2 + b^2 = 125$

Now, $(a - b)^2 = a^2 + b^2 - 2ab$

$= 125 - 2 \times -22$

$= 169$

$\therefore a - b = \pm 13$ **Ans.**

(ii) $a^2 - b^2 = (a + b)(a - b) = 9 \times \pm 13 = \pm 117$ **Ans.**

3 If $x \neq 0$ and $x + \frac{1}{x} = 2$, find : (i) $x^2 + \frac{1}{x^2}$ (ii) $x^4 + \frac{1}{x^4}$.

Solution :

(i) $x^2 + \frac{1}{x^2} = (x + \frac{1}{x})^2 - 2$

$= (2)^2 - 2$

$= 4 - 2 = 2$ **Ans.**

(ii) $x^4 + \frac{1}{x^4} = (x^2 + \frac{1}{x^2})^2 - 2$

$= (2^2) - 2$

$= 4 - 2 = 2$ **Ans.**

Alternative method :

(i) $x + \quad = 2$

$\Rightarrow (x + \frac{1}{x})^2 = (2)^2$

$\Rightarrow x^2 + \frac{1}{x^2} + 2 \times x \times \frac{1}{x} = 4$

$\Rightarrow x^2 + \frac{1}{x^2} = 4 - 2 = 2$ **Ans.**

(ii) $(x^2 + \frac{1}{x^2})^2 = (2)^2$

$\Rightarrow x^4 + \quad + 2 = 4$

$\Rightarrow x^4 + \frac{1}{x^4} = 2$ **Ans.**

4 Given : $a^2 + \frac{1}{a^2} = 7$ and $a \neq 0$, find :

(i) $a + \frac{1}{a}$ (ii) $a - \frac{1}{a}$ (iii) $a^2 - \frac{1}{a^2}$.

Solution :

(i) $\because \left(a + \frac{1}{a}\right)^2 = a^2 + \frac{1}{a^2} + 2 = 7 + 2 = 9 \Rightarrow a + \frac{1}{a} = \pm\sqrt{9} = \pm 3$ **Ans.**

(ii) $\because \left(a - \frac{1}{a}\right)^2 = a^2 + \frac{1}{a^2} - 2 = 7 - 2 = 5 \Rightarrow a - \frac{1}{a} = \pm\sqrt{5}$ **Ans.**

(iii) $a^2 - \frac{1}{a^2} = (a + \frac{1}{a})(a - \frac{1}{a}) = (\pm 3) \times (\pm\sqrt{5}) = \pm 3\sqrt{5}$ **Ans.**

Remember :

$(\pm a) \times (\pm b) = (+a) \times (+b)$ or $(-a) \times (+b)$ or $(+a) \times (-b)$ or $(-a) \times (-b)$

$= +ab$ or $-ab$ or $-ab$ or $+ab$

$= \pm ab$

$\therefore (\pm a) \times (\pm b) = \pm ab$ **Ans.**

5 If $a^2 - 5a + 1 = 0$ and $a \neq 0$, find : (i) $a + \frac{1}{a}$ (ii) $a^2 + \frac{1}{a^2}$.

Solution :

(i) $a^2 - 5a + 1 = 0$

$\Rightarrow \frac{a^2}{a} - \frac{5a}{a} + \frac{1}{a} = 0$ [Dividing each term by a]

$\Rightarrow a - 5 + \frac{1}{a} = 0 \Rightarrow a + \frac{1}{a} = 5$ **Ans.**

(ii) $a^2 + \frac{1}{a^2} = \left(a + \frac{1}{a}\right)^2 - 2 = 5^2 - 2 = 25 - 2 = 23$ **Ans.**

EXERCISE 4 (A)

1. Find the square of :
   (i) $2a + b$ (ii) $3a + 7b$
   (iii) $3a - 4b$ (iv) $\frac{3a}{2b} - \frac{2b}{3a}$
2. Use identities to evaluate :
   (i) $(101)^2$ (ii) $(502)^2$
   (iii) $(97)^2$ (iv) $(998)^2$

(iii) $(97)^2 = (100 - 3)^2$
$= (100)^2 - 2(100)(3) + (3)^2$
$= 10000 - 600 + 9 = 9409$

3. Evaluate :
   (i) $\left(\frac{7}{8}x + \frac{4}{5}y\right)^2$ (ii) $\left(\frac{2x}{7} - \frac{7y}{4}\right)^2$

4. Evaluate :

(i) $\left(\frac{a}{2b}+\frac{2b}{a}\right)^2-\left(\frac{a}{2b}-\frac{2b}{a}\right)^2-4$

(ii) $(4a + 3b)^2 - (4a - 3b)^2 + 48\ ab$.

5. If $a + b = 7$ and $ab = 10$; find $a - b$.

6. If $a - b = 7$ and $ab = 18$; find $a + b$.

7. If $x + y = \frac{7}{2}$ and $xy = \frac{5}{2}$; find :

(i) $x - y$ (ii) $x^2 - y^2$.

8. If $a - b = 0{\cdot}9$ and $ab = 0{\cdot}36$; find :

(i) $a + b$ (ii) $a^2 - b^2$.

9. If $a - b = 4$ and $a + b = 6$; find :

(i) $a^2 + b^2$ (ii) ab

10. If $a + \frac{1}{a} = 6$ and $a \neq 0$; find :

(i) $a - \frac{1}{a}$ (ii) $a^2 - \frac{1}{a^2}$

11. If $a - \frac{1}{a} = 8$ and $a \neq 0$; find :

(i) $a + \frac{1}{a}$ (ii) $a^2 - \frac{1}{a^2}$

12. If $a^2 - 3a + 1 = 0$ and $a \neq 0$; find :

(i) $a + \frac{1}{a}$ (ii) $a^2 + \frac{1}{a^2}$

13. If $a^2 - 5a - 1 = 0$ and $a \neq 0$; find :

(i) $a - \frac{1}{a}$ (ii) $a + \frac{1}{a}$

(iii) $a^2 - \frac{1}{a^2}$

14. If $3x + 4y = 16$ and $xy = 4$; find the value of $9x^2 + 16y^2$.

15. The number x is 2 more than the number y. If the sum of the squares of x and y is 34; find the product of x and y.

Given : $x - y = 2$ and $x^2 + y^2 = 34$
To find the value of xy.

16. The difference between two positive numbers is 5 and the sum of their squares is 73. Find the product of these numbers.

4.3 EXPANSIONS OF $(a \pm b)^3$

1. $(a + b)^3 = a^3 + 3a^2 b + 3ab^2 + b^3$
$= a^3 + b^3 + 3ab\ (a + b) \quad \Rightarrow \quad a^3 + b^3 = (a + b)^3 - 3ab\ (a + b)$

2. $(a - b)^3 = a^3 - 3a^2 b + 3ab^2 - b^3$
$= a^3 - b^3 - 3ab\ (a - b) \quad \Rightarrow \quad a^3 - b^3 = (a - b)^3 + 3ab\ (a - b)$

On combining result 1 and result 2, we get :

(a) $(a \pm b)^3 = a^3 \pm 3a^2 b + 3ab^2 \pm b^3$

(b) $a^3 \pm b^3 = (a \pm b)^3 \mp 3ab\ (a \pm b)$

3. $(a + \frac{1}{a})^3 = a^3 + \frac{1}{a^3} + 3\ (a + \frac{1}{a})$

$\Rightarrow \quad a^3 + \frac{1}{a^3} = (a + \frac{1}{a})^3 - 3\ (a + \frac{1}{a})$

4. $(a - \frac{1}{a})^3 = a^3 - \frac{1}{a^3} - 3\ (a - \frac{1}{a})$

$\Rightarrow \quad a^3 - \frac{1}{a^3} = (a - \frac{1}{a})^3 + 3\ (a - \frac{1}{a})$.

6 Evaluate : (i) $(2a + 3b)^3$ (ii) $(4a - 5b)^3$.

Solution :

(i) $(2a + 3b)^3 = (2a)^3 + 3\,(2a)^2\,(3b) + 3\,(2a)\,(3b)^2 + (3b)^3$

$= 8a^3 + 36\,a^2\,b + 54\,ab^2 + 27b^3$ **Ans.**

(ii) $(4a - 5b)^3 = (4a)^3 - 3\,(4a)^2\,(5b) + 3\,(4a)\,(5b)^2 - (5b)^3$

$= 64a^3 - 240a^2b + 300\,ab^2 - 125\,b^3$ **Ans.**

7 If $a^2 + \frac{1}{a^2} = 23$ and $a \neq 0$, find the value of $a^3 + \frac{1}{a^3}$.

Solution :

$a^2 + \frac{1}{a^2} = 23 \Rightarrow a^2 + \frac{1}{a^2} + 2 = 23 + 2$

$\Rightarrow \quad (a + \frac{1}{a})^2 = 25 \quad \Rightarrow \quad a + \frac{1}{a} = \pm 5$

When $a + \frac{1}{a} = 5$:

$a^3 + \frac{1}{a^3} = (a + \frac{1}{a})^3 - 3\,(a + \frac{1}{a})$

$= (5)^3 - 3 \times 5$

$= 110$

When $a = -5$:

$a^3 + \frac{1}{a^3} = (a + \frac{1}{a})^3 - 3\,(a + \frac{1}{a})$

$= (-5)^3 - 3 \times -5$

$= -125 + 15 = -110$ **Ans.**

8 If $a + b + c = 0$, show that : $a^3 + b^3 + c^3 = 3abc$.

Solution :

Given : $a + b + c = 0 \Rightarrow a + b = -c \Rightarrow (a + b)^3 = (-c)^3$

$\Rightarrow a^3 + b^3 + 3ab\,(a + b) = -c^3$

$\Rightarrow a^3 + b^3 + 3ab\,(-c) = -c^3$ [Since, $a + b = -c$]

$\Rightarrow a^3 + b^3 - 3abc = -c^3 \quad \therefore \; a^3 + b^3 + c^3 = 3abc$ **Ans.**

9 Use property to evaluate :

(i) $8^3 + (-5)^3 + (-3)^3$ (ii) $2^3 + 4^3 + (-6)^3$.

Solution :

Property required to be used is the result of example 8, given above.
That is, if $a + b + c = 0$, then $a^3 + b^3 + c^3 = 3abc$.

(i) Let $8 = a$, $-5 = b$ and $-3 = c$

$\therefore \quad a + b + c = 8 - 5 - 3 = 0$

$\Rightarrow \quad a^3 + b^3 + c^3 = 3abc$

$\Rightarrow \quad 8^3 + (-5)^3 + (-3)^3 = 3 \times 8 \times (-5) \times (-3)$

$= 360$ **Ans.**

(ii) Let $2 = a$, $4 = b$ and $-6 = c$

$\therefore \quad a + b + c = 2 + 4 - 6 = 0$

$\Rightarrow \quad a^3 + b^3 + c^3 = 3abc$

$\Rightarrow \quad 2^3 + 4^3 + (-6)^3 = 3 \times 2 \times 4 \times (-6) = -144$ **Ans.**

10 Expand :

(i) $(3x - 2y + 4)(3x - 2y - 4)$ (ii) $(5x - 3y + 2)(5x + 3y + 2)$.

Solution :

(i) $(3x - 2y + 4)(3x - 2y - 4)$

$= [(3x - 2y) + 4][(3x - 2y) - 4]$

$= (a + 4)(a - 4)$ [Taking $3x - 2y = a$]

$= a^2 - 16$ [$\because (a + 4)(a - 4) = a^2 - 4^2$]

$= (3x - 2y)^2 - 16$

$= (3x)^2 - 2 \times 3x \times 2y + (2y)^2 - 16 = 9x^2 - 12xy + 4y^2 - 16$ **Ans.**

(ii) $(5x - 3y + 2)(5x + 3y + 2)$

$= [(5x + 2) - 3y][(5x + 2) + 3y]$

$= (a - 3y)(a + 3y)$ [Taking $5x + 2 = a$]

$= a^2 - (3y)^2$

$= (5x + 2)^2 - 9y^2 = 25x^2 + 20x + 4 - 9y^2$ **Ans.**

EXERCISE 4 (B)

1. Find the cube of :

   (i) $3a - 2b$ (ii) $5a + 3b$

   (iii) $2a + \frac{1}{2a}$ $(a \neq 0)$ (iv) $3a - \frac{1}{a}$ $(a \neq 0)$

2. If $a^2 + \frac{1}{a^2} = 47$ and $a \neq 0$; find :

   (i) $a + \frac{1}{a}$ (ii) $a^3 + \frac{1}{a^3}$

3. If $a^2 + \frac{1}{a^2} = 18$ and $a \neq 0$; find :

   (i) $a - \frac{1}{a}$ (ii) $a^3 - \frac{1}{a^3}$

4. If $a + \frac{1}{a} = p$ and $a \neq 0$; then show that :

   $a^3 + \frac{1}{a^3} = p(p^2 - 3)$

5. If $a + 2b = 5$; then show that :

   $a^3 + 8b^3 + 30ab = 125$.

6. If $\left(a + \frac{1}{a}\right)^2 = 3$ and $a \neq 0$; then show that :

   $a^3 + \frac{1}{a^3} = 0$.

7. If $a + 2b + c = 0$; then show that :

   $a^3 + 8b^3 + c^3 = 6abc$.

8. Use property to evaluate :

   (i) $13^3 + (-8)^3 + (-5)^3$

   (ii) $7^3 + 3^3 + (-10)^3$

   (iii) $9^3 - 5^3 - 4^3$

   (iv) $38^3 + (-26)^3 + (-12)^3$

9. If $a \neq 0$ and $a - \frac{1}{a} = 3$; find :

   (i) $a^2 + \frac{1}{a^2}$ (ii) $a^3 - \frac{1}{a^3}$

10. If $a \neq 0$ and $a - \frac{1}{a} = 4$; find :

    (i) $a^2 + \frac{1}{a^2}$ (ii) $a^4 + \frac{1}{a^4}$ (iii) $a^3 - \frac{1}{a^3}$

11. If $x \neq 0$ and $x + \frac{1}{x} = 2$; then show that :

$$x^2 + \frac{1}{x^2} = x^3 + \frac{1}{x^3} = x^4 + \frac{1}{x^4}$$

12. If $2x - 3y = 10$ and $xy = 16$; find the value of $8x^3 - 27y^3$.

13. Expand :
   (i) $(3x + 5y + 2z)(3x - 5y + 2z)$
   (ii) $(3x - 5y - 2z)(3x - 5y + 2z)$

14. The sum of two numbers is 9 and their product is 20. Find the sum of their :
   (i) squares (ii) cubes.

15. Two positive numbers x and y are such that $x > y$. If the difference of these numbers is 5 and their product is 24, find :
   (i) sum of these numbers.
   (ii) difference of their cubes.
   (iii) sum of their cubes.

4.4 EXPANSION OF $(x \pm a)(x \pm b)$

1. $(x + a)(x + b) = x^2 + ax + bx + ab$
   $= x^2 + (a + b)x + ab$
2. $(x + a)(x - b) = x^2 + ax - bx - ab$
   $= x^2 + (a - b)x - ab$
3. $(x - a)(x + b) = x^2 - ax + bx - ab$
   $= x^2 - (a - b)x - ab$
4. $(x - a)(x - b) = x^2 - ax - bx + ab$
   $= x^2 - (a + b)x + ab$

1. $(x + 3)(x + 5) = x^2 + (3 + 5)x + 3 \times 5$
   $= x^2 + 8x + 15$
2. $(x + 3)(x - 5) = x^2 + (3 - 5)x + 3 \times -5$
   $= x^2 - 2x - 15$
3. $(x - 3)(x + 5) = x^2 - (3 - 5)x - 3 \times 5$
   $= x^2 + 2x - 15$
4. $(x - 3)(x - 5) = x^2 - (3 + 5) + 3 \times 5$
   $= x^2 - 8x + 15$

4.5 EXPANSION OF $(a \pm b \pm c)^2$

$$(a \pm b \pm c)^2 = a^2 + b^2 + c^2 \pm 2ab \pm 2bc \pm 2ca$$

1. $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$
   $= a^2 + b^2 + c^2 + 2(ab + bc + ca)$
2. $(a + b - c)^2 = a^2 + b^2 + c^2 + 2ab - 2bc - 2ca$
   $= a^2 + b^2 + c^2 + 2(ab - bc - ca)$
3. $(a - b + c)^2 = a^2 + b^2 + c^2 - 2ab - 2bc + 2ca$
   $= a^2 + b^2 + c^2 - 2(ab + bc - ca)$
4. $(a - b - c)^2 = a^2 + b^2 + c^2 - 2ab + 2bc - 2ca$
   $= a^2 + b^2 + c^2 - 2(ab - bc + ca)^2$

11 Expand :
(i) $(2x + 3y - 4z)^2$ (ii) $(3x - 2y + 5z)^2$ (iii) $(4a - 5b - 2c)^2$

Solution :

(i) $(a + b - c)^2 = a^2 + b^2 + c^2 + 2ab - 2bc - 2ca$

$\Rightarrow$ $(2x + 3y - 4z)^2 = (2x)^2 + (3y)^2 + (4z)^2 + 2 \times 2x \times 3y - 2 \times 3y \times 4z - 2 \times 4z \times 2x$
$= 4x^2 + 9y^2 + 16z^2 + 12xy - 24yz - 16zx$
$= 4x^2 + 9y^2 + 16z^2 + 4(3xy - 6yz - 4zx)$ **Ans.**

(ii) $(a - b + c)^2 = a^2 + b^2 + c^2 - 2ab - 2bc + 2ca$

$\Rightarrow$ $(3x - 2y + 5z)^2 = (3x)^2 + (2y)^2 + (5z)^2 + 2 \times 3x \times 2y - 2 \times 2y \times 5z + 2 \times 5z \times 3x$

$= 9x^2 + 4y^2 + 25z^2 - 12xy - 20yz + 30zx$

$= 9x^2 + 4y^2 + 25z^2 - 2(6xy + 10yz - 15zx)$ **Ans.**

(iii) $(a - b - c)^2 = a^2 + b^2 + c^2 - 2ab + 2bc - 2ca$

$\Rightarrow$ $(4a - 5b - 2c)^2 = (4a)^2 + (5b)^2 + (2c)^2 - 2 \times 4a \times 5b + 2 \times 5b \times 2c - 2 \times 2c \times 4a$

$= 16a^2 + 25b^2 + 4c^2 - 40ab + 20bc - 16ca$

$= 16a^2 + 25b^2 + 4c^2 - 4(10ab - 5bc + 4ca)$ **Ans.**

12 (i) If $a^2 + b^2 + c^2 = 29$ and $a + b + c = 9$, find : $ab + bc + ca$.

(ii) If $a + b - c = 4$ and $a^2 + b^2 + c^2 = 38$; find : $ab - bc - ca$.

(iii) If $a - b - c = 3$ and $a^2 + b^2 + c^2 = 77$; find : $ab - bc + ca$.

Solution :

(i) Since, $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$

$\therefore$ $(9)^2 = 29 + 2(ab + bc + ca)$

$\Rightarrow$ $81 - 29 = 2(ab + bc + ca)$

$\therefore$ $ab + bc + ca = \frac{52}{2} = 26$ **Ans.**

(ii) $(a + b - c)^2 = a^2 + b^2 + (-c^2) + 2[a \times b + b \times (-c) + (-c) \times a]$

$= a^2 + b^2 + c^2 + 2(ab - bc - ca)$

$\Rightarrow$ $4^2 = 38 + 2(ab - bc - ca)$

$\Rightarrow$ $16 - 38 = 2(ab - bc - ca)$ *i.e.* $-22 = 2(ab - bc - ca)$

$\therefore$ $ab - bc - ca = -11$ **Ans.**

(iii) $(a - b - c)^2 = a^2 + (-b)^2 + (-c^2) + 2[a \times (-b) + (-b) \times (-c) + (-c) \times a]$

$= a^2 + b^2 + c^2 + 2(-ab + bc - ca)$

$\Rightarrow$ $3^2 = 77 - 2(ab - bc + ca)$

$\Rightarrow$ $2(ab - bc + ca) = 77 - 9 = 68$

$\therefore$ $ab - bc + ca = \frac{68}{2} = 34$ **Ans.**

EXERCISE 4 (C)

1. Expand :

(i) $(x + 8)(x + 10)$ (ii) $(x + 8)(x - 10)$

(iii) $(x - 8)(x + 10)$ (iv) $(x - 8)(x - 10)$

2. Expand :

(i) $\left(2x - \frac{1}{x}\right)\left(3x + \frac{2}{x}\right)$

(ii) $\left(3a + \frac{2}{b}\right)\left(2a - \frac{3}{b}\right)$

3. Expand :

(i) $(x + y - z)^2$ (ii) $(x - 2y + 2)^2$

(iii) $(5a - 3b + c)^2$ (iv) $(5x - 3y - 2)^2$

(v) $\left(x - \frac{1}{x} + 5\right)^2$

4. If $a + b + c = 12$ and $a^2 + b^2 + c^2 = 50$; find $ab + bc + ca$.

5. If $a^2 + b^2 + c^2 = 35$ and $ab + bc + ca = 23$; find $a + b + c$.

6. If $a + b + c = p$ and $ab + bc + ca = q$; find $a^2 + b^2 + c^2$.

7. If $a^2 + b^2 + c^2 = 50$ and $ab + bc + ca = 47$, find $a + b + c$.

8. If $x + y - z = 4$ and $x^2 + y^2 + z^2 = 30$, then find the value of $xy - yz - zx$.

4.6 USING EXPANSIONS

13 If $x^3 + y^3 + z^3 = 3xyz$ and $x + y + z = 0$; find the value of :

$$\frac{(x+y)^2}{xy} + \frac{(y+z)^2}{yz} + \frac{(z+x)^2}{zx}$$

Solution :

(i) $x + y + z = 0 \Rightarrow x + y = -z,\ y + z = -x$ and $z + x = -y$

$$\therefore \frac{(x+y)^2}{xy} + \frac{(y+z)^2}{yz} + \frac{(z+x)^2}{zx} = \frac{(-z)^2}{xy} + \frac{(-x)^2}{yz} + \frac{(-y)^2}{zx}$$

$$= \frac{z^2}{xy} + \frac{x^2}{yz} + \frac{y^2}{zx}$$

$$= \frac{z^3 + x^3 + y^3}{xyz} = \frac{3xyz}{xyz} = 3$$ **Ans.**

14 If $a + \frac{1}{a} = m$, $a - \frac{1}{a} = n$ and $a \neq 0$; find the relation between m and n

Solution :

$$\left(a+\frac{1}{a}\right)^2 - \left(a-\frac{1}{a}\right)^2 = 4$$

$\Rightarrow$ $m^2 - n^2 = 4$; *which is the required relation.* **Ans.**

15 In the expansion of $(5x - 3)(x + 2)^2$; find : (i) coefficients of x^2 and x (ii) constant term.

Solution :

$$(5x - 3)(x + 2)^2 = (5x - 3)(x^2 + 4x + 4)$$

$$= 5x^3 + 20x^2 + 20x - 3x^2 - 12x - 12 = 5x^3 + 17x^2 + 8x - 12$$

(i) **Coefficient of x^2 = 17** and **coefficient of x = 8** **Ans.**

(ii) **Constant term = –12** **Ans.**

16 If each of a, b and c is a non-zero number and $\frac{a}{b} = \frac{b}{c}$, *show that :*

$(a + b + c)(a - b + c) = a^2 + b^2 + c^2$

Solution :

$$\frac{a}{b} = \frac{b}{c} \Rightarrow ac = b^2$$

Now, $(a + b + c)(a - b + c) = (a+c + b)(a+c - b)$

$= (a + c)^2 - b^2$

$= a^2 + 2ac + c^2 - b^2$

$= a^2 + 2b^2 + c^2 - b^2$ $[\because ac = b^2]$

$= a^2 + b^2 + c^2$

Hence the required result

If the sum of two numbers is 5 and the sum of their cubes is 35; find the sum of their squares.

Solution :

Let the numbers be x and y.

Given : $x + y = 5$ and $x^3 + y^3 = 35$

Required : $x^2 + y^2$

$x^3 + y^3 = 35 \Rightarrow (x + y)^3 - 3xy(x + y) = 35$

$\Rightarrow 5^3 - 3xy \times 5 = 35$

$\Rightarrow 15xy = 90$ and $xy = 6$

$\therefore x^2 + y^2 = (x + y)^2 - 2xy$

$= 5^2 - 2 \times 6 = 13$ **Ans.**

EXERCISE 4 (D)

1. If $x + 2y + 3z = 0$ and $x^3 + 4y^3 + 9z^3 = 18xyz$; evaluate :

   $\frac{(x+2y)^2}{xy} + \frac{(2y+3z)^2}{yz} + \frac{(3z+x)^2}{zx}$

2. If $a + \frac{1}{a} = m$ and $a \neq 0$; find in terms of 'm'; the value of:

   (i) $a - \frac{1}{a}$ (ii) $a^2 - \frac{1}{a^2}$

3. In the expansion of $(2x^2 - 8)(x - 4)^2$; find the value of :

   (i) coefficient of x^3 (ii) coefficient of x^2

   (iii) constant term.

4. If $x > 0$ and $x^2 + \frac{1}{9x^2} = \frac{25}{36}$, find : $x^3 + \frac{1}{27x^3}$.

5. If $2(x^2 + 1) = 5x$, find :

   (i) $x - \frac{1}{x}$ (ii) $x^3 - \frac{1}{x^3}$

6. If $a^2 + b^2 = 34$ and $ab = 12$; find :

   (i) $3(a + b)^2 + 5(a - b)^2$

   (ii) $7(a - b)^2 - 2(a + b)^2$

7. If $3x - \frac{4}{x} = 4$ and $x \neq 0$; find : $27x^3 - \frac{64}{x^3}$.

8. If $x^2 + \frac{1}{x^2} = 7$ and $x \neq 0$; find the value of :

   $7x^3 + 8x - \frac{7}{x^3} - \frac{8}{x}$.

9. If $x = \frac{1}{x-5}$ and $x \neq 5$, find : $x^2 - \frac{1}{x^2}$.

10. If $x = \frac{1}{5-x}$ and $x \neq 5$, find : $x^3 + \frac{1}{x^3}$.

11. If $3a + 5b + 4c = 0$, show that :

    $27a^3 + 125b^3 + 64c^3 = 180\,abc$.

12. The sum of two numbers is 7 and the sum of their cubes is 133, find the sum of their squares.

13. In each of the following, find the value of 'a':

    (i) $4x^2 + ax + 9 = (2x + 3)^2$

    (ii) $4x^2 + ax + 9 = (2x - 3)^2$

    (iii) $9x^2 + (7a - 5)x + 25 = (3x + 5)^2$

14. If $\frac{x^2+1}{x} = 3\frac{1}{3}$ and $x > 1$; find

(i) $x - \frac{1}{x}$ (ii) $x^3 - \frac{1}{x^3}$

15. The difference between two positive numbers is 4 and the difference between their cubes is 316. Find :

(i) their product.

(ii) the sum of their squares.

4.7 SPECIAL PRODUCTS

1. $(x + a)(x + b)(x + c) = [x^2 + (a + b)x + ab](x + c)$

$= x^3 + (a + b) \times x^2 + abx + cx^2 + (a + b) \times cx + abc$

$= x^3 + (a + b + c)x^2 + (ab + bc + ca)x + abc$

2. $(a + b)(a^2 - ab + b^2) = a^3 + b^3$

3. $(a - b)(a^2 + ab + b^2) = a^3 - b^3$

4. $(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

$= a^3 + b^3 + c^3 - 3abc$

18 If $a = 2$, $b = 3$ and $c = 4$, find the value of : $\frac{ab+bc+ca-a^2-b^2-c^2}{3abc-a^3-b^3-c^3}$, using suitable identity.

Solution :

$$= \frac{a^2+b^2+c^2-ab-bc-c}{a^3+b^3+c^3-3abc}$$ [Changing the sign of each term in numerator and denominator]

$$= \frac{a^2+b^2+c^2-ab-bc-ca}{(a+b+c)(a^2+b^2+c^2-ab-bc-ca)}$$

$$= \frac{1}{a+b+c} = \frac{1}{2+3+4} = \mathbf{\frac{1}{9}}$$ **Ans.**

19 Evaluate : $\frac{0{\cdot}6\times0{\cdot}6\times0{\cdot}6-0{\cdot}3\times0{\cdot}3\times0{\cdot}3}{0{\cdot}6\times0{\cdot}6+0{\cdot}6\times0{\cdot}3+0{\cdot}3\times0{\cdot}3}$, using suitable identity.

Solution :

On taking $0{\cdot}6 = a$ and $0{\cdot}3 = b$, the given expression becomes

$$= \frac{a^3-b^3}{a^2+ab+b^2} = \frac{(a-b)(a^2+ab+b^2)}{a^2+ab+b^2}$$

$$= a - b = 0{\cdot}6 - 0{\cdot}3 = \mathbf{0{\cdot}3}$$ **Ans.**

EXERCISE 4 (E)

1. Simplify :

   (i) $(x + 6)(x + 4)(x - 2)$

   (ii) $(x - 6)(x - 4)(x + 2)$

   (iii) $(x - 6)(x - 4)(x - 2)$

   (iv) $(x + 6)(x - 4)(x - 2)$

2. Simplify using following identity :

   $(a \pm b)(a^2 \mp ab + b^2) = a^3 \quad b^3$

   (i) $(2x + 3y)(4x^2 - 6xy + 9y^2)$

   (ii) $\left(3x - \frac{5}{x}\right)\left(9x^2 + 15 + \frac{25}{x^2}\right)$

   (iii) $\left(\frac{a}{3} - 3b\right)\left(\frac{a^2}{9} + ab + 9b^2\right)$

3. Using suitable identity, evaluate :

   (i) $(104)^3$ (ii) $(97)^3$

4. Simplify : $\frac{(x^2 - y^2)^3 + (y^2 - z^2)^3 + (z^2 - x^2)^3}{(x - y)^3 + (y - z)^3 + (z - x)^3}$.

   If $a + b + c = 0$; we have :

   $a^3 + b^3 + c^3 = 3abc$

   Since, $(x^2 - y^2) + (y^2 - z^2) + (z^2 - x^2) = 0$

   $\Rightarrow (x^2 - y^2)^3 + (y^2 - z^2)^3 + (z^2 - x^2)^3$

   $= 3(x^2 - y^2)(y^2 - z^2)(z^2 - x^2)$ I

   Also, $(x - y) + (y - z) + (z - x) = 0$

   $\Rightarrow (x - y)^3 + (y - z)^3 + (z - x)^3$

   $= 3(x - y)(y - z)(z - x)$ II

   Dividing result I by result II, we get the required answer.

5. Evaluate :

   (i) $\frac{0{\cdot}8 \times 0{\cdot}8 \times 0{\cdot}8 + 0{\cdot}5 \times 0{\cdot}5 \times 0{\cdot}5}{0{\cdot}8 \times 0{\cdot}8 - 0{\cdot}8 \times 0{\cdot}5 + 0{\cdot}5 \times 0{\cdot}5}$

   (ii) $\frac{1{\cdot}2 \times 1{\cdot}2 + 1{\cdot}2 \times 0{\cdot}3 + 0{\cdot}3 \times 0{\cdot}3}{1{\cdot}2 \times 1{\cdot}2 \times 1{\cdot}2 - 0{\cdot}3 \times 0{\cdot}3 \times 0{\cdot}3}$

6. If $a - 2b + 3c = 0$; state the value of $a^3 - 8b^3 + 27c^3$.

7. If $x + 5y = 10$; find the value of $x^3 + 125y^3 + 150xy - 1000$.

8. If $x = 3 + 2\sqrt{2}$, find :

   (i) $\frac{1}{x}$ (ii) $x - \frac{1}{x}$

   (iii) $\left(x - \frac{1}{x}\right)^3$ (iv) $x^3 - \frac{1}{x^3}$

   (iv) $\left(x - \frac{1}{x}\right)^3 = x^3 - \frac{1}{x^3} - 3\left(x - \frac{1}{x}\right)$

9. If $a + b = 11$ and $a^2 + b^2 = 65$; find $a^3 + b^3$.

   Using $(a + b)^2 = a^2 + b^2 + 2ab$, find ab, then apply :

   $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$ to get $a^3 + b^3$.

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