3

Polygons

A closed plane shape bounded by three or more line segments is called a polygon.

Some polygons have names based on the number of sides, as shown in the table. Otherwise, a polygon with n sides is called n-gon. For example a polygon that has 15 sides is called 15-gon.

Table 3.1 Names of polygons

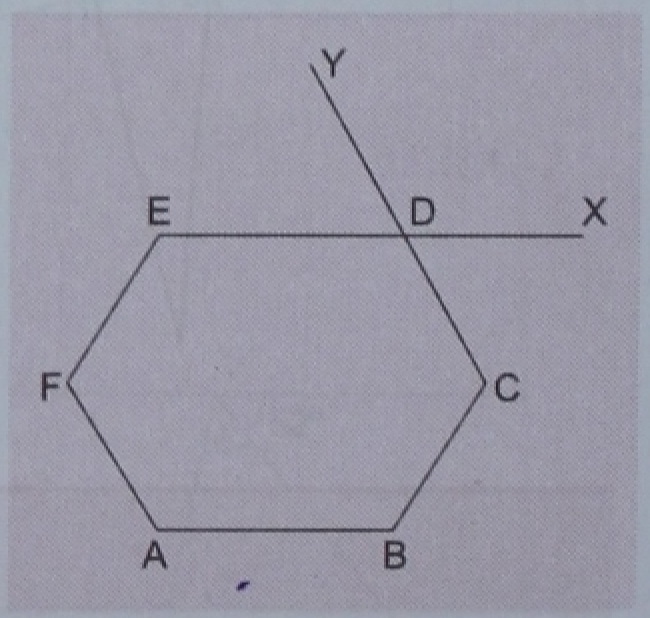
Number of sides	Name
3	Triangle
4	Quadrilateral
5	Pentagon
6	Hexagon
7	Heptagon
8	Octagon
9	Nonagon
10	Decagon

When all the sides of a polygon are equal in length and all its angles are equal in magnitude, it is called a regular polygon.

Examples   An equilateral triangle and a square are regular polygons.

Angles of a polygon

A polygon has as many angles as it has sides. For example, in the adjoining figure, the hexagon $ABCDEF$ has six (interior) angles $\angle A, \angle B, \angle C, \angle D, \angle E$ and $\angle F$.



If we extend the side ED to X, we get the exterior angle CDX at the vertex D. And if we extend the side CD to Y, we get the exterior angle EDY at D, and $\angle CDX =$ vertically opposite $\angle EDY$.

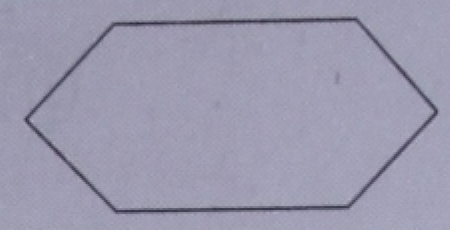
Also, $\angle CDX + \angle CDE = \angle EDY + \angle CDE = 180°$ (straight angle).

We can generalise this for any polygon as follows.

1. At each vertex of a polygon, there are two equal exterior angles.
2. The sum of an exteror angle and adjacent interior angle at a vertex = 180°.

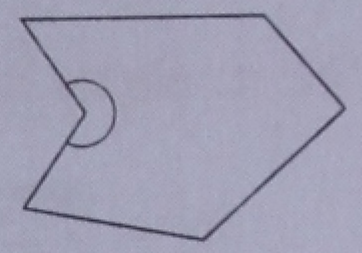
Convex polygon

If all the interior angles of a polygon are less than 180°, the polygon is said to be convex. All regular polygons are convex.


Convex polygon

Concave (or re-entrant) polygon

If one or more of the interior angles of a polygon is greater than 180° (that is, a reflex angle), the polygon is said to be concave.


Concave polygon

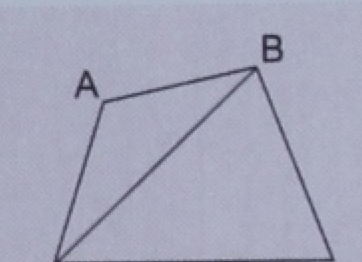
Angle properties of a polygon

PROPERTY 1 The sum of the interior angles of a polygon with n sides $= (2n - 4)$ right angles.

(i) The sum of the (interior) angles of a triangle $= (2 \times 3 - 4)$ right angles $= 2$ right angles $= 180°$, which we know is true.

(ii) The sum of the (interior) angles of a quadrilateral $= (2 \times 4 - 4)$ right angles $= 4$ right angles.

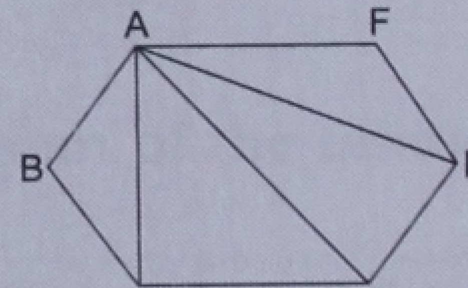
In the adjoining figure, the sum of the angles of the quadrilateral $ABCD$ = the sum of the angles of $\triangle ABD$ + the sum of the angles of $\triangle BCD$ = 2 right angles + 2 right angles = 4 right angles.



So, the property is true for a quadrilateral.

(iii) The sum of the (interior) angles of a hexagon $= (2 \times 6 - 4)$ right angles $= 8$ right angles.

In the adjoining figure, the sum of the angles of the hexagon $ABCDEF$ = the sum of the angles of the triangles ABC, ACD, ADE and AEF $= 4 \times 2$ right angles = 8 right angles.



So, the property holds for a hexagon.

It can be verified easily that it holds for other polygons as well.

EXAMPLE **Find the magnitude of each interior angle of a regular octagon.**

Solution The sum of the eight angles of an octagon $= (2 \times 8 - 4)$ right angles $= 12 \times 90°$.

Now, in a regular polygon, all the angles are equal.

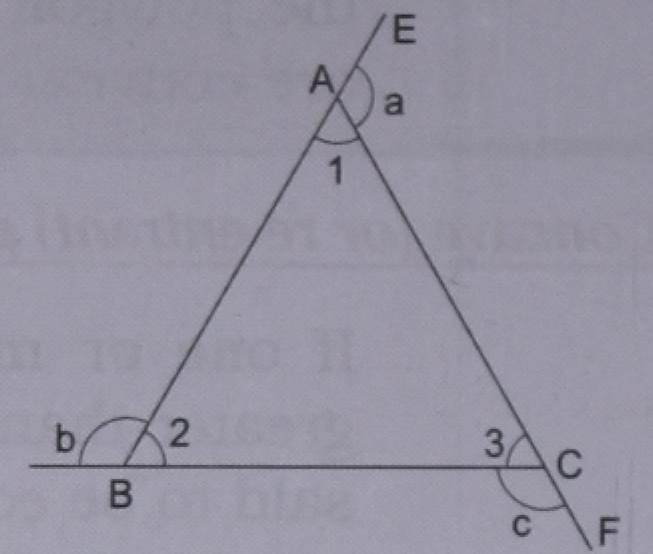
$\therefore$ each angle of a regular octagon $= \dfrac{12 \times 90°}{8} = 135°$. We can generalise this:

> An interior angle of a regular polygon of n sides $= \dfrac{1}{n}(2n - 4)$ right angles

Note When we speak of angle of a polygon we mean interior angle.

PROPERTY 2 The sum of the exterior angles of a convex polygon $= 360°$.

(i) In the adjoining figure,



$\angle a + \angle 1 = 180°$, $\angle b + \angle 2 = 180°$,
$\angle c + \angle 3 = 180°$.

$\therefore \quad \angle a + \angle 1 + \angle b + \angle 2 + \angle c + \angle 3$
$= 180° + 180° + 180° = 540°$

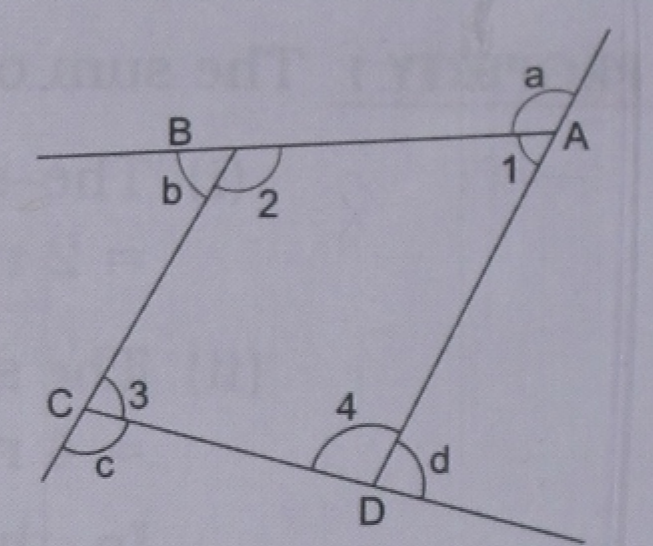
$\Rightarrow \angle a + \angle b + \angle c + (\angle 1 + \angle 2 + \angle 3) = 540°$.

But, $\angle 1 + \angle 2 + \angle 3 = 180°$, being angles of a triangle.

$\therefore \quad \angle a + \angle b + \angle c = 540° - 180° = 360°$.

So, the property is true for a triangle.

(ii) In the adjoining figure,



$\angle a + \angle 1 = 180°$, $\angle b + \angle 2 = 180°$,
$\angle c + \angle 3 = 180°$, $\angle d + \angle 4 = 180°$.

$\therefore \quad \angle a + \angle 1 + \angle b + \angle 2 + \angle c + \angle 3 + \angle d + \angle 4$
$= 180° + 180° + 180° + 180° = 720°$

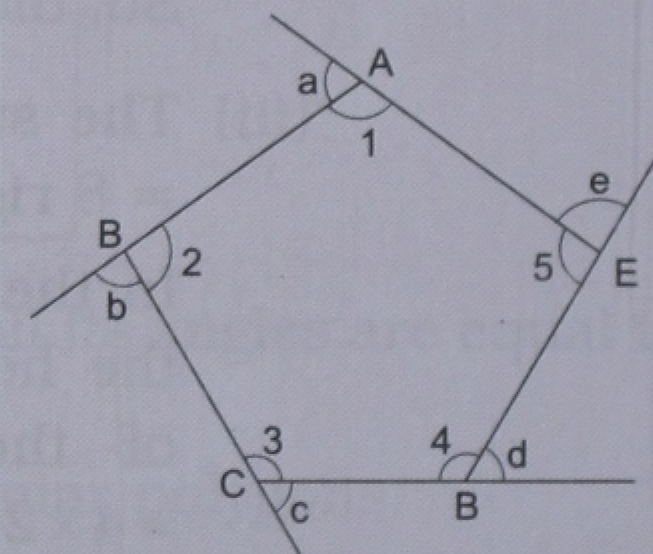
$\Rightarrow \angle a + \angle b + \angle c + \angle d + (\angle 1 + \angle 2 + \angle 3 + \angle 4) = 720°$.

But the sum of the interor angles of a quadrilateral $= 360°$.

$\therefore \quad \angle a + \angle b + \angle c + \angle d = 720° - 360° = 360°$.

So, the property holds for a quadrilateral.

(iii) In the adjoining figure,



$\angle a + \angle 1 = 180°$, $\angle b + \angle 2 = 180°$,
$\angle c + \angle 3 = 180°$, $\angle d + \angle 4 = 180°$,
$\angle e + \angle 5 = 180°$.

$\therefore \quad \angle a + \angle 1 + \angle b + \angle 2 + \angle c + \angle 3 + \angle d + \angle 4$
$+ \angle e + \angle 5 = 5 \times 180° = 900°$.

$\Rightarrow \angle a + \angle b + \angle c + \angle d + \angle e + (\angle 1 + \angle 2 + \angle 3 + \angle 4 + \angle 5) = 900°$.

But the sum of interor angles of a pentagon
$= (2 \times 5 - 4)$ right angles $= 6 \times 90° = 540°$.

$\therefore \quad \angle a + \angle b + \angle c + \angle d + \angle e = 900° - 540° = 360°$.

So, the property is true for a pentagon.

You can easily verify that the property holds for other polygons as well.

EXAMPLE 1 **Find the measure of each exterior angle of a regular hexagon.**

Solution The sum of the exterior angles of a polygon $= 360°$.

The interior angles of a regular poygon are equal, so its exterior angles are also equal.

The number of exterior angles in a hexagon $= 6$.

So, each exterior angle of a regular hexagon $= \frac{360°}{6} = 60°$.

We can generalise this result as follows.

The exterior angle of a regular polygon of n sides $= \frac{360°}{n}$

Also, the number of sides of a regular polygon $= \frac{360°}{\text{an exterior angle}}$

EXAMPLE 2 **Calculate the number of sides of a regular polygon if each exterior angle is 24°.**

Solution The number of sides of the regular polygon $= \frac{360°}{\text{an exterior angle}} = \frac{360}{24} = 15.$

Solved Examples

EXAMPLE 1 **The sum of the interior angles of a polygon is 2340°. How many sides does it have?**

Solution Let the number of sides of the polygon $= n$.

Then the sum of its interior angles $= (2n-4)$ right angles $= 2340°$ (given).

$\therefore \quad (2n-4) \times 90° = 2340° \Rightarrow 2n-4 = \frac{2340}{90} = 26 \Rightarrow 2n = 30.$

$\therefore \quad n = 15$. So, the polygon has 15 sides.

EXAMPLE 2 **Each interior angle of a regular polygon is 168°. How many sides does it have?**

Solution Each exterior angle of the regular polygon $= 180° -$ an interior angle

$= 180° - 168° = 12°.$

$\therefore \quad$ number of sides of the regular polygon $= \frac{360°}{\text{an exterior angle}} = \frac{360}{12} = 30.$

Alternatively

If the number of sides of the regular polygon be n then the sum of the interior angles $= 168° \times n = (2n-4) \times 90°$ (by property)

$\Rightarrow \quad 360° = 180° \times n - 168° \times n \Rightarrow 12° \times n = 360° \Rightarrow n = \frac{360}{12} = 30.$

EXAMPLE 3 **Find the measure of each interior angle of a regular 20-gon.**

Solution Each exterior angle of a regular 20-gon $= \frac{360°}{20} = 18°.$

$\therefore \quad$ each interior angle $= 180° -$ each exterior angle $= 180° - 18° = 162°.$

EXAMPLE 4 **Is it possible to have a polygon in which the sum of the interior angles is 810°?**

Solution Let the number of sides of the polygon $= n$. Given, sum of interior angles $= 810°$.

$\Rightarrow \quad (2n-4) \times 90 = 810 \Rightarrow 2n-4 = \frac{810}{90} = 9 \Rightarrow n = \frac{13}{2} = 6\frac{1}{2}.$

A polygon cannot have $6\frac{1}{2}$ sides. So, such a polygon is not possible.

EXAMPLE 5 **Is it possible to have a regular polygon with exterior angles of 30°?**

Solution $\therefore$ the number of sides of the polygon $= \dfrac{360°}{\text{an exterior angle}} = \dfrac{360}{30} = 12$.

A polygon can have 12 sides. So, such a regular polygon is possible.

EXAMPLE 6 **The angles of a hexagon are in the ratio 3 : 5 : 7 : 9 : 2 : 4. Find the angles.**

Solution Let the angles of the hexagon be $3x, 5x\ 7x, 9x, 2x$ and $4x$.
Then the sum of the angles $= 3x + 5x + 7x + 9x + 2x + 4x = 30x$.
Also, the sum of the angles of the hexagon = $(2 \times 6 - 4)$ right angles $= 8 \times 90°$.

$\therefore \quad 30x = 8 \times 90° \Rightarrow x = \dfrac{8 \times 90°}{30} = 24°$.

$\therefore \quad 3x = 3 \times 24° = 72°, \quad 5x = 5 \times 24° = 120°, \quad 7x = 7 \times 24° = 168°,$
$9x = 9 \times 24° = 216°, \quad 2x = 2 \times 24° = 48°, \quad 4x = 4 \times 24° = 96°.$

Hence, the angles of the hexagon are 72°, 120°, 168°, 216°, 48° and 96°.

EXAMPLE 7 **If each interior angle of a regular polygon is eleven times an exterior angle, find the number of sides of the polygon.**

Solution Let an exterior angle $= x$. Then an interior angle $= 11x$.

$\therefore \quad x + 11x = 180° \Rightarrow 12x = 180° \Rightarrow x = \dfrac{180}{12} = 15°$.

$\therefore$ an exterior angle = 15°.

$\therefore$ number of sides of the polygon $= \dfrac{360°}{\text{an exterior angle}} = \dfrac{360}{15} = 24$.

EXAMPLE 8 **Four of the angles of a pentagon are equal and the fifth is 20° greater than each of the equal angles. Find the angles.**

Solution Let each of the four equal angles $= x$. Then the fifth angle $= x + 20°$.

$\therefore$ the sum of the angles of the pentagon $= 4x + (x + 20°) = 5x + 20°$

$\Rightarrow \quad (2 \times 5 - 4)$ right angles $= 5x + 20° \Rightarrow 6 \times 90° = 5x + 20°$

$\Rightarrow \quad 5x = 540° - 20° = 520°$.

So, $x = \dfrac{520°}{5} = 104°$.

Thus, each of the four equal angles = 104° and the fifth angle $= 104° + 20° = 124°$.

EXAMPLE 9 **Five angles of a polygon are 172° each. The remaining angles are 160° each. Calculate the number of sides of the polygon.**

Solution Let the number of sides of the polygon $= n$.
Then the sum of the angles of the polygon $= 5 \times 172° + (n - 5) \times 160°$
$= (2n - 4)$ right angles $= (2n - 4) \times 90°$.

$\therefore \quad (2n - 4) \times 90 = 5 \times 172 + (n - 5) \times 160 \Rightarrow 180n - 360 = 860 + 160n - 800$

$\Rightarrow \quad 180n - 160n = 860 - 800 + 360 = 420 \Rightarrow 20n = 420 \Rightarrow n = 21$.

Hence, the polygon has 21 sides.

Remember These

1. At a vertex of a polygon, exterior angle + adjacent interior angle = 180°.

2. The sum of the interior angles of a polygon of n sides $= (2n - 4)$ right angles $= (2n-4) \times 90°$.

3. An interior angle of a regular n-gon $= \frac{2n-4}{n}$ right angles.

4. The sum of the exterior angles of any convex polygon $= 360°$.

5. An exterior angle of a regular n-gon $= \frac{360°}{n}$.

6. The number of sides of a regular n-gon $= \frac{360°}{\text{an exterior angle}}$.

EXERCISE 3

1. Find the sum of the interior angles of a polygon of
(i) 7 sides (ii) 9 sides (iii) 10 sides (iv) 14 sides

2. Find the magnitude of each interior angle of a regular
(i) hexagon (ii) heptagon (iii) 12-gon (iv) 15-gon

3. Find the number of sides of a polygon if the sum of its angles is
(i) 2520° (ii) 3240° (iii) 20 straight angles (iv) 20 right angles

4. Find the number of sides of a regular polygon if each interior angle is
(i) 108° (ii) 144° (iii) 168° (iv) 172°
(v) $171\frac{3}{7}°$

5. Find the number of sides of a regular polygon if each exterior angle is
(i) 15° (ii) 30° (iii) 120° (iv) 90°
(v) $22\frac{1}{2}°$

6. Is it possible to have a polygon in which the sum of the interior angles is
(i) 1890° (ii) 2250° (iii) 3690° (iv) 3780°

7. Can a regular polygon have interior angles of
(i) 40° (ii) 110° (iii) 135° (iv) 148°

8. Can a regular polygon have exterior angles of
(i) 54° (ii) 100° (iii) 48° (iv) $27\frac{9}{13}°$

9. (i) The angles of a quadrilateral are in the ratio 7 : 5 : 9 : 15. Find the smallest angle.
(ii) The interior angles of a pentagon are in the ratio 4 : 5 : 11 : 13 : 12. Find the largest angle.

10. The angles of a hexagon are $x+10°$, $2x+20°$, $2x-20°$, $3x-50°$, $x+40°$ and $x+20°$. Find x.

11. Calculate the number of sides of a regular polygon if
(i) an interior angle is five times an exterior angle
(ii) the ratio of an exterior angle to an interior angle is 2 : 7
(iii) an exterior angle exceeds an interior angle by 60°

12. In an octagon, four of the angles are equal and each of the others is 20° greater than each of the first four. Find the angles.

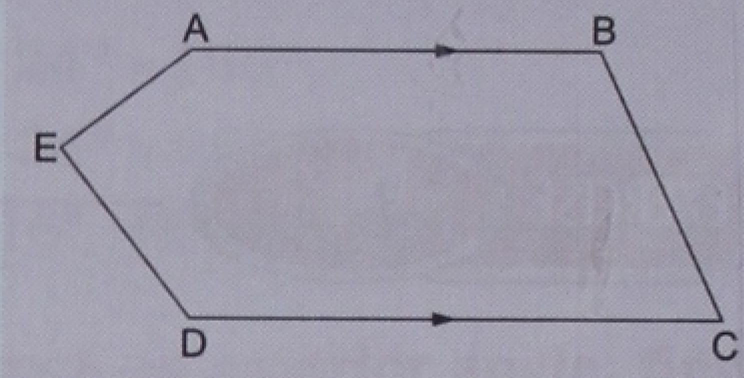
13. In a pentagon, two angles are 40° and 60°, and the rest are in the ratio 1 : 3 : 7. Find the biggest angle of the pentagon.

14. A heptagon has two equal angles of 120° and five other equal angles. Find the equal angles.

15. (i) Three angles of a polygon are 80° each. The remaining angles are 160° each. Calculate the number of sides of the polygon.

(ii) Four angles of a polygon are 120° each. The remaining angles are 150° each. Calculate the number of sides of the polygon.

16. In the adjoining figure, $AB \parallel DC$, $\angle B = 3\angle C$ and $\angle A : \angle E : \angle D = 2 : 3 : 4$. Find the angles.



ANSWERS

1. (i) 900° (ii) 1260° (iii) 1440° (iv) 2160° **2.** (i) 120° (ii) $128\frac{4}{7}°$ (iii) 150° (iv) 156°

3. (i) 16 (ii) 20 (iii) 22 (iv) 12 **4.** (i) 5 (ii) 10 (iii) 30 (iv) 45 (v) 42

5. (i) 24 (ii) 12 (iii) 3 (iv) 4 (v) 16 **6.** (i) No (ii) No (iii) No (iv) Yes

7. (i) No (ii) No (iii) Yes (iv) No **8.** (i) No (ii) No (iii) No (iv) Yes

9. (i) 50° (ii) 156° **10.** 70° **11.** (i) 12 (ii) 9 (iii) 3 **12.** 125°, 145°

13. 280° **14.** 132° **15.** (i) 6 (ii) 8

16. $\angle A = 80°$, $\angle B = 135°$, $\angle C = 45°$, $\angle D = 160°$, $\angle E = 120°$

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