Chapter 22

TRIANGLES

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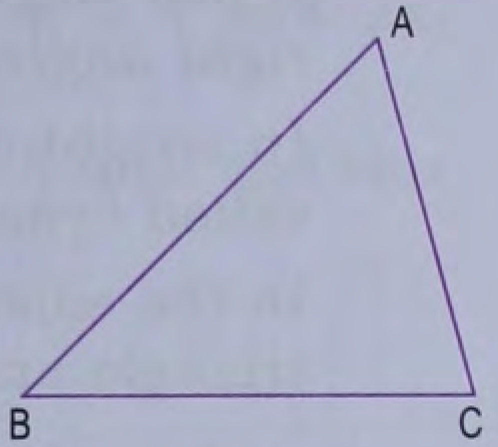
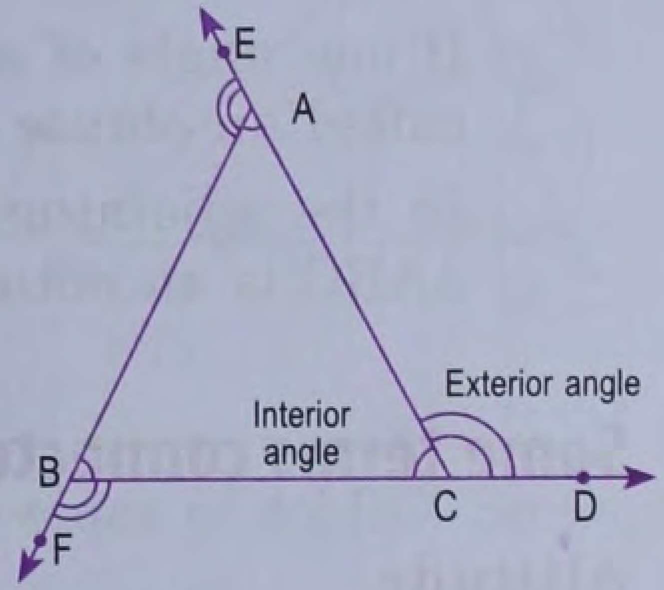
*A **triangle** is a closed figure bounded by three line segments.*



The adjoining figure shows a triangle ABC. The line segments AB, BC and CA are called its *sides* and the angles CAB, ABC and BCA are called its *interior angles* or simply the *angles*. The points A, B and C are called its *vertices*. Usually, the triangle ABC is written as Δ ABC. The three sides and the three angles of a triangle are called its *six elements*.

Exterior angles of a triangle



Let ABC be a triangle and its sides BC, CA and AB be produced to D, E and F respectively, then ∠ACD, ∠EAB and ∠FBC are called *exterior angles* at C, A and B respectively. The two interior angles of ΔABC that are opposite to the exterior ∠ACD are called its *interior opposite angles*. Thus, ∠A and ∠B are interior opposite angles of exterior ∠ACD.

Notice that ∠BCA and ∠ACD form a linear pair, so ∠BCA + ∠ACD = 180°. Thus, we have :

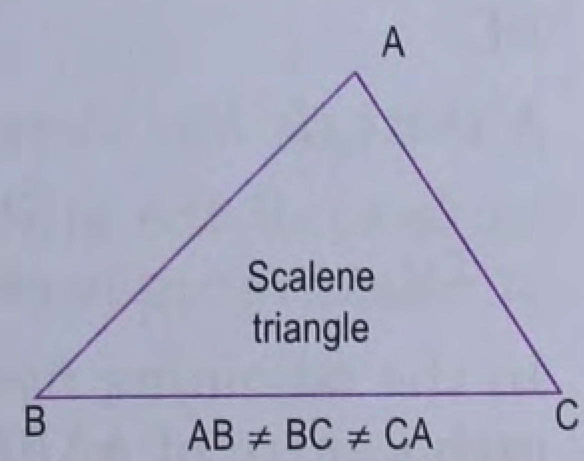
An exterior angle + adjacent interior angle = 180°

Classification of triangles on the basis of sides

(*i*) **Scalene triangle**

If all the three sides of a triangle are unequal, it is called a *scalene triangle*.

In the adjoining figure, AB ≠ BC ≠ CA, so ΔABC is a scalene triangle.

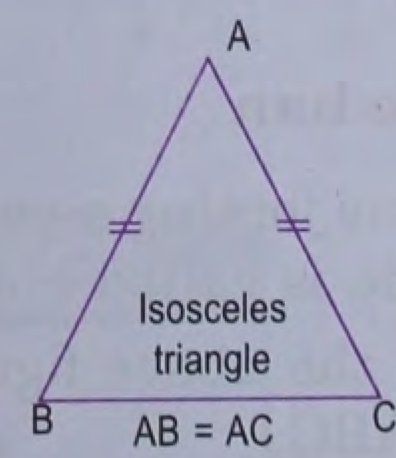


(*ii*) **Isosceles triangle**

If any two sides of a triangle are equal, it is called an *isosceles triangle*.

In the adjoining figure, AB = AC, so ΔABC is an isosceles triangle.

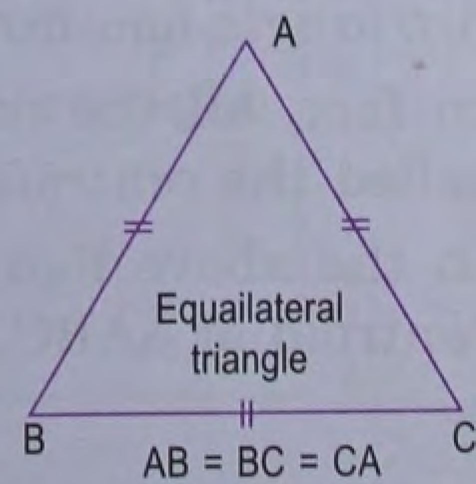
Usually, equal sides are indicated by putting marks on each of them.



(*iii*) **Equilateral triangle**

If all the three sides of a triangle are equal, it is called an *equilateral triangle*.

In the adjoining figure, AB = BC = CA, so ΔABC is an equilateral triangle.

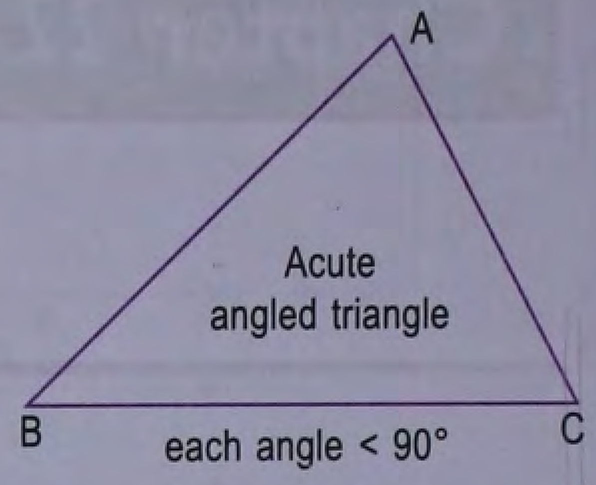


Classification of Triangles On the Basis of Angles

(*i*) **Acute angled triangle**

If all the three angles of a triangle are acute (less than 90°), it is called an *acute angled triangle*.

In the adjoining figure, each angle is less than 90°, so ΔABC is an acute angled triangle.

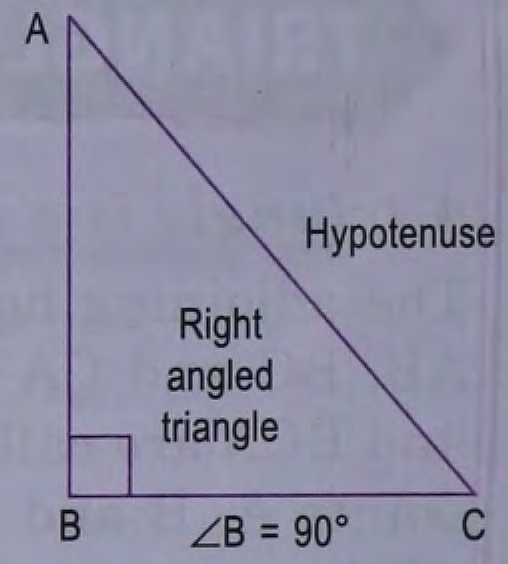


(*ii*) **Right angled triangle**

If one angle of a triangle is right angle (= 90°), it is called a *right angled triangle*.

In a right angled triangle, the side opposite to right angle is called *hypotenuse*.

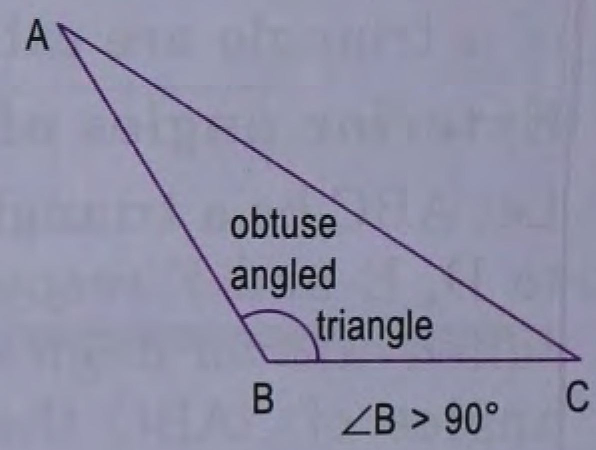
In the adjoining figure, $\angle B = 90°$, so ΔABC is a right angled triangle and side AC is the hypotenuse.



(*iii*) **Obtuse angled triangle**

If one angle of a triangle is obtuse (greater than 90°), it is called an *obtuse angled triangle*.

In the adjoining figure, $\angle B$ is obtuse (greater than 90°), so ΔABC is an obtuse angled triangle.



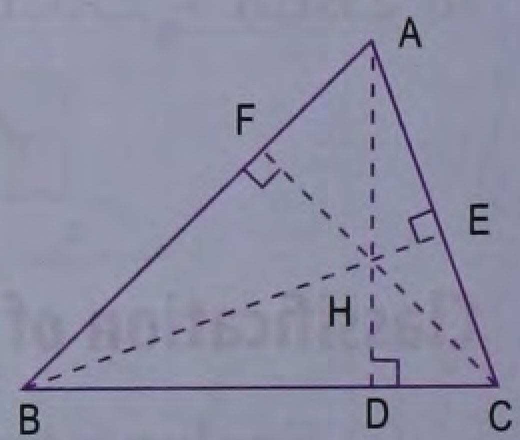
Some terms connected with a triangle

Altitude

Perpendicular from a vertex of a triangle to the opposite side is called an *altitude* of the triangle.

In the adjoining figure, $AD \perp BC$, so AD is an altitude to the side BC.

A triangle has three altitudes.



In fact, all the altitudes pass through the same point and the point of concurrence is called the *orthocentre* of the triangle.

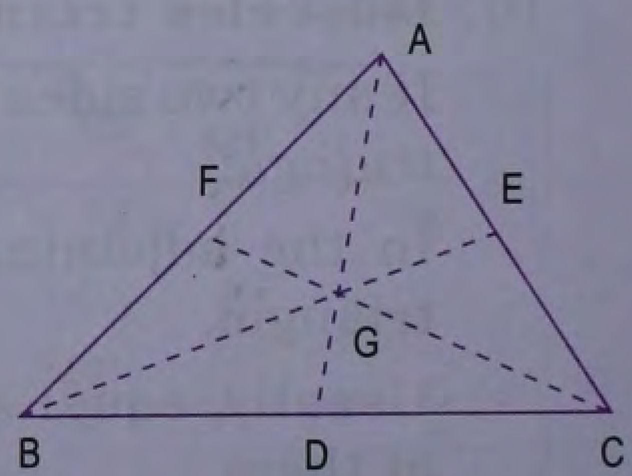
In the adjoining figure, AD, BE and CF are the altitudes of ΔABC and the point H is the orthocentre of ΔABC.

Median

Line joining a vertex of a triangle to the mid-point of the opposite side is called a *median* of the triangle.

In the above figure, D is mid-point of BC, so AD is a median of ΔABC.

A triangle has three medians.

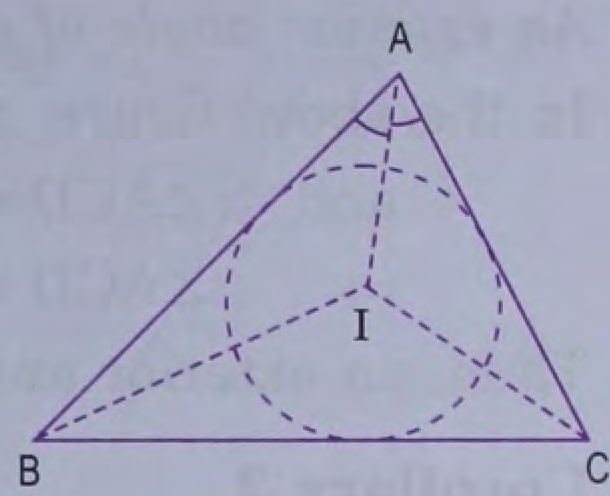


In fact, all the medians pass through the same point and the point of concurrence is called the *centroid* of the triangle.

In the above figure, AD, BE and CF are the medians of ΔABC and the point G is the centroid of ΔABC.

Incentre and incircle

Line bisecting an (interior) angle of a triangle is called the (internal) *bisector* of the angle of the triangle.



In the adjoining figure, ∠BAI = ∠IAC, so AI is the (internal) bisector of ∠A.

A triangle has three internal bisectors of its angles.

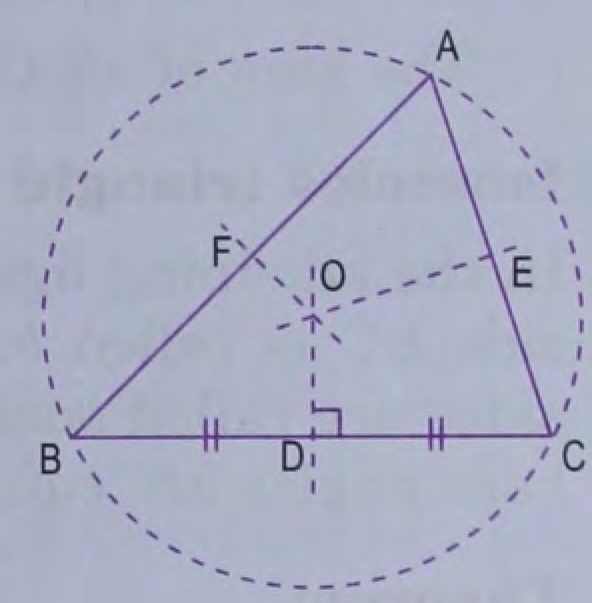
In fact, all the (internal) bisectors of the angles of a triangle pass through the same point and the point of concurrence is called the *incentre* of the triangle.

In the above figure, IA, IB and IC are the (internal) bisectors of ∠A, ∠B and ∠C respectively of ΔABC. So I is the incentre of ΔABC.

Moreover, incentre is the centre of a circle which touches all the sides of Δ ABC and this circle is called *incircle* of ΔABC.

Circumcentre and circumcircle

Line bisecting a side of a triangle and perpendicular to it is called the *right bisector* of the side of the triangle.



In the adjoining figure, D is mid-point of BC and OD ⊥ BC, so OD is the right bisector of the side BC.

A triangle has three right bisectors of its sides.

In fact, all right bisectors of the sides of a triangle pass through the same point and the point of concurrence is called the *circumcentre* of the triangle.

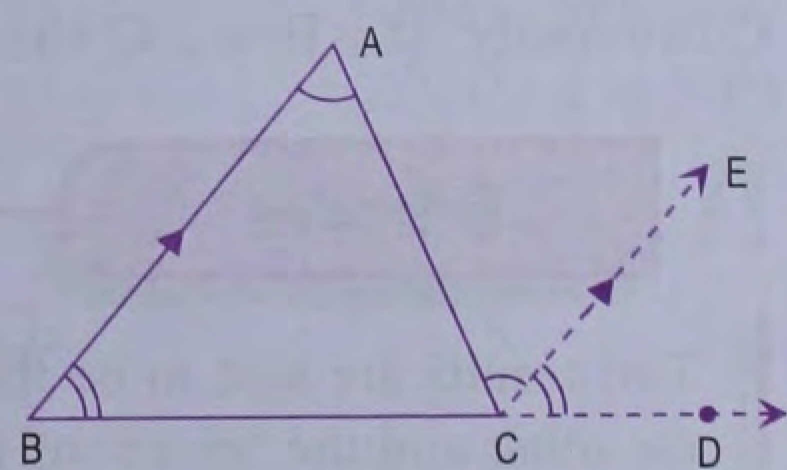
In the above figure, OD, OE and OF are the right bisectors of the sides of ΔABC. So O is the circumcentre of ΔABC.

Moreover, circumcentre is the centre of a circle which passes through the vertices of ΔABC and this circle is called *circumcircle* of ΔABC.

Angles Property of a Triangle

Theorem

The sum of angles of a triangle is 180°.



Given. A triangle ABC.

To prove. ∠A + ∠B + ∠C = 180°

Construction. Produce BC to D and through C draw a line CE parallel to BA.

Proof.

Statements	*Reasons*
1. ∠ACE = ∠A	1. CE ∥ BA, alt. ∠*s* are equal
2. ∠ECD = ∠B	2. CE ∥ BA, corres. ∠*s* are equal
3. ∠C + ∠ACE + ∠ECD = 180°	3. BCD is a straight line, sum of angles at a point on one side of a straight line = 180°
4. ∠C + ∠A + ∠B = 180° **Q.E.D.**	4. Using 1 and 2

Corollary 1

An exterior angle of a triangle = sum of its two interior opposite angles.

In the above figure, notice that ∠ACD is an exterior angle of Δ ABC

and ∠ACD = ∠ACE + ∠ECD

⇒ ∠ACD = ∠A + ∠B.

Thus, an exterior angle = sum of its two interior opposite angles.

Corollary 2

An exterior angle of a triangle is greater than either of the two interior opposite angles.

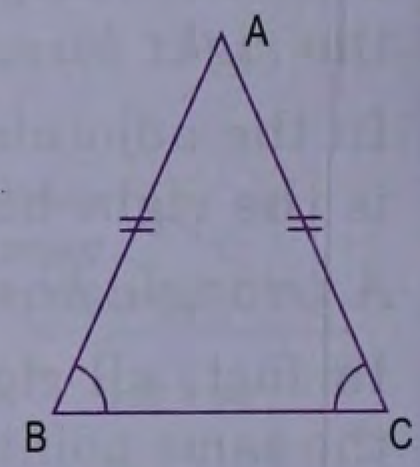
In the above figure, ∠ACD = ∠A + ∠B, therefore, ∠ACD being the sum of ∠A and ∠B is greater than ∠A as well as ∠B.

Corollary 3

If two angles of a triangle are equal to two angles of another triangle, then the third angles of both the triangles are also equal.

(∵ The sum of all the three angles of a triangle is 180°.)

Isosceles triangle



In the adjoining figure, AB = AC, so ΔABC is an isosceles triangle. The side BC is called *base* of ΔABC. The angles which are opposite equal sides are called *base angles* and ∠A is called *vertical angle*. In fact, the base angles are equal in size *i.e.* ∠B = ∠C.

Theorem

If two sides of a triangle are equal, then the angles opposite to them are also equal.

(For proof, see example 6 page 271)

Converse is also true.

If two angles of a triangle are equal, then the sides opposite to them are also equal.

(For proof, see example 7 page 271)

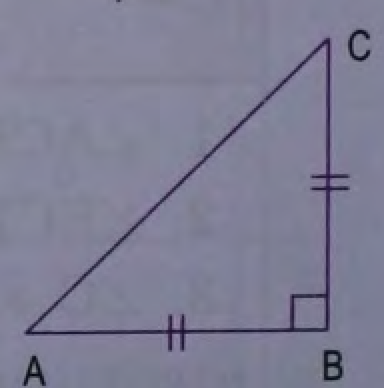
In the above figure, AC = AB, so ∠B = ∠C.

Conversely, if ∠B = ∠C then AC = AB.

Remark

Two results are said to be the converse of each other if the 'given' of one is the 'conclusion' of the other and the 'given' of the other is the 'conclusion' of the first.

Isosceles right angled triangle



In the adjoining diagram, AB = BC and ∠B = 90°, so ΔABC is an isosceles right angled triangle.

As BC = AB, ∠A = ∠C

(angles opposite equal sides are equal)

Since sum of angles of a triangle is 180°,

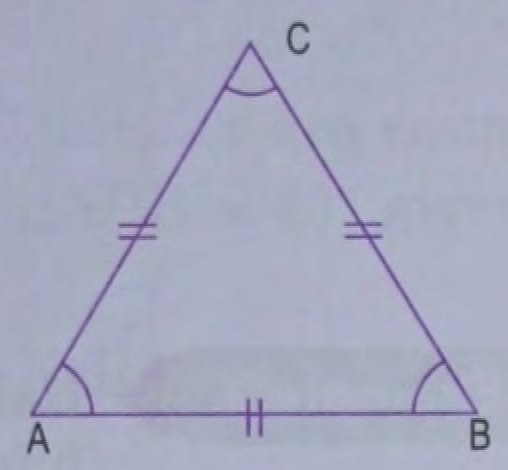
∠A + ∠C + 90° = 180°

⇒ ∠A + ∠C = 180° – 90° = 90°

But ∠A = ∠C, it follows that ∠A = ∠C = 45°

Hence, the angles of an isosceles right angled triangle are 45°, 45° and 90°.

Equilateral triangle



In the adjoining figure, AB = BC = CA, so ΔABC is an equilateral triangle.

As BC = CA = AB, $\angle A = \angle B = \angle C$

(angles opposite equal sides are equal)

Since sum of angles of a triangle is 180°,

$\angle A + \angle B + \angle C = 180°$.

But $\angle A = \angle B = \angle C$, it follows that

$\angle A = \angle B = \angle C = 60°$

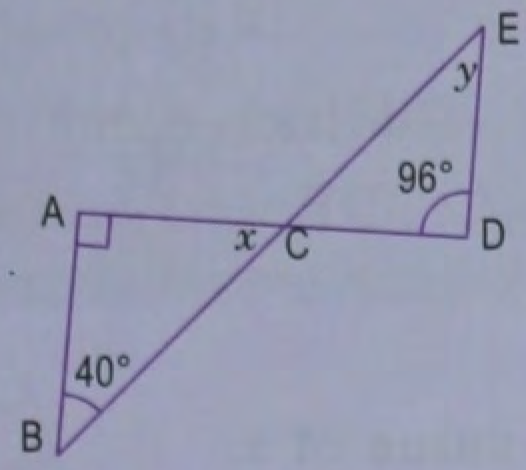
Thus, we have :

In an equilateral triangle, all the three angles are equal and each angle = 60°.

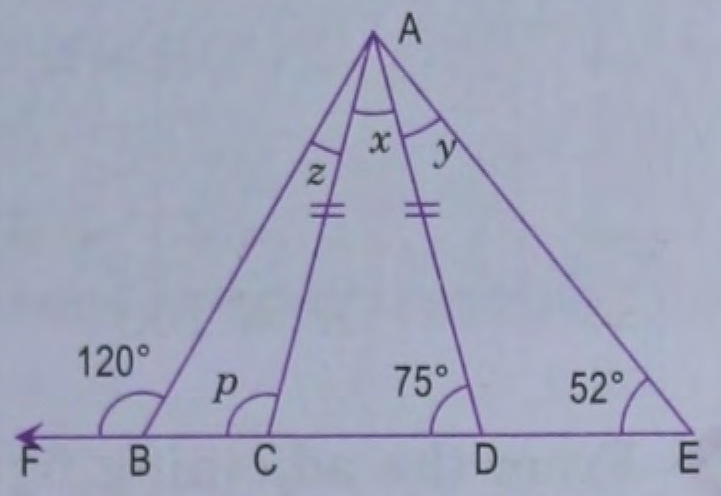
Converse is also true.

In a triangle, if all the three angles are equal (each = 60°) then it is an equilateral triangle.

Example 1. Calculate the measure of each lettered angle in the following figures:



(*i*)



(*ii*)

Solution.

State which geometrical fact is used to calculate angle

(*i*) In ΔABC, sum of angles = 180°,

$\therefore\ x + 90° + 40° = 180°$

$\Rightarrow\ x = 180° - 90° - 40° = 50°$

$\angle ECD = x$ (vert. opp. $\angle s$)

$\Rightarrow\ \angle ECD = 50°$ ($\because x = 50°$)

In Δ CDE, sum of angles = 180°,

$\therefore\ 50° + 96° + y = 180°$

$\Rightarrow\ y = 180° - 50° - 96° = 34°$.

(*ii*) In Δ ACD, AD = AC (given)

$\Rightarrow\ \angle ACD = \angle ADC$ (angles opposite equal sides are equal)

$\Rightarrow\ \angle ACD = 75°$ ($\because \angle ADC = 75°$ given)

$\therefore\ x + 75° + 75° = 180°$ (sum of angles in a triangle = 180°)

$\Rightarrow\ x = 180° - 75° - 75° = 30°$.

For ΔADE, $\angle ADC$ is an exterior angle.

$\therefore\ 75° = y + 52°$ (exterior angle = sum of two int. opp. $\angle s$)

$\Rightarrow\ y = 75° - 52° = 23°$.

As BCD is a straight line, $\angle BCA$ and $\angle ACD$ form linear pair.

$\therefore p + \angle ACD = 180°$

$\Rightarrow\ p + 75° = 180°$ ($\because \angle ACD = 75°$, proved above)

$\Rightarrow\ p = 180° - 75° = 105°$.

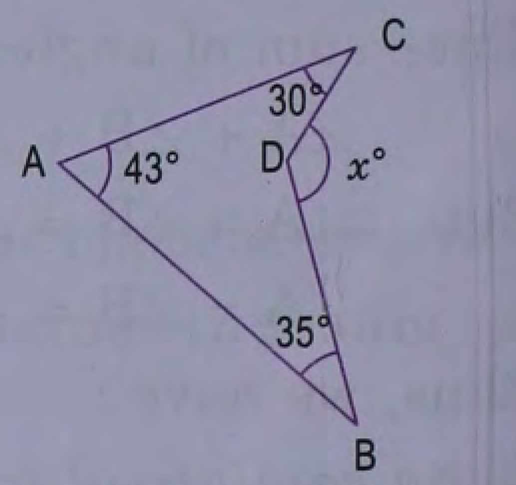
For ΔABC, $\angle ABF$ is an exterior angle.

$\therefore \quad 120° = z + p$ (ext. $\angle$ = sum of two int. opp. $\angle s$)

$\Rightarrow \quad 120° = z + 105°$ ($\because p = 105°$)

$\Rightarrow \quad z = 120° - 105° = 15°$.

Example 2. Using the information given in the adjoining figure, find the value of x.

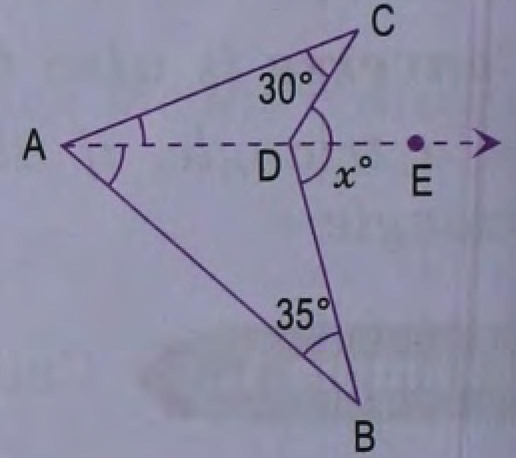


Solution. Join AD and produce it to E.

For ΔCAD, $\angle CDE$ is an exterior angle,

$\therefore \quad \angle CDE = \angle CAD + 30°$ ...(*i*)

(ext. $\angle$ = sum of two int. opp. $\angle s$)



For ΔBAD, $\angle EDB$ is an exterior angle,

$\therefore \quad \angle EDB = \angle DAB + 35°$ ...(*ii*)

(ext. $\angle$ = sum of two int. opp. $\angle s$)

On adding (*i*) and (*ii*), we get

$\angle CDE + \angle EDB = \angle CAD + \angle DAB + 30° + 35°$

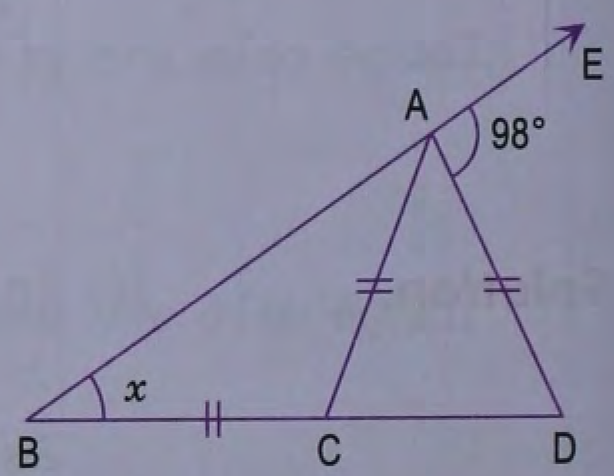
$\Rightarrow \quad \angle CDB = \angle CAB + 65°$

(Addition axiom of adjacent angles)

$\Rightarrow \quad x° = 43° + 65°$ ($\angle CDB = x°$ and $\angle CAB = 43°$ given)

$\Rightarrow \quad x° = 108° \quad \Rightarrow \quad x = 108$.

Example 3. From the adjoining figure, find the value of x.



Solution. In Δ ABC, BC = AC (given)

$\therefore \quad \angle BAC = x$

(angles opposite equal sides are equal)

For Δ ABC, $\angle ACD$ is an exterior angle

$\therefore \quad \angle ACD = x + x$ (ext. $\angle$ = sum of two int. opp. $\angle s$)

$\Rightarrow \quad \angle ACD = 2x$

In Δ ACD, AC = AD (given)

$\therefore \quad \angle ADC = 2x$ (angles opposite equal sides are equal)

$\Rightarrow \quad \angle ADB = 2x$

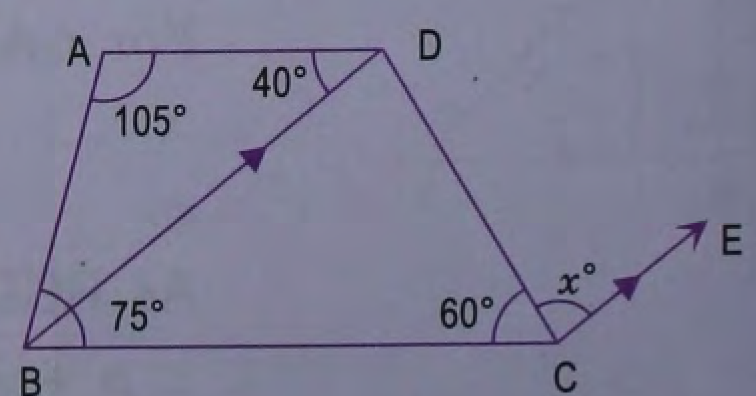
For Δ ABD, $\angle EAD$ is an exterior angle

$\therefore \quad \angle EAD = \angle ABD + \angle ADB$ (ext. $\angle$ = sum of two int. opp. $\angle s$)

$\Rightarrow \quad 98° = x + 2x \Rightarrow 3x = 98°$

$$\Rightarrow \quad x = \left(\frac{98}{3}\right)^{\circ} = 32°\ 40'.$$

Example 4. From the adjoining diagram, find the value of x. Also assign a special name to quadrilateral ABCD.



Solution. In the given figure,

$\angle A + \angle B = 105° + 75° = 180°$

Notice that the transversal AB cuts two lines AD and BC such that

$\angle A + \angle B = 180°$ *i.e.* the sum of co-interior angles = 180°, therefore, the lines AD and BC are parallel lines.

$\angle DBC = \angle ADB$ (AD || BC, alt. $\angle s$ are equal)

$\Rightarrow \quad \angle DBC = 40°$ ($\because \angle ADB = 40°$ given)

Also BD and CE are parallel (given),

$\therefore \quad \angle BDC = x°$ (alt. $\angle s$ are equal)

In Δ BDC, sum of angles = 180°,

$\therefore \quad \angle BDC + \angle DBC + 60° = 180°$

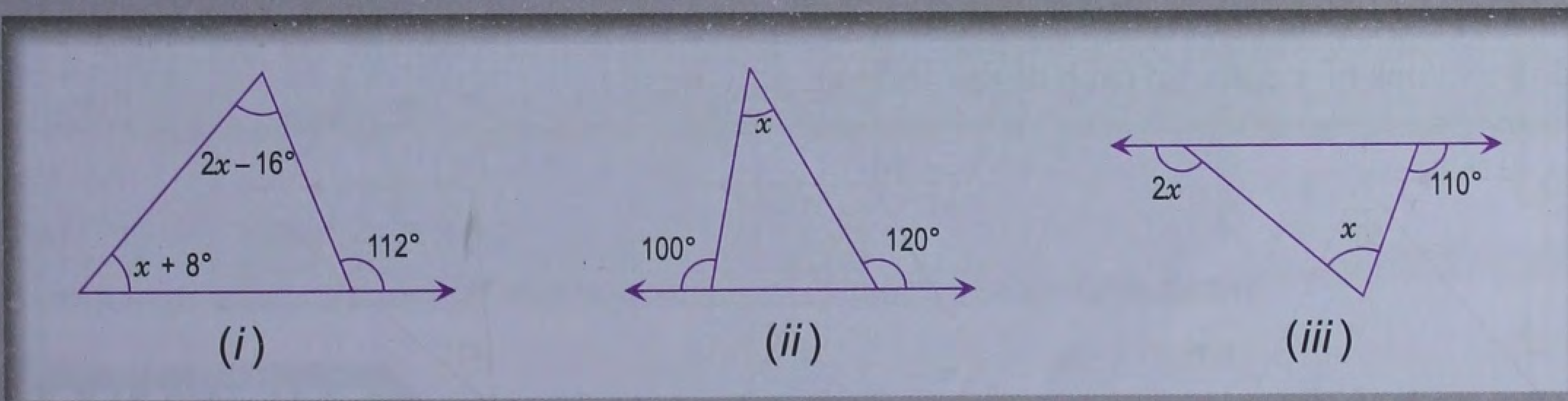
$\Rightarrow \quad x° + 40° + 60° = 180°$

$\Rightarrow \quad x° = 180° - 60° - 40° = 80° \quad \Rightarrow \quad x = 80.$

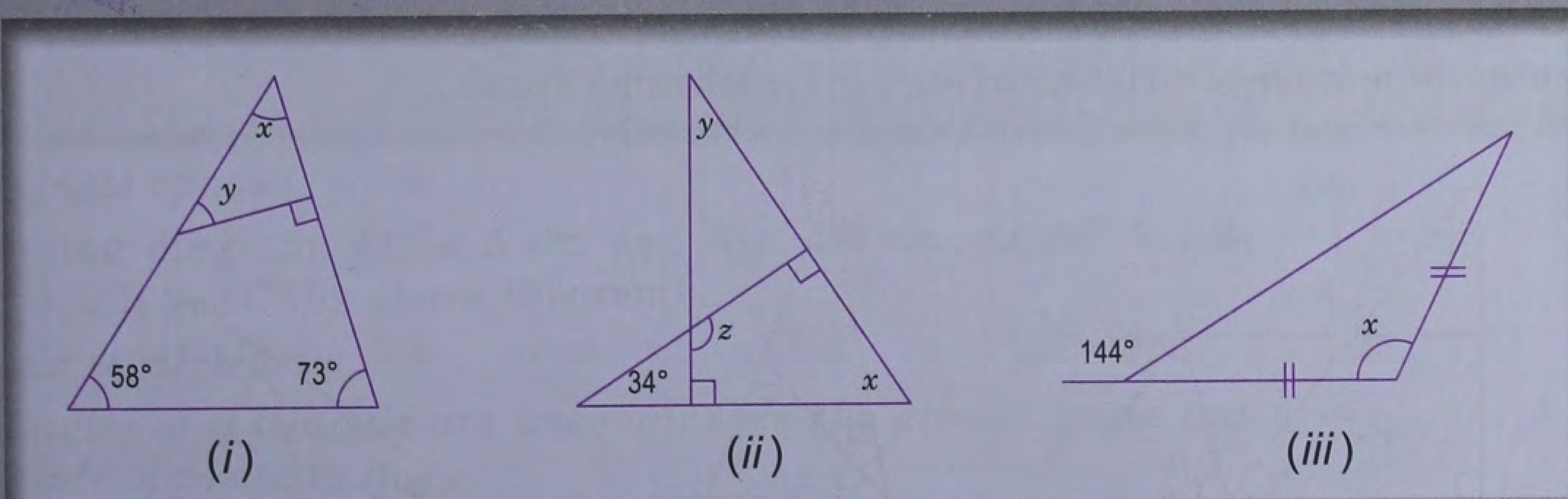
Since AD and BC are parallel, ABCD is a trapezium.

Exercise 22.1

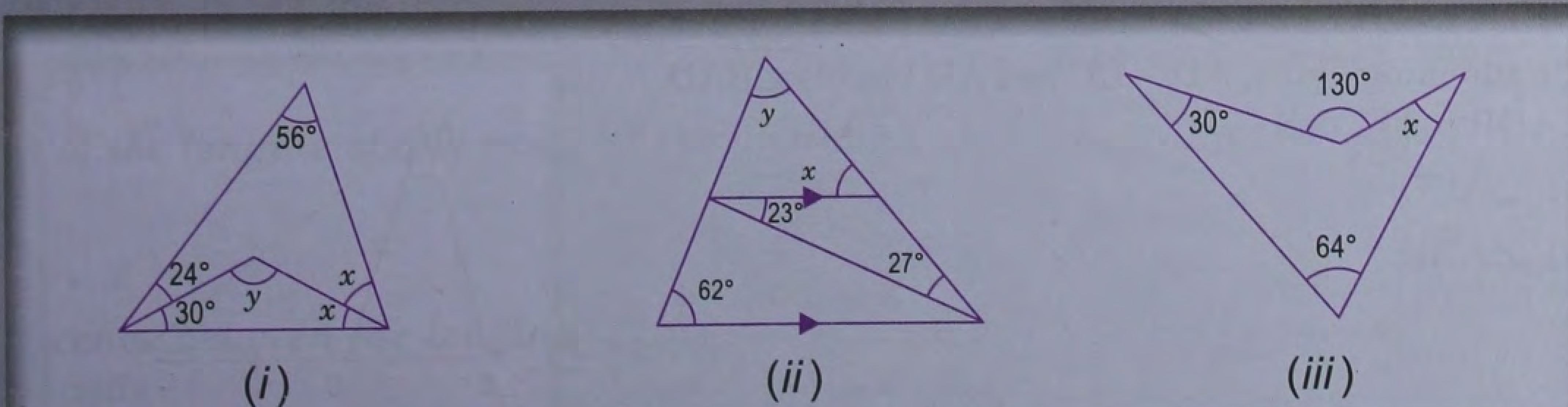
1. Fill in the following blanks :
   (*i*) In a triangle, angles opposite equal sides are ...
   (*ii*) In a triangle, sides opposite equal angles are ...
   (*iii*) The angles of an isosceles right angled triangle are ..., ..., ...
   (*iv*) Each angle of an equilateral triangle = ... degrees.
2. Find the value of x in each of the following figures :



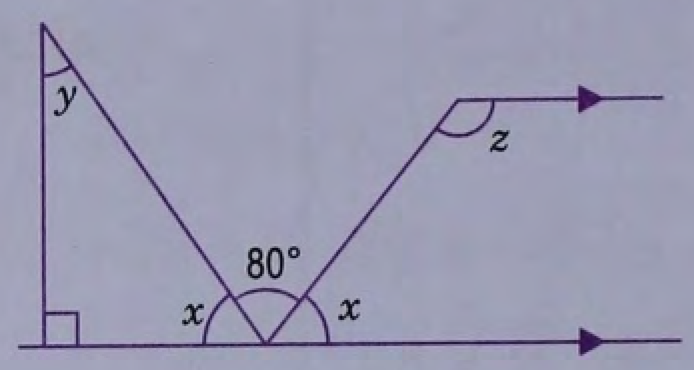
3. Calculate the size of each lettered angle in the following figures :



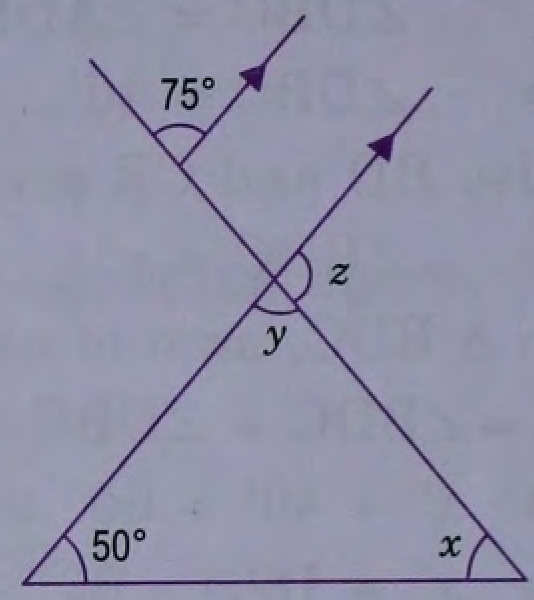
4. Calculate the size of each lettered angle in the following figures :



5. Calculate the size of each lettered angle in the following figures :

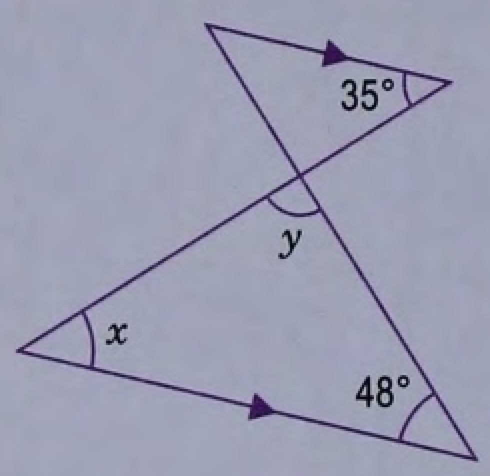


(i)

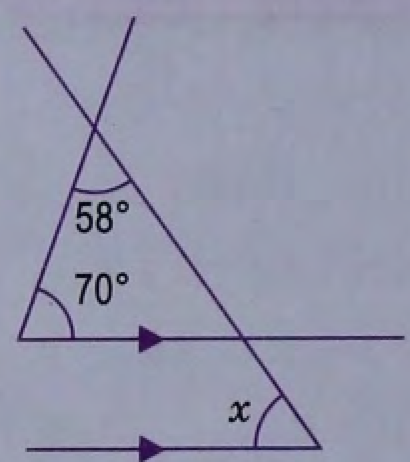


(ii)

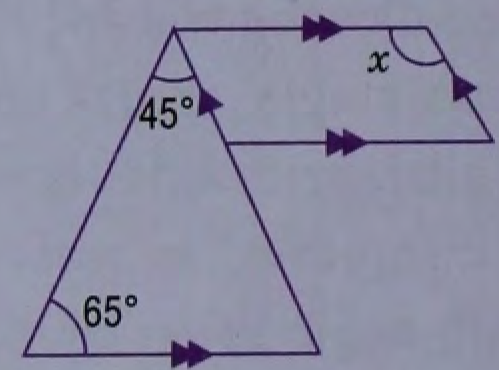
6. Calculate the size of each lettered angle in the following figures :



(i)

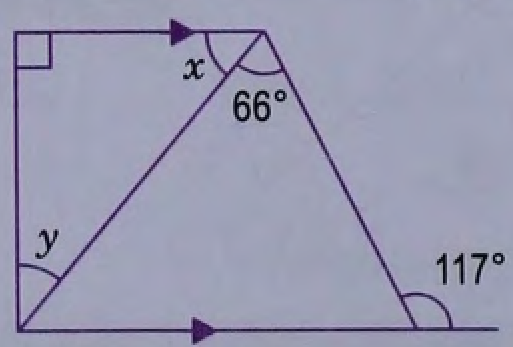


(ii)

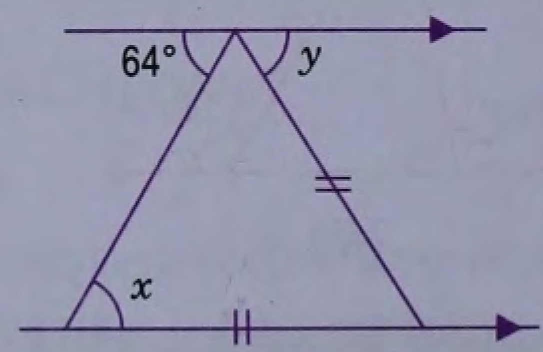


(iii)

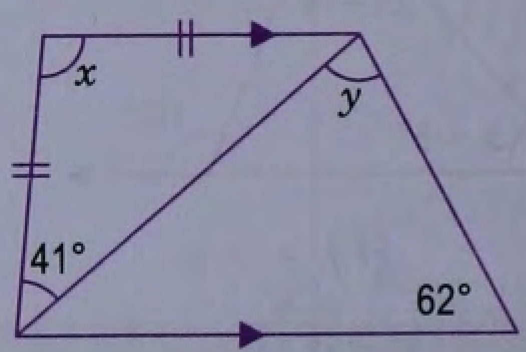
7. Find the values of x and y in each of the following figures :



(i)

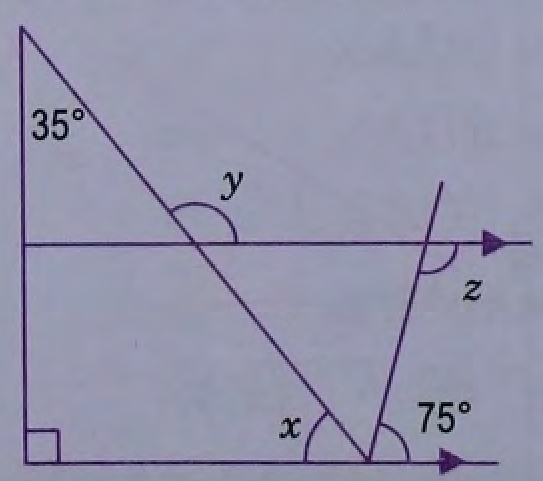


(ii)

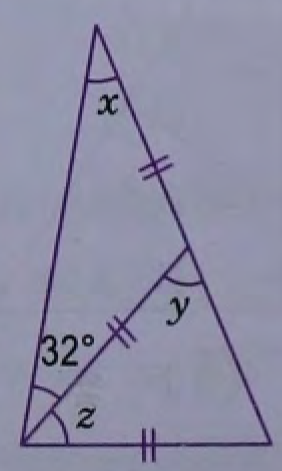


(iii)

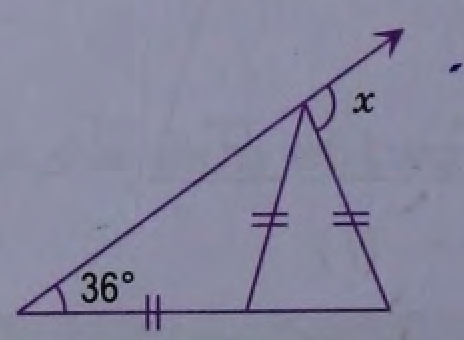
8. Calculate the measure of each lettered angle in the following figures :



(i)



(ii)

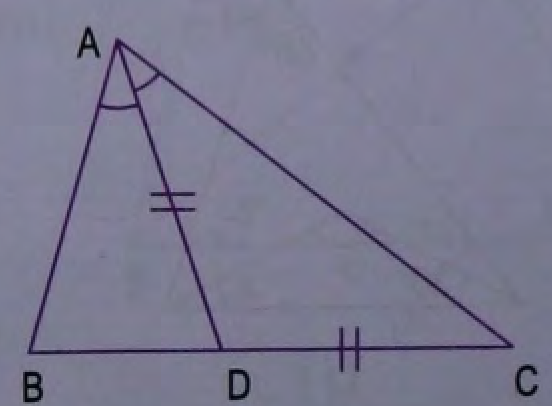


(iii)

9. In the adjoining figure, AD = DC and AD bisects ∠BAD. If ∠ADB = 70°, find :

(i) ∠ACD

(ii) ∠ABD.



10. If the angles of a triangle are in the ratio 9 : 10 : 11, find the angles.

11. If the two acute angles of a right angled triangle are in the ratio 11 : 4, find these angles.

12. In a ΔABC, the bisectors of $\angle B$ and $\angle C$ meet at the point I. Prove that $\angle BIC = 90° + \frac{1}{2} \angle A$.

[Hint. Since BI and CI are bisectors of $\angle B$ and $\angle C$ respectively,

$\angle IBC = \frac{1}{2} \angle B$ and $\angle ICB = \frac{1}{2} \angle C$

In ΔIBC, $\angle IBC + \angle ICB + \angle BIC = 180°$

$\Rightarrow \frac{1}{2} \angle B + \frac{1}{2} \angle C + \angle BIC = 180°$

$\Rightarrow \angle BIC = 180° - \frac{1}{2} (\angle B + \angle C) = 180° - \frac{1}{2} (180° - \angle A)$

because in ΔABC, $\angle A + \angle B + \angle C = 180°$.**]**

13. If the exterior angles of a triangle are $(3x + 5)°$, $(2x + 27)°$ and $(3x - 24)°$, find the value of x.

[Hint. The interior angles are $180° - (3x + 5)°$, $180° - (2x + 27)°$ and $180° - (3x - 24)°$.**]**

14. In an isosceles triangle, a base angle is four times its vertical angle. Find all the angles of the triangle.

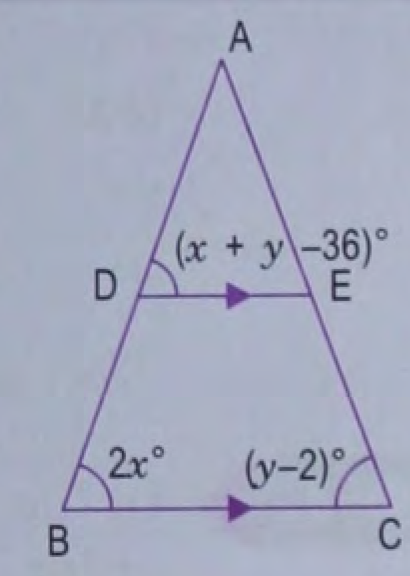
15. In an isosceles triangle, the vertical angle is 15° more than each of its base angles. Find the angles of the triangle.

16. The ratio between the vertical angle and a base angle of an isosceles triangle is 4 : 3. Find all the angles of the triangle.

17. In the adjoining ΔABC, AB = AC and DE || BC. Find

(*i*) x (*ii*) y (*iii*) $\angle BAC$.

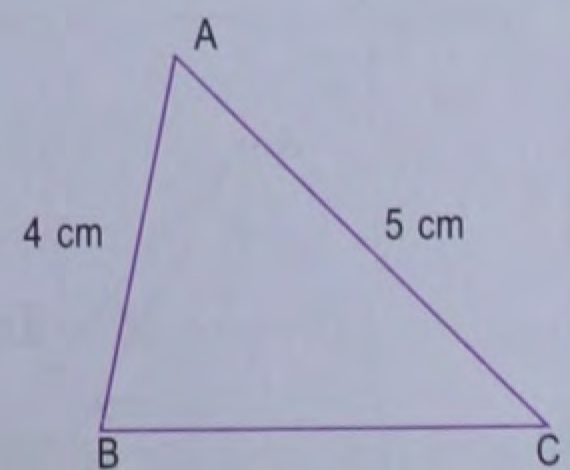
[Hint. $2x = y - 2$ and $2x = x + y - 36$.**]**



INEQUALITIES

Theorem

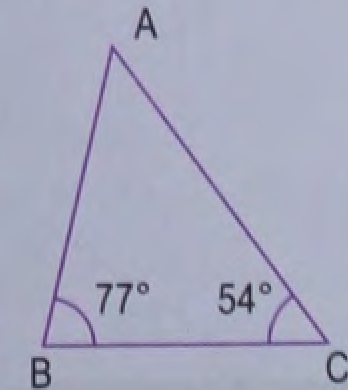
If two sides of a triangle are unequal, then the greater side has greater angle opposite to it.



In adjoining diagram, AC = 5 cm and AB = 4 cm. As AC > AB, therefore, $\angle B > \angle C$ (by above theorem).

Converse is also true.

If two angles of a triangle are unequal, then the greater angle has greater side opposite to it.



In adjoining diagram, $\angle B = 77°$ and $\angle C = 54°$. So $\angle B > \angle C$, therefore, AC > AB.

Corollary 1

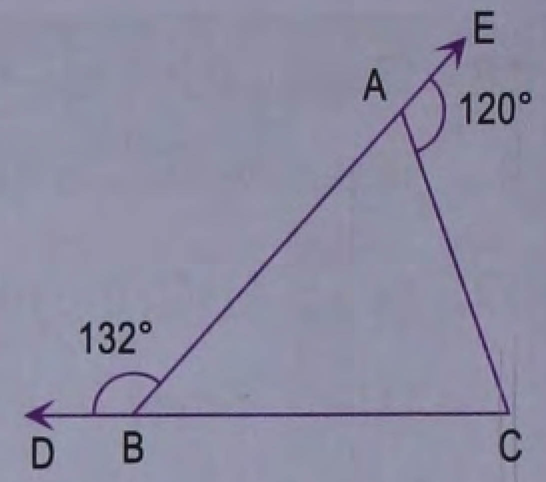
The sum of the lengths of any two sides of a triangle is greater than the length of the third side.

Corollary 2

The difference between the lengths of any two sides of a triangle is less than the length of the third side.

(Proofs of the above results are deferred to the next class.)

Example 1. In the adjoining figure, $\angle DBA = 132^\circ$ and $\angle EAC = 120^\circ$. Show that $AB > AC$.



Solution. As DBC is a straight line,

$132^\circ + \angle ABC = 180^\circ$

$\Rightarrow \quad \angle ABC = 180^\circ - 132^\circ = 48^\circ$

For Δ ABC, $\angle EAC$ is an exterior angle

$120^\circ = \angle ABC + \angle BCA$ (ext. $\angle$ = sum of two int. opp. $\angle s$)

$\Rightarrow \quad 120^\circ = 48^\circ + \angle BCA$

$\Rightarrow \quad \angle BCA = 120^\circ - 48^\circ = 72^\circ$

Thus, we find that $\angle BCA > \angle ABC$

$\Rightarrow \quad AB > AC.$ (side opposite to greater angle is greater).

Example 2. In the adjoining diagram, $AB = AC$ and D is any point on BC. Prove that $AB > AD$.

Solution. Given $AB = AC \Rightarrow \angle B = \angle C$

(angles opposite equal sides are equal)

For Δ ADC, $\angle ADB$ is an exterior angle

$\therefore \quad \angle ADB = \angle C + \angle DAC$

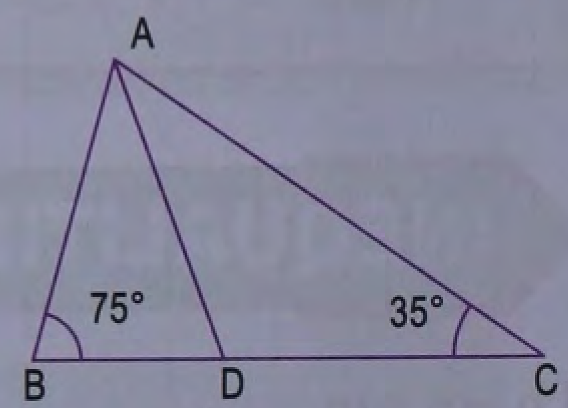
(ext. $\angle$ = sum of two opp. int. $\angle s$)

$\Rightarrow \quad \angle ADB = \angle B + \angle DAC$ ($\because \angle C = \angle B$)

$\Rightarrow \quad \angle ADB > \angle B$

$\Rightarrow \quad AB > AD.$ (side opposite to greater angle is greater)

Example 3. In the adjoining diagram, AD bisects $\angle A$. Arrange AB, BD and DC in ascending order.



Solution. $\angle A + 75^\circ + 35^\circ = 180^\circ$

(sum of angles in a triangle = 180°)

$\Rightarrow \quad \angle A = 180^\circ - 75^\circ - 35^\circ = 70^\circ$

Since AD bisects $\angle A$,

$\angle BAD = \angle DAC = \frac{1}{2} \cdot 70^\circ = 35^\circ$

$\angle ADB = \angle DAC + \angle C$ (ext. $\angle$ = sum of two int. opp. $\angle s$)

$= 35^\circ + 35^\circ = 70^\circ$

$\therefore$ In Δ ABD, $\angle BAD < \angle ADB < \angle ABD$

$\Rightarrow \quad BD < AB < AD$ ...(*i*)

(side opposite to smaller angle is smaller)

Also in Δ ADC, $\angle DAC = 35^\circ = \angle C$

$\Rightarrow \quad AD = DC$ ...(*ii*)

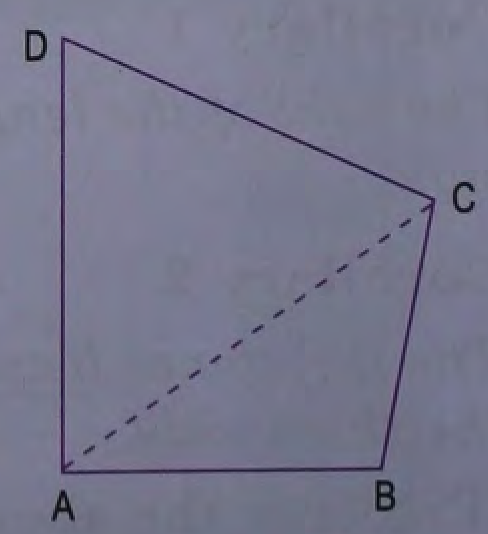
(sides opposite equal angles are equal)

From (*i*) and (*ii*), we get $BD < AB < DC$.

Example 4. From the adjoining figure, prove that

(*i*) $AB + BC + CD > DA$

(*ii*) $AB + BC + CD + DA > 2\,AC$



Solution. (*i*) In Δ ABC, $AB + BC > AC$ ...(1)

(sum of lengths of two sides of a Δ > length of third side)

In Δ ACD, AC + CD > DA ...(2) (same reason)

On adding (1) and (2), we get

AB + BC + AC + CD > AC + DA

⇒ AB + BC + CD > DA. (∵ AC is common to both sides)

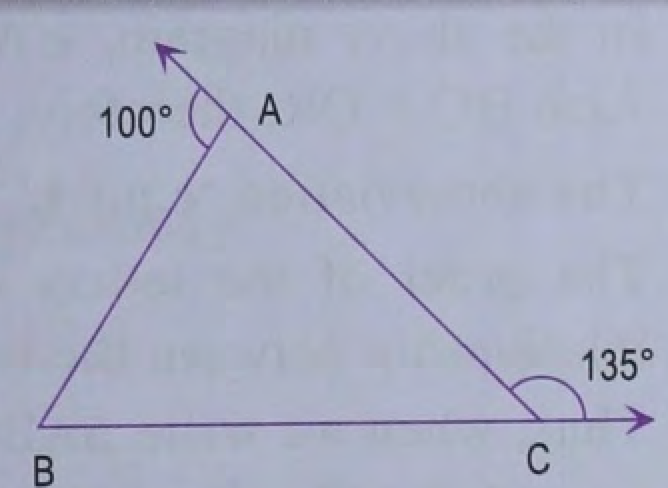
(*ii*) In Δ ACD, CD + DA > AC ...(3)

On adding (1) and (3), we get

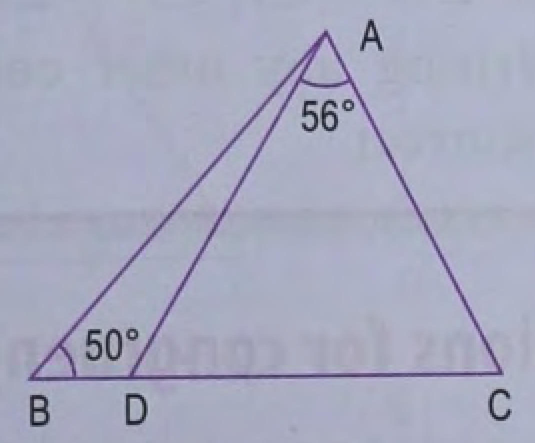
AB + BC + CD + DA > 2 AC.

Exercise 22.2

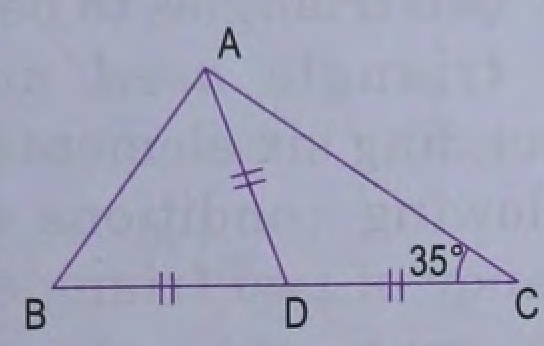
1. In Δ ABC, ∠A = 48° and ∠B = 62°. Name its
   (*i*) smallest side (*ii*) largest side.
2. In Δ PQR, ∠Q = 49° and ∠R = 85°. Name its
   (*i*) largest side (*ii*) smallest side.
   Also write its sides in ascending order.
3. In Δ ABC, BC = 7 cm, CA = 5 cm and AB = 6·2 cm. Which is the
   (*i*) greatest angle (*ii*) smallest angle?
   Also arrange its angles in descending order.
4. In a triangle ABC, ∠C = 90°. Which is the greatest side?
5. In Δ PQR, ∠Q = 105°. Which is the greatest side?
6. In Δ ABC, ∠A : ∠B : ∠C = 3 : 5 : 7. Without finding the angles of the triangle, name its
   (*i*) greatest side (*ii*) smallest side.
7. In Δ ABC, AB : BC : CA = 3 : 8 : 5. Name its
   (*i*) smallest angle (*ii*) largest angle.
8. In the adjoining figure, show that AC > AB. Also arrange the sides of ΔABC in ascending order.



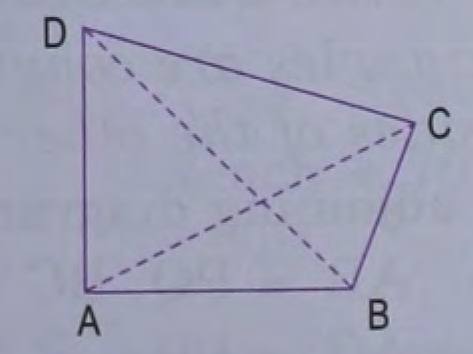
9. In a Δ ABC, ∠A = 53° and ∠B = 49°. The bisector of ∠C meets AB at D. Arrange the sides of Δ ADC in the descending order of their lengths.
10. In the adjoining figure, AC = AD.
    Prove that :
    (*i*) AD > DC
    (*ii*) AD < AB.
    Also name the greatest side and the smallest side of ΔABC.



11. Using the information given in the adjoining figure, show that :
    (*i*) AC > DC
    (*ii*) AB > AD.

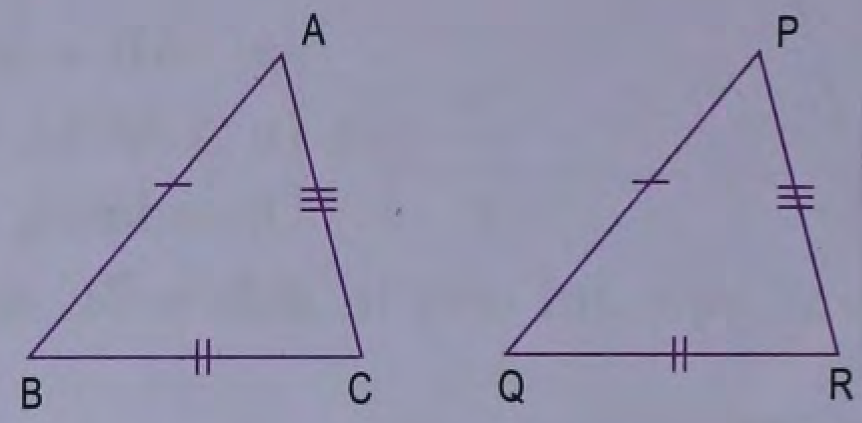


12. From the adjoining diagram, show that :
    (*i*) AB + BC + CD + DA > 2BD
    (*ii*) AB + BC + CD + DA > AC + BD.



CONGRUENT TRIANGLES

Two triangles are called ***congruent triangles*** *if and only if they have exactly the same shape and same size.*



In the adjoining figure, two triangles ABC and PQR are congruent. It means that the sketch of one triangle can be slided onto the sketch of the other, so that they fit each other exactly.

Notice that these triangles are such that

AB = PQ, BC = QR, CA = RP, ∠A = ∠P, ∠B = ∠Q and ∠C = ∠R.

Thus, these triangles have the same shape and same size, so they are congruent triangles.

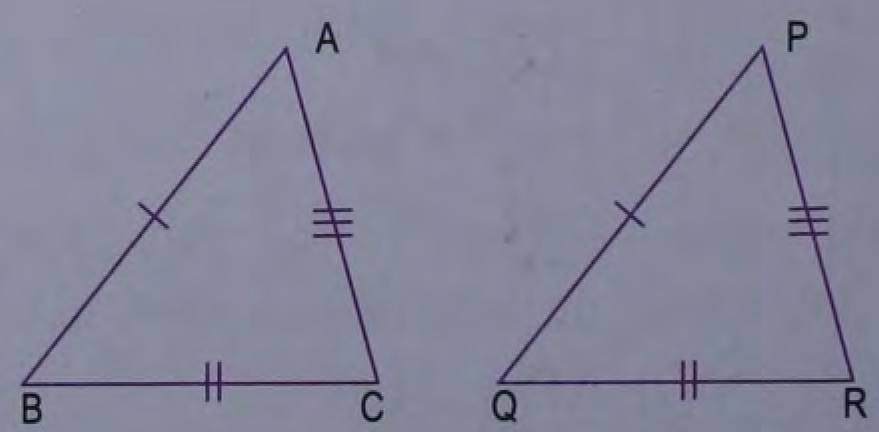
We use the symbol ≡ or ≅ (read as 'congruent to') to indicate the congruency of triangles.

Remarks

- Congruent triangles are 'equal in all respects' *i.e.* they are exact duplicate of each other.
- If two triangles are congruent triangles, then any one can be **superposed** on the other to cover it exactly.
- In congruent triangles, the sides and the angles which coincide by superposition are called **corresponding sides** and **corresponding angles**.
- The corresponding sides lie opposite to the equal angles and the corresponding angles lie opposite to the equal sides.
  In the above diagram, ∠A = ∠P, therefore, the corresponding sides BC and QR are equal. Also BC = QR, therefore, the corresponding angles A and P are equal etc.
- The abbreviation **'c.p.c.t.'** will be used for 'corresponding parts of congruent triangles'.
- The order of the letters in the names of congruent triangles displays the corresponding relationship between the two triangles.
  Thus, when we write ΔABC ≅ ΔPQR, it means that A lies on P, B lies on Q and C lies on R *i.e.* ∠A = ∠P, ∠B = ∠Q, ∠C = ∠R and BC = QR, CA = RP, AB = PQ.
  Writing any other correspondence *i.e.* ΔABC ≅ ΔPRQ, ΔABC = ΔRPQ etc. will be incorrect.

Conditions for congruency of triangles

For any two triangles to be congruent, the **six elements** of one triangle *need not be proved* equal to the corresponding six elements of the other triangle. In fact, the following conditions are sufficient to ensure the congruency of two triangles.



S.S.S. (Side-Side-Side) axiom of congruency

Two triangles are congruent triangles if the three sides of one triangle are equal to the three sides of the other triangle.

In the adjoining diagram,

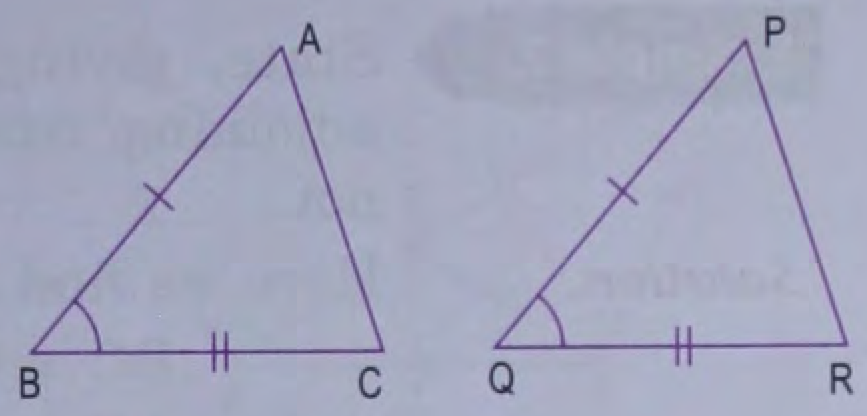
AB = PQ, BC = QR and

AC = PR

∴ Δ ABC ≅ Δ PQR.

- **S.A.S. (Side-Angle-Side) axiom of congruency**

Two triangles are congruent triangles if two sides and the included angle of one triangle are equal to two sides and the included angle of the other triangle.



In the adjoining diagram,

AB = PQ, BC = QR and

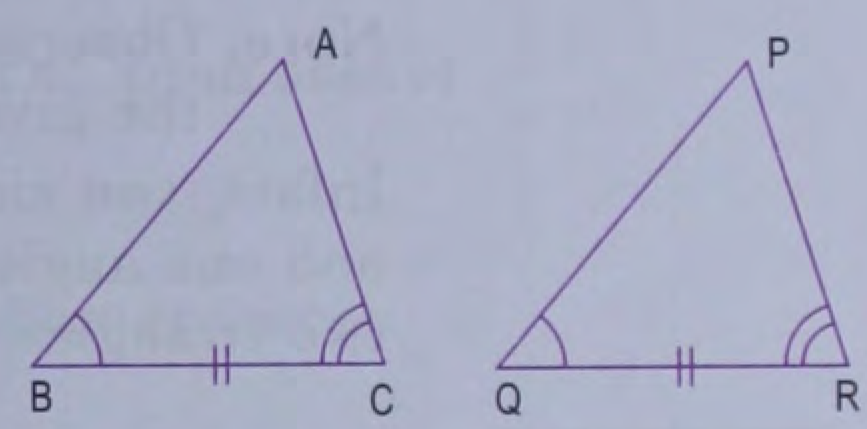
∠B = ∠Q

∴ ΔABC ≅ ΔPQR

Note. The equality of the included angle is essential.

- **A.S.A. (Angle-Side-Angle) axiom of congruency**

Two triangles are congruent triangles if two angles and the included side of one triangle are equal to two angles and the included side of the other triangle.



In the adjoining diagram,

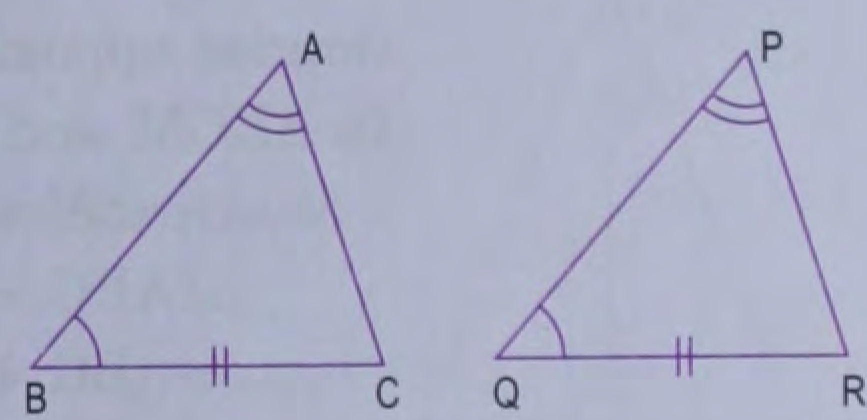
∠B = ∠Q, ∠C = ∠R and

BC = QR

∴ ΔABC ≅ ΔPQR.

- **A.A.S. (Angle-Angle-Side) axiom of congruency**

Two triangles are congruent triangles if any two angles and a (non-included) side of one triangle are equal to two angles and the corresponding side of the other triangle.



In the adjoining diagram,

∠A = ∠P, ∠B = ∠Q and

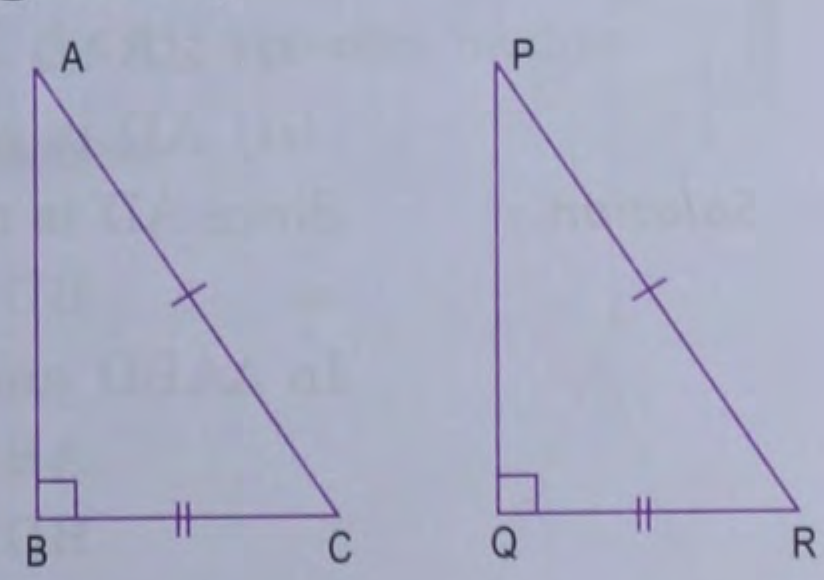
BC = QR

∴ ΔABC ≅ ΔPQR.

Note. The equality of corresponding sides is essential.

- **R.H.S. (Right angle-Hypotenuse-Side) axiom of congruency**

Two right angled triangles are congruent triangles if the hypotenuse and one side of one triangle are equal to the hypotenuse and one side of the other triangle.



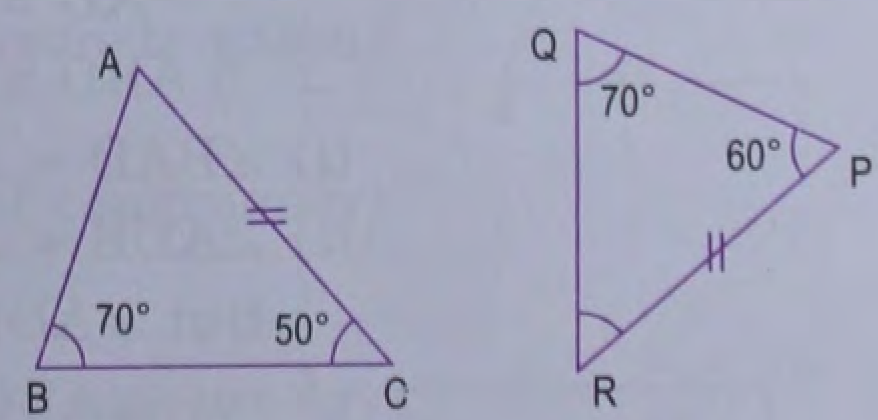
In the adjoining diagram,

∠B = a right angle = ∠Q,

AC = PR and BC = QR

∴ ΔABC ≅ ΔPQR.

Example 1. State whether the adjoining triangles are congruent triangles or not.



Solution. Since the sum of angles of a triangle is 180°,

∠A = 180° – (70° + 50°) = 60°.

In ΔABC and ΔPQR

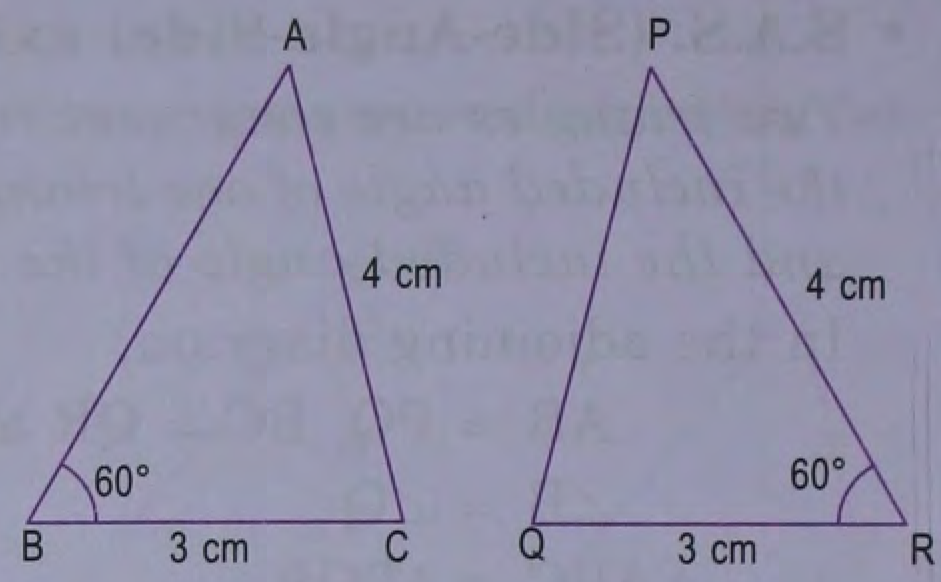
∠A = ∠P (each = 60°)

∠B = ∠Q (each = 70°, given)

and the corresponding sides AC = PR (given).

∴ ΔABC ≅ ΔPQR. (A.A.S. axiom of congruency)

Example 2. State, giving reasons, whether the adjoining triangles are congruent or not.



Solution. Here we find that

BC = QR (each = 3 cm)

and AC = PR (each = 4 cm)

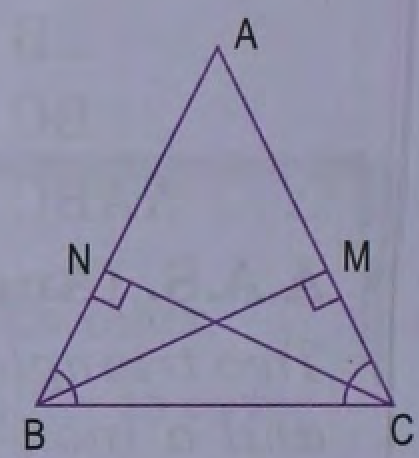
but the included angles are not equal.

So the given triangles ABC and PQR are not congruent.

Note. Observe that ∠B = ∠R (each = 60°) but ∠B is not included between the given sides.

Infact, two sides of one triangle equal to two sides of the other triangle and one angle of both triangles equal may not ensure the congruency of two triangles.

Example 3. In the adjoining figure, AB = AC. Prove that BM = CN.



Solution. In ΔABC, AB = AC (given)

∴ ∠ABC = ∠ACB

(angles opposite equal sides are equal)

In ΔBCM and ΔCBN,

∠N = ∠M (each = 90°)

∠ABC = ∠ACB (from above)

BC = BC (common)

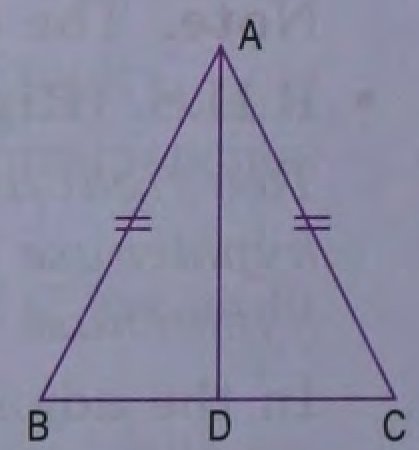
∴ ΔBCM ≅ ΔCBN (A.A.S. axiom of congruency)

⇒ BM = CN. (c.p.c.t.)

Example 4. In the adjoining figure, AB = AC and AD is median of Δ ABC. Prove that

(*i*) ∠BAD = ∠CAD.

(*ii*) AD is perpendicular to BC.



Solution. Since AD is median of Δ ABC, D is mid-point of BC

⇒ BD = DC.

In ΔABD and ΔACD

AB = AC (given)

BD = DC (from above)

AD = AD (common)

∴ Δ ABD ≅ Δ ACD (S.S.S. axiom of congruency)

(*i*) ∠BAD = ∠CAD. (c.p.c.t.)

(*ii*) ∠ADB = ∠ADC (c.p.c.t.)

But ∠ADB + ∠ADC = 180° (linear pair)

⇒ ∠ADC + ∠ADC = 180°

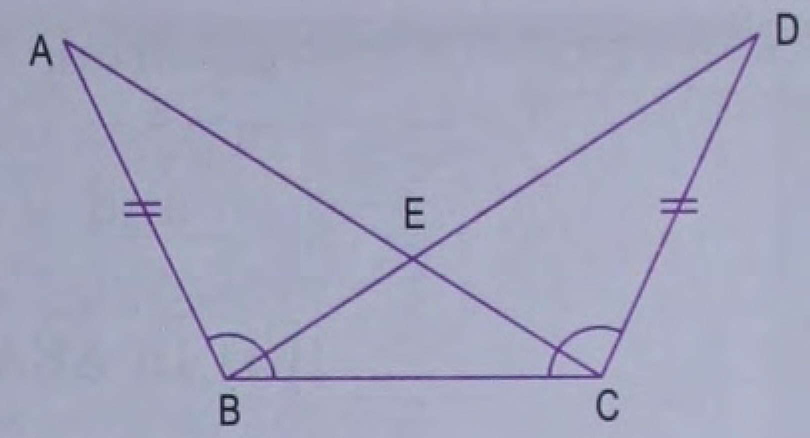
⇒ 2 ∠ADC = 180° ⇒ ∠ADC = 90°

Hence, AD is perpendicular to BC.

Example 5. In the figure given on next page, AB = CD and ∠ABC = ∠BCD. Prove that

(*i*) AC = BD (*ii*) BE = CE.

Solution. In ΔABC and ΔDCB,



$AB = CD$ (given)

$\angle ABC = \angle BCD$ (given)

$BC = BC$ (common)

$\therefore$ $\Delta ABC \cong \Delta DCB$ (S.A.S. axiom of congruency)

(*i*) $AC = BD$ (c.p.c.t.)

Also $\angle BAC = \angle BDC$ (c.p.c.t.)

(*ii*) In ΔABE and ΔDCE

$\angle BAE = \angle EDC$ ($\because \angle BAC = \angle BDC$, from above)

$\angle AEB = \angle CED$ (vert. opp. $\angle s$)

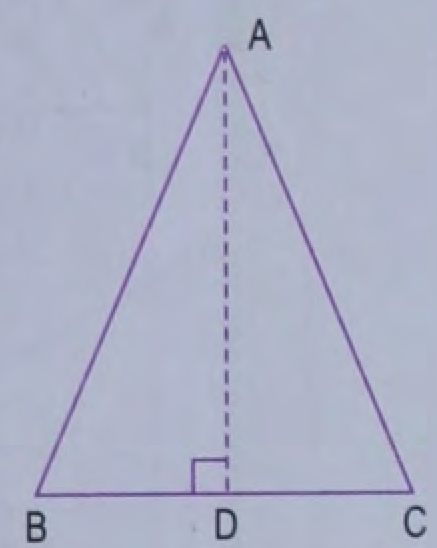
$AB = CD$ (given)

$\therefore$ $\Delta ABE \cong \Delta DCE$ (A.A.S. axiom of congruency)

$\Rightarrow$ $BE = CE$ (c.p.c.t.)

Example 6. In a Δ ABC, AB = AC. Prove that $\angle B = \angle C$.

Solution. *From A, draw AD perpendicular to BC.*



In ΔABD and ΔACD,

$AB = AC$ (given)

$\angle ADB = \angle ADC$ (each = 90°, by construction)

$AD = AD$ (common)

$\therefore$ $\Delta ABD \cong \Delta ACD$ (R.H.S. axiom of congruency)

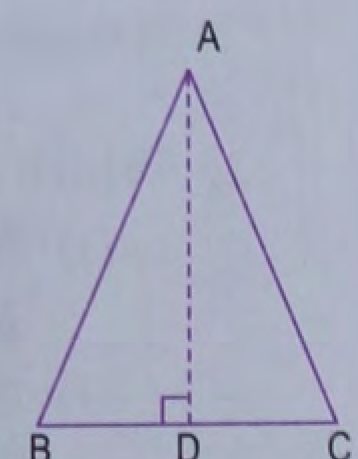
$\Rightarrow$ $\angle B = \angle C$. (c.p.c.t.)

Remark

In the above example, BD = DC (c.p.c.t.) *i.e.* D is mid-point of BC. So AD becomes median.

Example 7. In a ΔABC, $\angle B = \angle C$. Prove that AB = AC.

Solution. *From A, draw AD perpendicular to BC.*



In ΔABD and ΔACD,

$\angle B = \angle C$ (given)

$\angle ADB = \angle ADC$ (each = 90°, by construction)

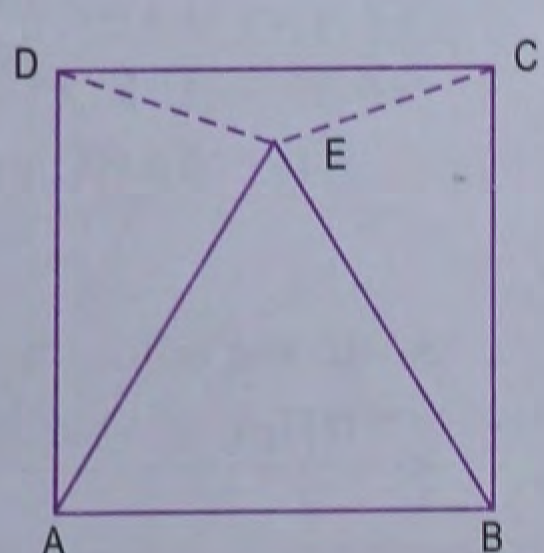
$AD = AD$ (common)

$\therefore$ $\Delta ABD \cong \Delta ACD$ (A.A.S. axiom of congruency)

$\Rightarrow$ $AB = AC$. (c.p.c.t)

Example 8. In the adjoining figure, ABCD is a square and ABE is an equilateral triangle. Prove that

(*i*) $\angle EAD = \angle EBC$ (*ii*) DE = CE.



Solution. Given ABCD is a square and ABE is an equilateral triangle. We know that each angle in a square = 90° and each angle in an equilateral triangle = 60°.

(*i*) From figure,

$\angle EAD = \angle DAB - \angle EAB = 90° - 60° = 30°$

and $\angle EBC = \angle CBA - \angle EBA = 90° - 60° = 30°$

$\therefore$ $\angle EAD = \angle EBC$.

(*ii*) In ΔEAD and ΔEBC,

$EA = EB$ (sides of equilateral triangle)

$AD = BC$ (sides of a square)

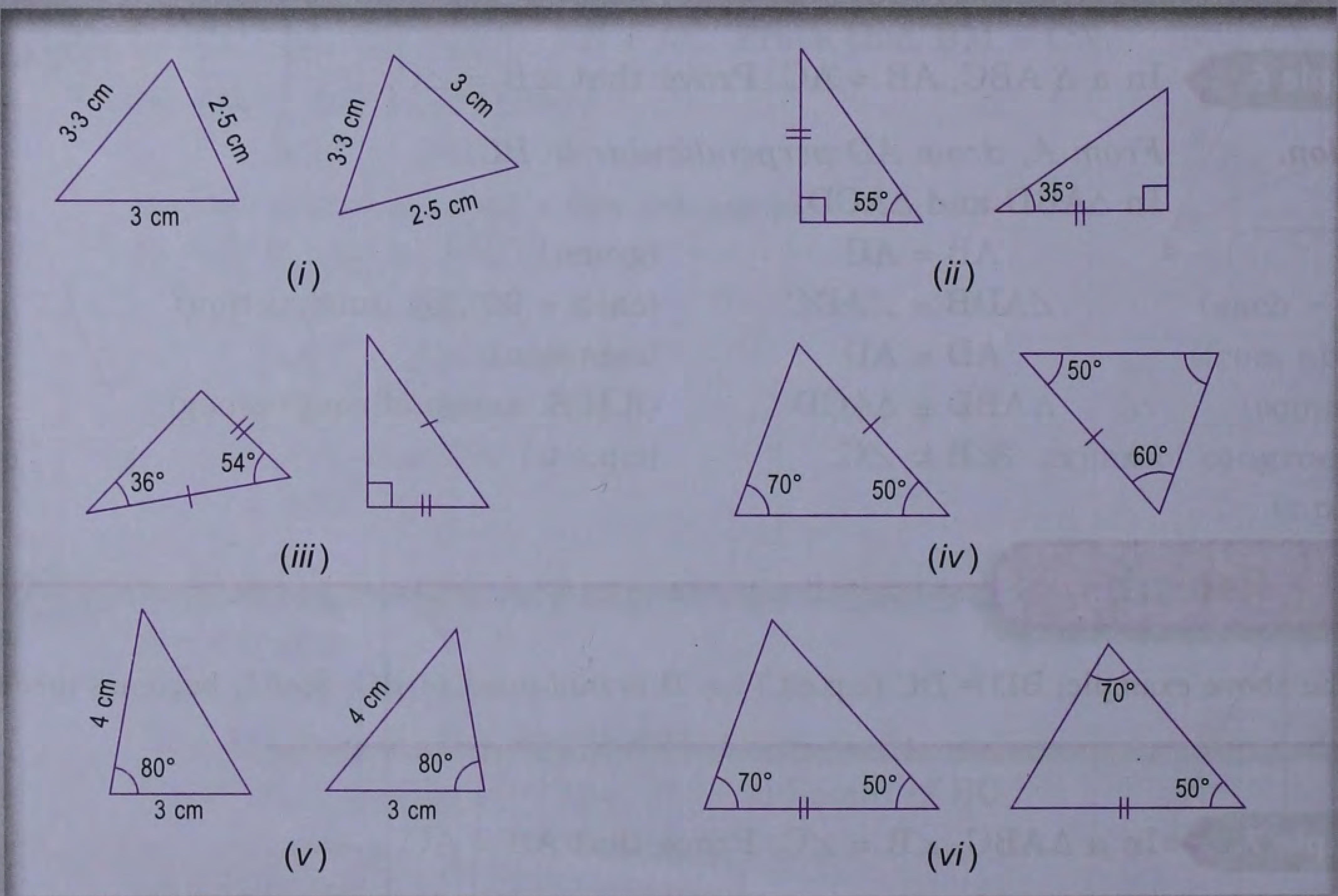
$\angle EAD = \angle EBC$ (proved in (*i*))

$\therefore$ $\Delta EAD \cong \Delta EBC$ (S.A.S. axiom of congruency)

$\Rightarrow$ $DE = CE$. (c.p.c.t.)

Exercise 22.3

1. State, giving reasons, whether the following pairs of triangles are congruent or not.



2. Which of the following pairs of triangles are congruent? Give reasons.

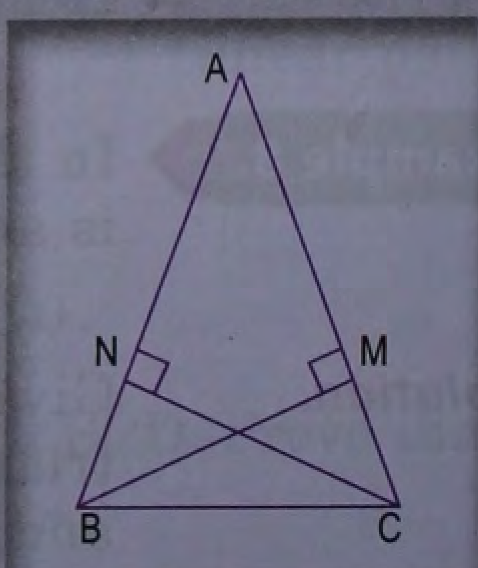
(*i*) ΔABC : $AB = 4$ cm, $BC = 5$ cm, $\angle B = 72°$
ΔQRP : $QR = 4$ cm, $RP = 5$ cm, $\angle R = 72°$

(*ii*) ΔABC : $AB = 4$ cm, $BC = 5$ cm, $\angle B = 72°$
ΔPQR : $PQ = 4$ cm, $RP = 5$ cm, $\angle R = 72°$

(*iii*) ΔABC : $BC = 6$ cm, $\angle A = 90°$, $\angle C = 50°$
ΔPQR : $QR = 6$ cm, $\angle R = 50°$, $\angle Q = 40°$

(*iv*) ΔABC : $AB = 5$ cm, $\angle A = 90°$, $BC = 8$ cm
ΔPRQ : $PR = 5$ cm, $\angle P = 90°$, $QR = 8$ cm.

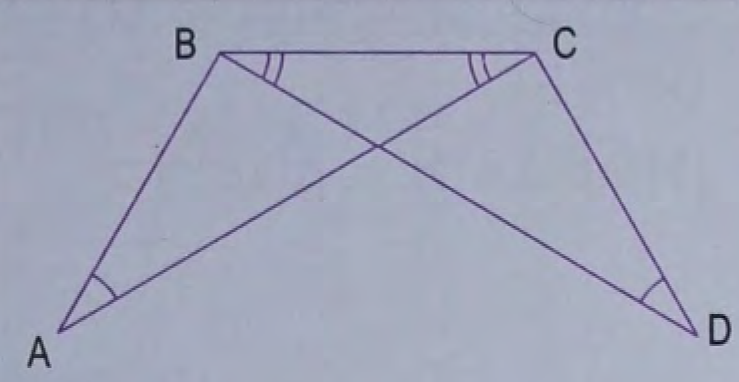
(*v*) ΔABC and ΔADC in which $AB = AD$ and $BC = CD$



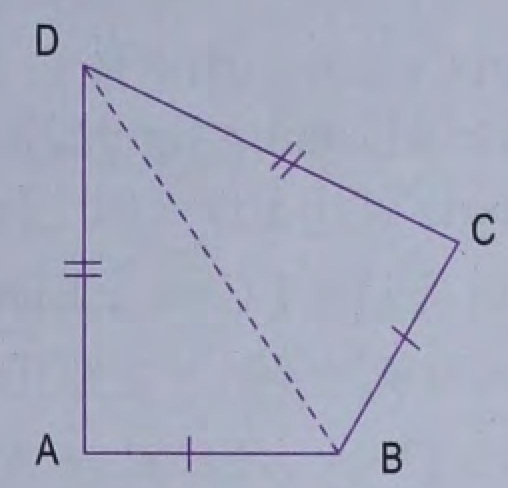
3. In the adjoining diagram, $BM = CN$. Prove that ΔABC is isosceles.
[Hint. $\Delta ANC \cong \Delta AMB$.**]**

4. In the adjoining diagram, ∠BAC = ∠BDC and ∠ACB = ∠DBC. Prove that AC = BD.

[Hint. ΔABC ≅ ΔDCB.**]**



5. In the adjoining figure, AB = BC and AD = CD. Prove that ∠A = ∠C.



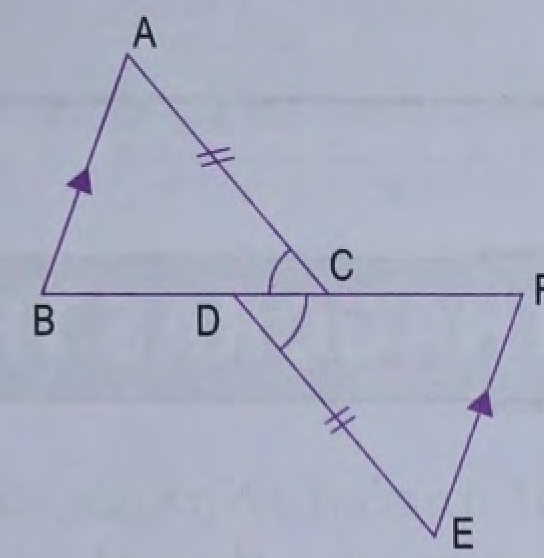
6. In the adjoining figure, ∠ACB = ∠EDF, BA || EF and AC = DE.
Prove that

(*i*) AB = EF

(*ii*) BD = CF

[Hint. (*ii*) BC = DF

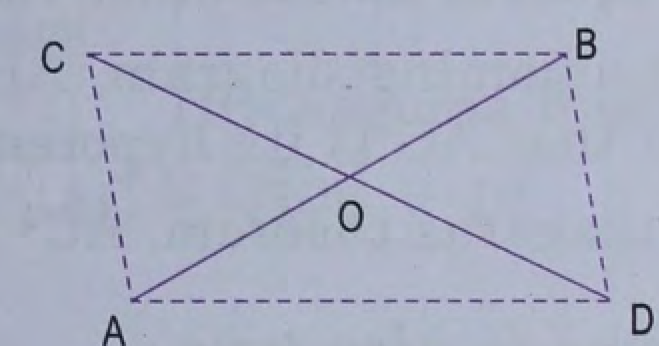
⇒ BD + DC = DC + CF.**]**



7. In the adjoining figure, line segments AB and CD bisect each other at O. Prove that

(*i*) AC = DB

(*ii*) AD = CB

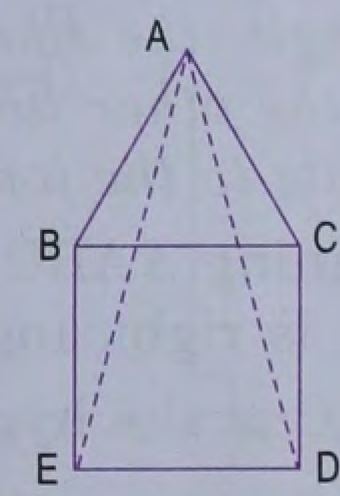


8. In the adjoining figure, ABC is an equilateral triangle and BCDE is a square. Prove that

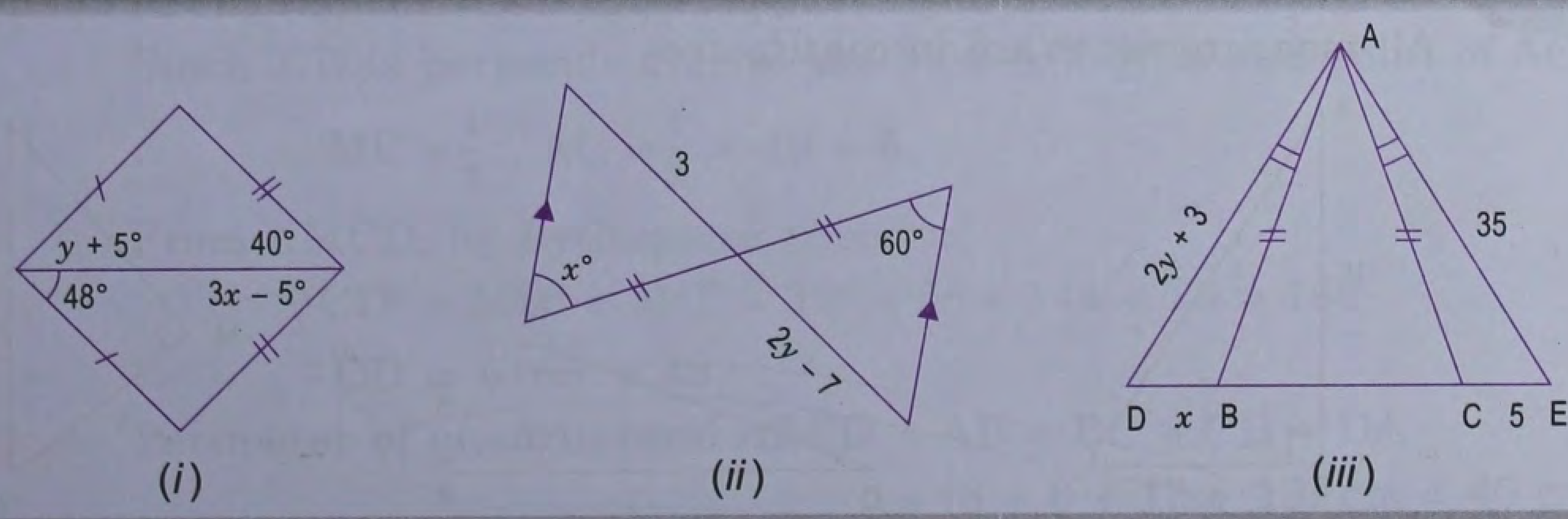
(*i*) ∠BAE = 15°

(*ii*) AE = AD

[Hint. AB = BE and ∠ABE = 60° + 90°.**]**



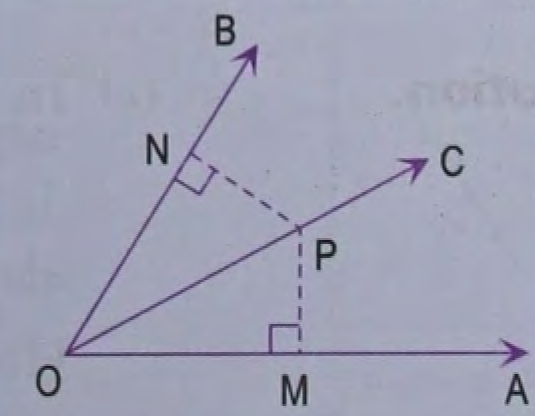
9. In each of the following figures, find the values of x and y:



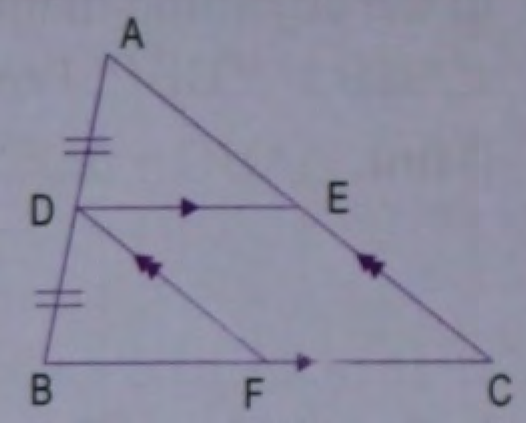
[Hint. (*iii*) AB = AC ⇒ ∠ABC = ∠ACB ⇒ ∠ABD = ∠ACE, why?**]**

10. OC is bisector of ∠AOB, P is any point on OC. From P, PM and PN are drawn perpendiculars on OA and OB respectively. Prove that MP = NP.

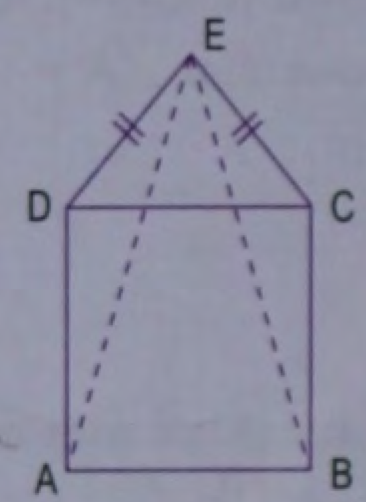
[Hint. Δ OMP ≅ ΔONP.**]**



11. In the adjoining figure, AD = BD, DE || BC and FD || CE. Prove that DE = BF.
[Hint. $\Delta ADE \cong \Delta DBF$.**]**

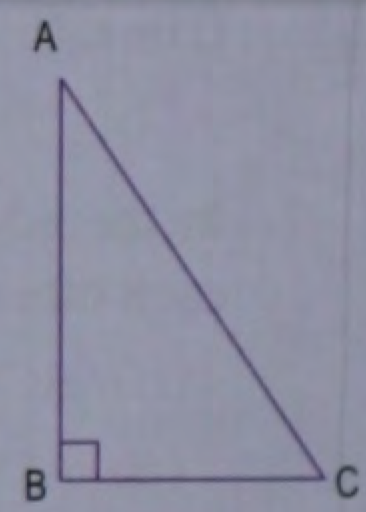


12. In the adjoining figure, ABCD is a square and EDC is an isosceles triangle with ED = EC. Prove that AE = EB.
[Hint. $ED = EC \Rightarrow \angle EDC = \angle ECD$
$\Rightarrow \angle EDC + 90° = \angle ECD + 90°$
$\Rightarrow \angle EDA = \angle ECB \Rightarrow \Delta EDA \cong \Delta ECB$.**]**



PYTHAGORAS THEOREM

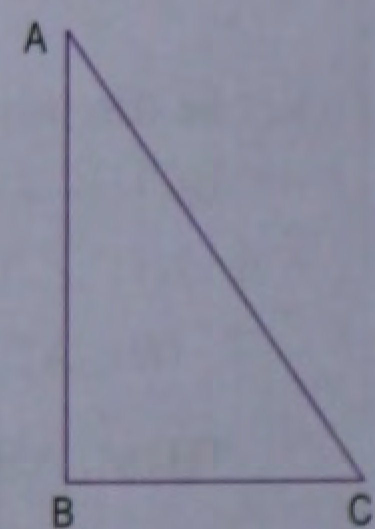
In a right angled triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides.



In the adjoining diagram, ABC is a right angled triangle at B *i.e.* $\angle B = 90°$, so that AC is its hypotenuse.

By Pythagoras theorem, $AC^2 = AB^2 + BC^2$.

Converse is also true.

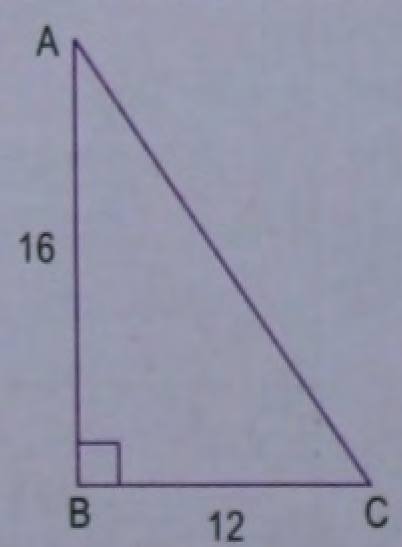


If in a triangle, the square on the longest side is equal to the sum of the squares on the other two sides, then the triangle is right angled and the angle opposite to the longest side is right angle.

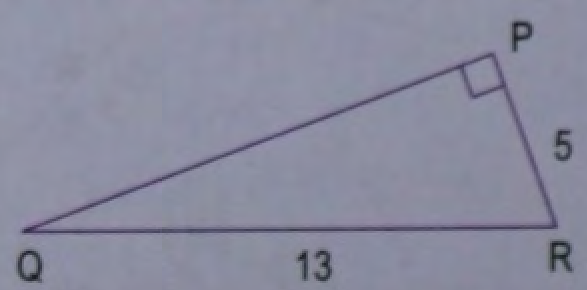
In the adjoining ΔABC, AC is the largest side and if $AC^2 = AB^2 + BC^2$, then Δ ABC is right angled triangle at B *i.e.* $\angle B = 90°$.

Note that AC is the hypotenuse.

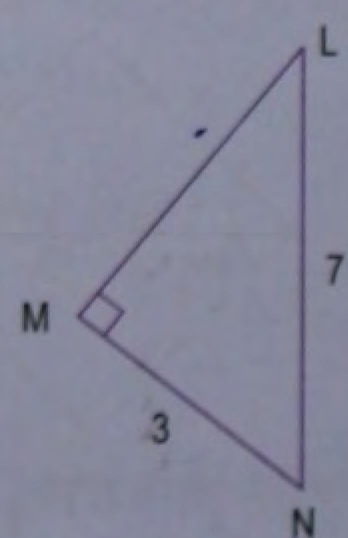
Example 1. For each of the following triangles, calculate the length of the unknown side. All measurements are in centimetres.



(*i*)



(*ii*)



(*iii*)

Solution. (*i*) In ΔABC, $\angle B = 90°$, so AC is the hypotenuse.

$$AC^2 = AB^2 + BC^2 \quad \text{(Pythagoras theorem)}$$

$$\Rightarrow \quad AC^2 = 16^2 + 12^2 = 256 + 144 = 400 \Rightarrow AC = \sqrt{400} = 20$$

Hence, the length of AC = 20 cm.

(*ii*) In ΔPQR, $\angle P = 90°$, so QR is the hypotenuse.

$$QR^2 = PQ^2 + PR^2 \quad \text{(Pythagoras theorem)}$$

$$\Rightarrow \quad PQ^2 = QR^2 - PR^2 = 13^2 - 5^2 = 169 - 25 = 144$$

$$\Rightarrow \quad PQ = \sqrt{144} = 12.$$

Hence, the length of PQ = 12 cm.

(*iii*) In ΔLMN, $\angle M = 90°$, so LN is the hypotenuse.

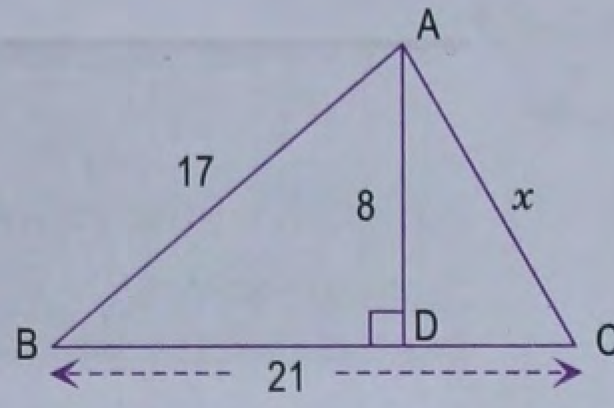
$$LN^2 = ML^2 + MN^2 \quad \text{(Pythagoras theorem)}$$

$$\Rightarrow \quad ML^2 = LN^2 - MN^2 = 7^2 - 3^2 = 49 - 9 = 40$$

$$\Rightarrow \quad ML = \sqrt{40} = 2\sqrt{10}$$

Hence, the length of ML = $2\sqrt{10}$ cm.

Example 2. In the adjoining diagram, all measurements are in centimetres. Find the value of x.



Solution. In ΔABD, $\angle ADB = 90°$,

so AB is the hypotenuse.

$$AB^2 = BD^2 + AD^2 \quad \text{(Pythagoras theorem)}$$

$$\Rightarrow \quad BD^2 = AB^2 - AD^2 = 17^2 - 8^2 = 289 - 64 = 225$$

$$\Rightarrow \quad BD = \sqrt{225} = 15$$

From figure, DC = BC – BD = 21 – 15 = 6

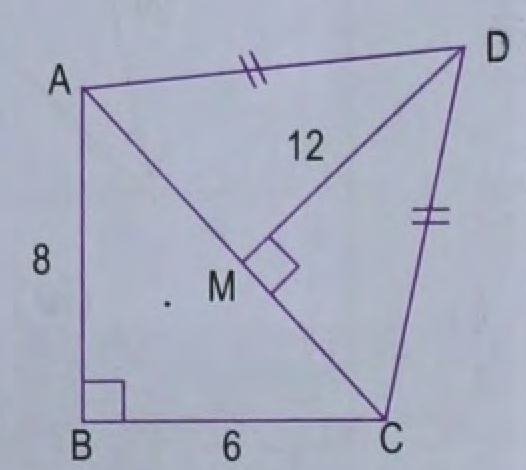
In ΔADC, $\angle ADC = 90°$, so AC is the hypotenuse.

$$AC^2 = AD^2 + DC^2 \quad \text{(Pythagoras theorem)}$$

$$= 8^2 + 6^2 = 64 + 36 = 100 \quad \Rightarrow \quad AC = \sqrt{100} = 10$$

Hence, $x = 10$.

Example 3. In the adjoining figure, all measurements are in centimetres. Find the perimeter of the quadrilateral ABCD.



Solution. From ΔABC, by Pythagoras theorem

$$AC^2 = AB^2 + BC^2$$

$$= 8^2 + 6^2 = 64 + 36 = 100$$

$$\Rightarrow \quad AC = \sqrt{100} = 10.$$

In ΔACD, AD = CD (given), so it is an isosceles triangle.

Since MD is perpendicular to the base AC, M is mid-point of AC.

$$\therefore \quad MC = \frac{1}{2} . AC = \frac{1}{2} \times 10 = 5.$$

From ΔMCD, by Pythagoras theorem

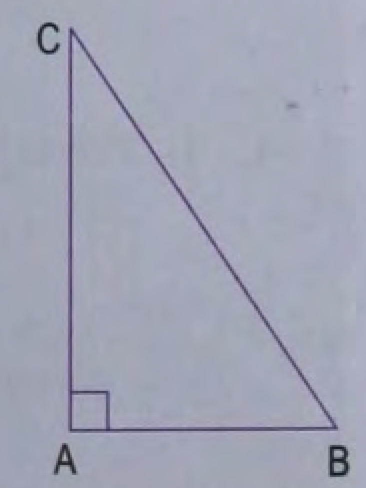
$$CD^2 = MD^2 + MC^2 = 12^2 + 5^2 = 144 + 25 = 169$$

$$\Rightarrow \quad CD = \sqrt{169} = 13.$$

Perimeter of quadrilateral ABCD = AB + BC + CD + DA

= (8 + 6 + 13 + 13) cm = 40 cm.

Example 4. In the adjoining figure, ABC is a right angled triangle at A. If the sides AB and AC are in the ratio 3 : 4 and the area of the triangle ABC is 96 cm^2, find the length of the hypotenuse of the triangle.



Solution. Since the sides AB and AC are in the ratio 3 : 4, let the sides of the trianlge be $3x$ cm and $4x$ cm respectively. Given that the area of ΔABC = 96 cm^2

$\Rightarrow \frac{1}{2} \times 3x \times 4x = 96 \quad \Rightarrow \quad 6x^2 = 96$

$\Rightarrow \quad x^2 = 16 \quad \Rightarrow \quad x = \sqrt{16} = 4$

$\therefore$ AB = 3 × 4 cm = 12 cm and AC = 4 × 4 cm = 16 cm.

As ΔABC is right angled at A, by Pythagoras theorem,

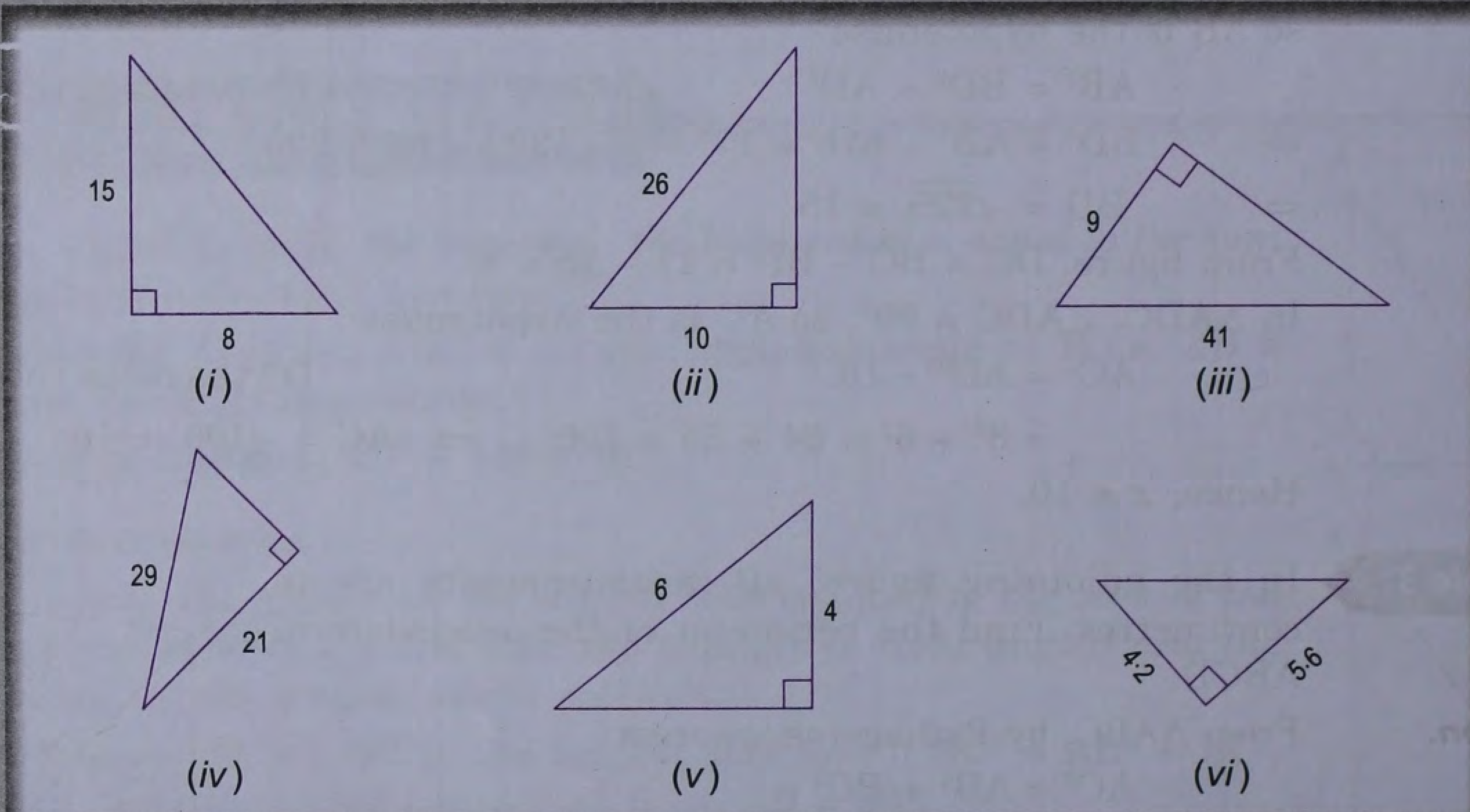
$BC^2 = AB^2 + AC^2 = (12)^2 + (16)^2 = 144 + 256 = 400$

$\Rightarrow \quad BC = \sqrt{400}$ cm = 20 cm

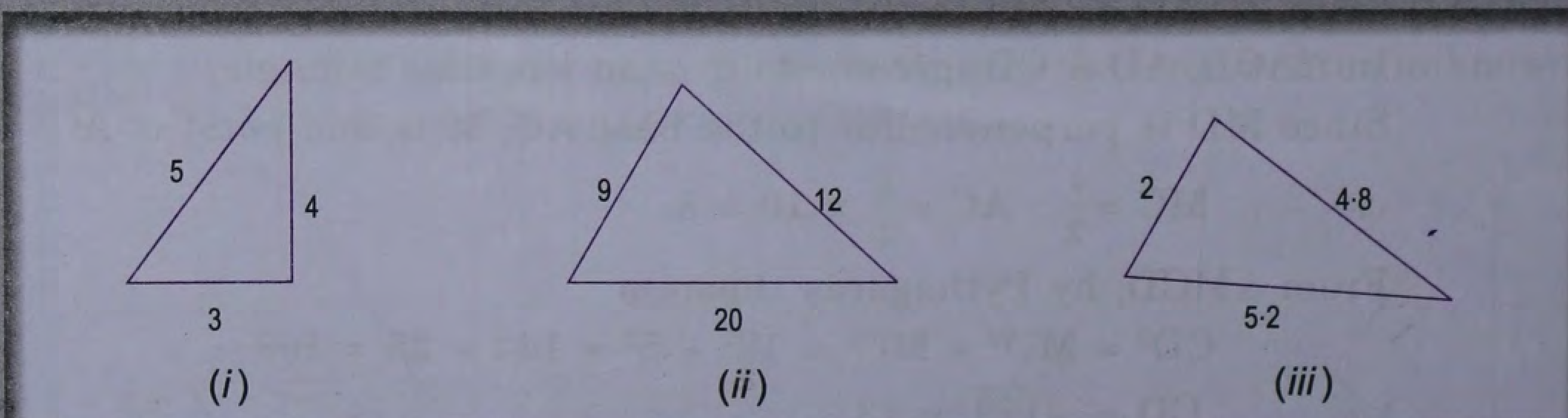
Hence, the hypotenuse of ΔABC = 20 cm.

Exercise 22.4

1. For each of the following triangles, calculate the length of the unknown side. All measurements are in centimetres.

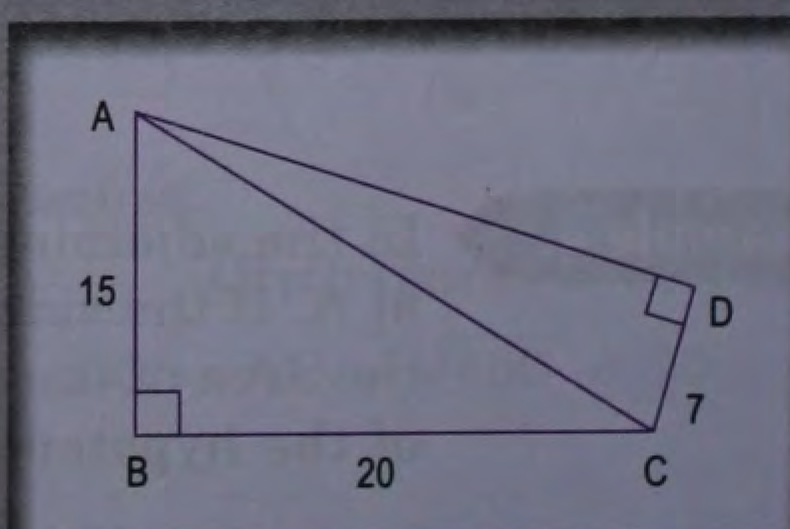


2. State whether or not the following triangles are right angled. All measurements are in centimetres.



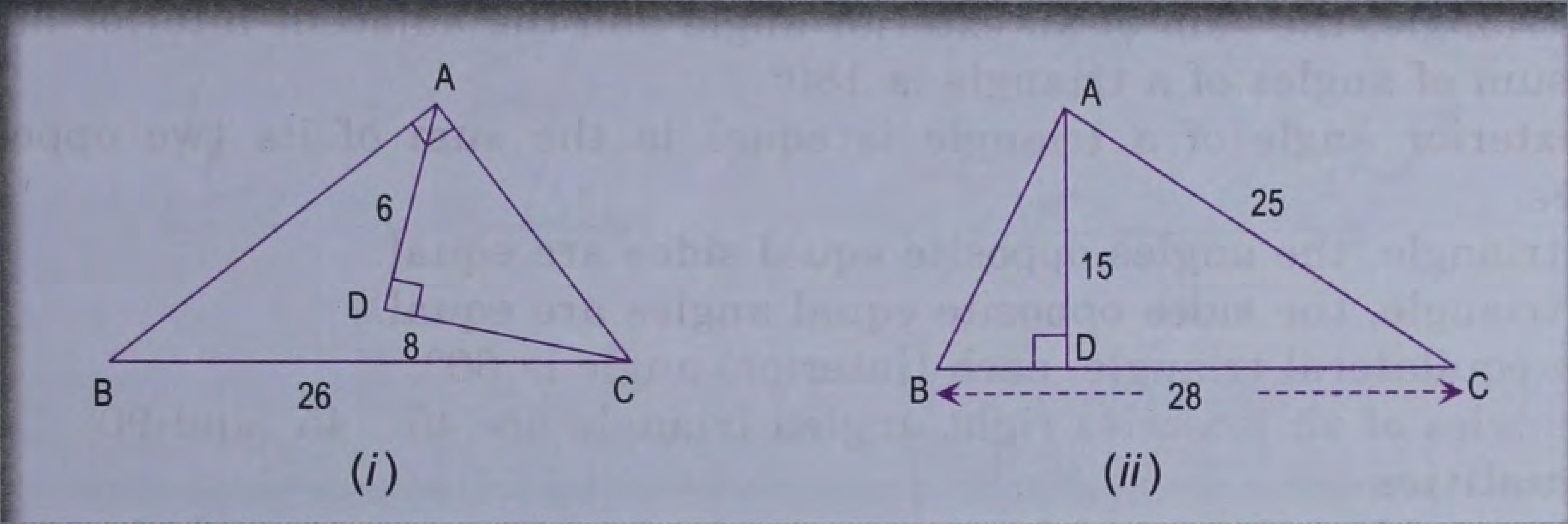
3. A ladder 13 m long rests with its foot 5 m from the base of a vertical wall. How high up the wall will the ladder reach?

4. In the adjoining figure, all measurements are in metres. Find
   (*i*) AC
   (*ii*) AD.



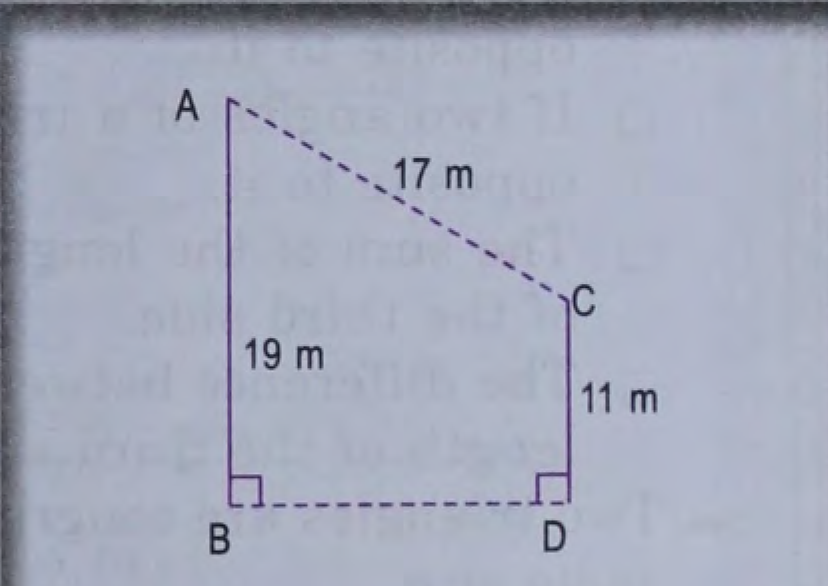
5. (*a*) In the figure (*i*) given below, ∠BAC = 90°, ∠ADC = 90° and all measurements are in centimetres. Find the length of AB.

(*b*) In the figure (*ii*) given below, all measurements are in centimetres. If AD is perpendicular to BC, find the length of AB.



6. In the adjoining figure, AB and CD are two vertical poles of height 19 m and 11 m respectively. If the shortest distance between their tops is 17 m, find how far apart they are?

[Hint. Through C, draw a line parallel to BD.**]**

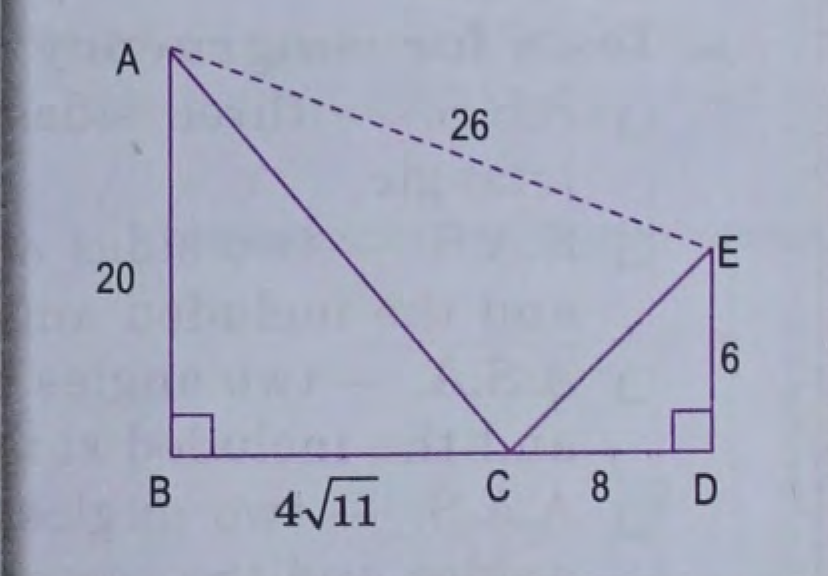


7. In the adjoining figure, all measurements are in centimetres. Find

(*i*) AC (*ii*) CE.

Hence prove that $AE^2 = AC^2 + CE^2$.

State the size of ∠ACE.



8. The base BC of an isosceles triangle ABC is 20 cm long and each of its equal side is of length 26 cm. Find the length of the altitude through the vertex A.

[Hint. From A, draw AD perpendicular to BC, then D is mid-point of BC.**]**

9. The diagonals of a rhombus are 30 cm and 16 cm. Find

(*i*) the length of one side

(*ii*) perimeter of the rhombus.

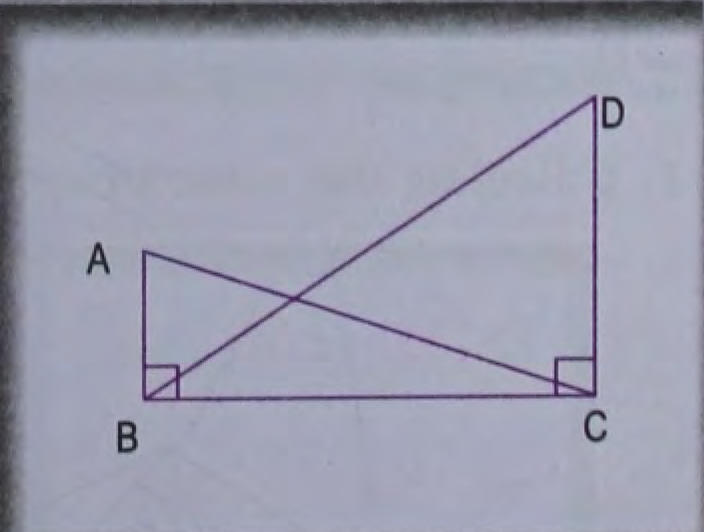
[Hint. The diagonals of a rhombus bisect each other at right angles.**]**

10. In the adjoining figure,

∠ABC = 90° and ∠BCD = 90°

Prove that

$AC^2 - AB^2 = BD^2 - CD^2$.



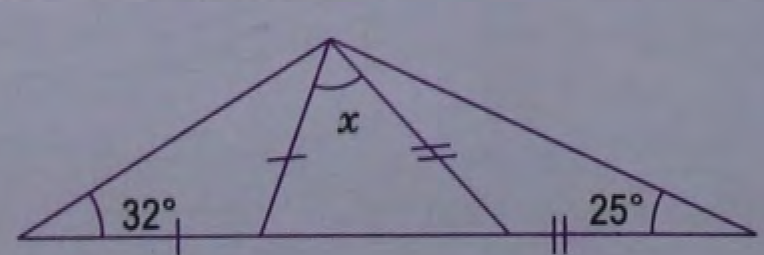
11. In a triangle ABC, ∠B = 90° and M is the mid-point of BC. Prove that $AC^2 = AM^2 + 3BM^2$.

Summary

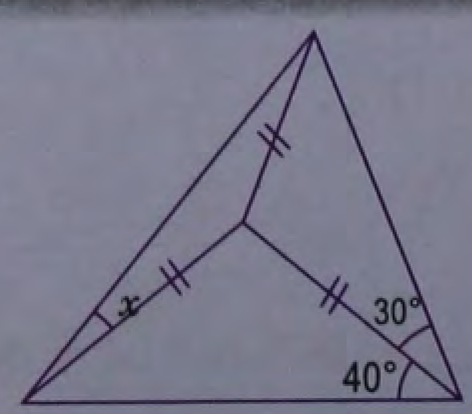
- A triangle is a closed figure bounded by three line segments. It has three sides, three (interior) angles and three vertices.
- In a triangle, the sum of an exterior angle and the adjacent interior angle is 180°.
- The sum of angles of a triangle is 180°.
- An exterior angle of a triangle is equal to the sum of its two opposite interior angles.
- In a triangle, the angles opposite equal sides are equal.
- In a triangle, the sides opposite equal angles are equal.
- In an equilateral triangle, each (interior) angle is 60°.
- The angles of an isosceles right angled triangle are 45°, 45° and 90°.
- **Inequalities**
  - If two sides of a triangle are unequal, then the greater side has greater angle opposite to it.
  - If two angles of a triangle are unequal, then the greater angle has greater side opposite to it.
  - The sum of the lengths of any two sides of a triangle is greater than the length of the third side.
  - The difference between the lengths of any two sides of a triangle is less than the length of the third side.
- Two triangles are congruent if and only if they have exactly the same shape and the same size.
- **Tests for congruency of two triangles**
  - S.S.S. — three sides of one triangle are equal to the three sides of the other triangle.
  - S.A.S. — two sides and the included angle of one triangle are equal to two sides and the included angle of the other triangle.
  - A.S.A. — two angles and the included side of one triangle are equal to two angles and the included side of the other triangle.
  - A.A.S. — two angles and a (non-included) side of one triangle are equal to two angles and the *corresponding side* of the other triangle.
  - R.H.S. — the hypotenuse and a side of one right angled triangle are equal to the hypotenuse and a side of the other right angled triangle.
- **Pythagoras theorem**
  - In a right angled triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides.
  - If in a triangle, the square on the longest side is equal to the sum of the squares on the other two sides, then the triangle is right angled and the angle opposite to the longest side is the right angle.

Check Your Progress

1. Calculate the value of x in each of the following sketches :

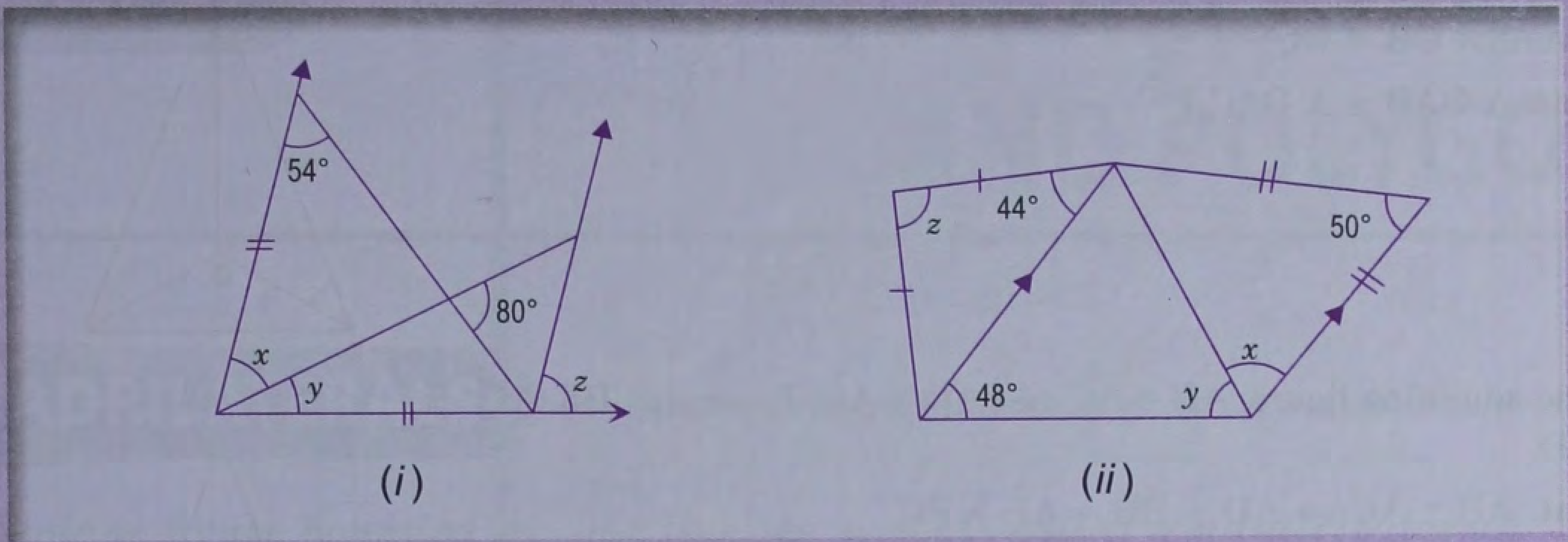


(*i*)

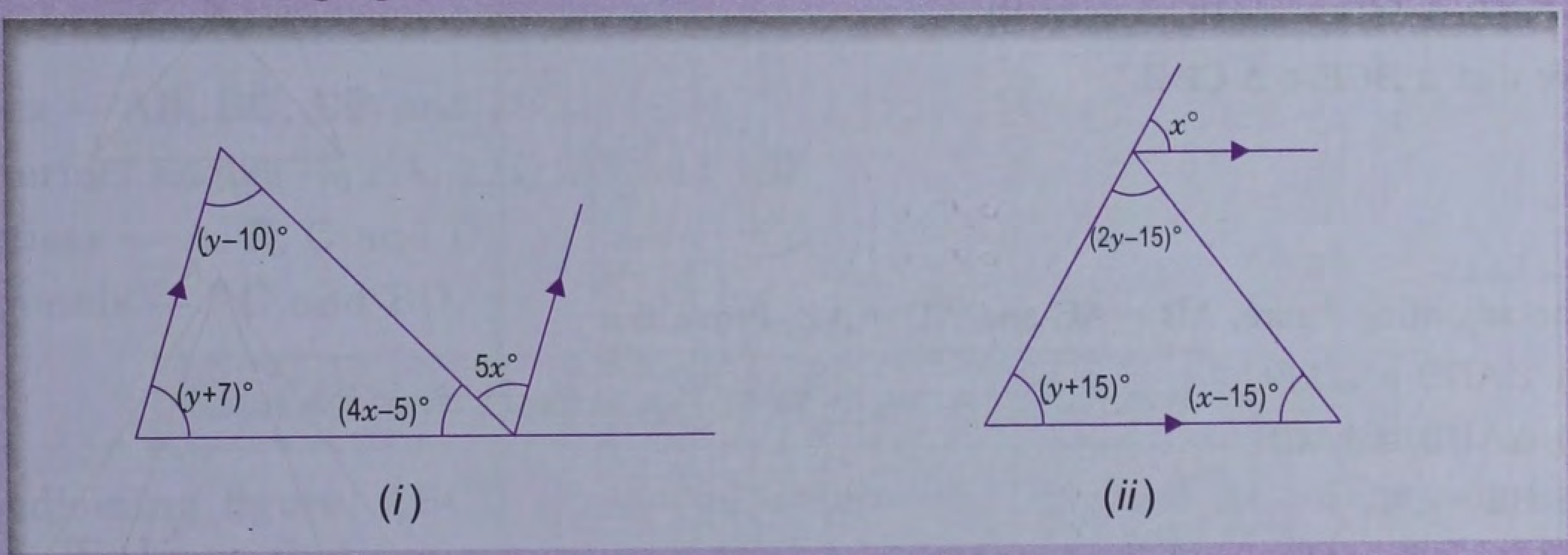


(*ii*)

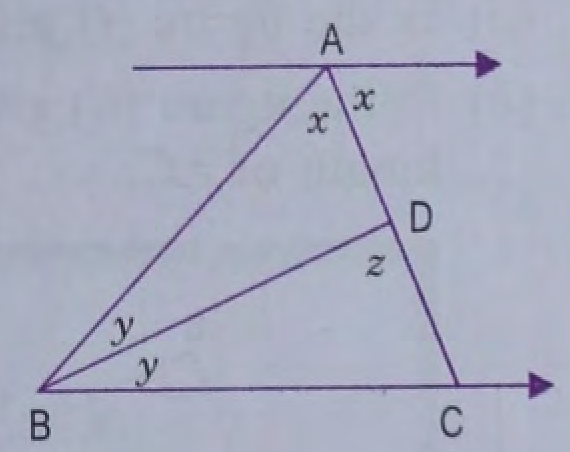
2. Calculate the size of each lettered angle in the following figures :



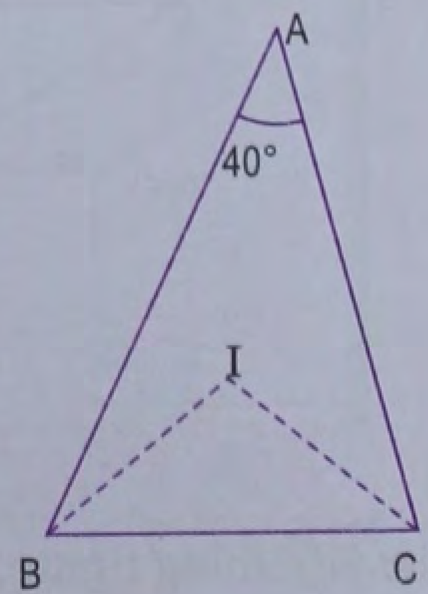
3. From the following figures, find the values of x and y :



4. The vertical angle of an isosceles triangle is three times the sum of its base angles. Find all the angles of the triangle.



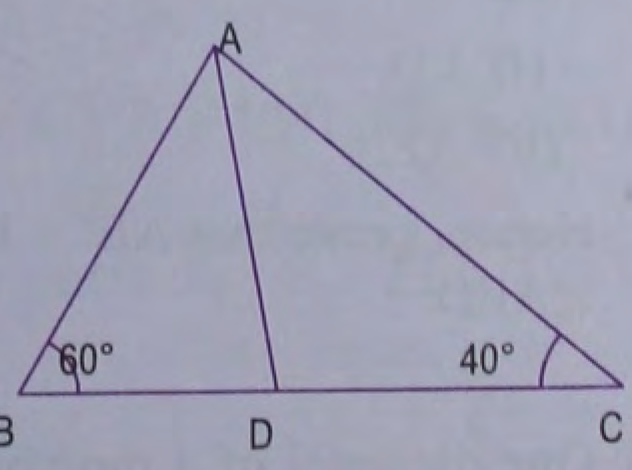
5. From the adjoining diagram, prove that
   (*i*) $x + y = 90°$
   (*ii*) $z = 90°$
   (*iii*) AB = BC.



6. In the adjoining figure, BI and CI are bisectors of ∠B and ∠C respectively.
   Find ∠BIC.
   [Hint. $\angle B + \angle C + 40° = 180°$
   $\Rightarrow \quad \angle B + \angle C = 140°$
   $\Rightarrow \quad \frac{1}{2}\angle B + \frac{1}{2}\angle C = 70°$
   Also $\frac{1}{2}\angle B + \frac{1}{2}\angle C + \angle BIC = 180°$.**]**

7. PQR is a right angled triangle at Q and PQ : QR = 3 : 2. Which is the least angle?

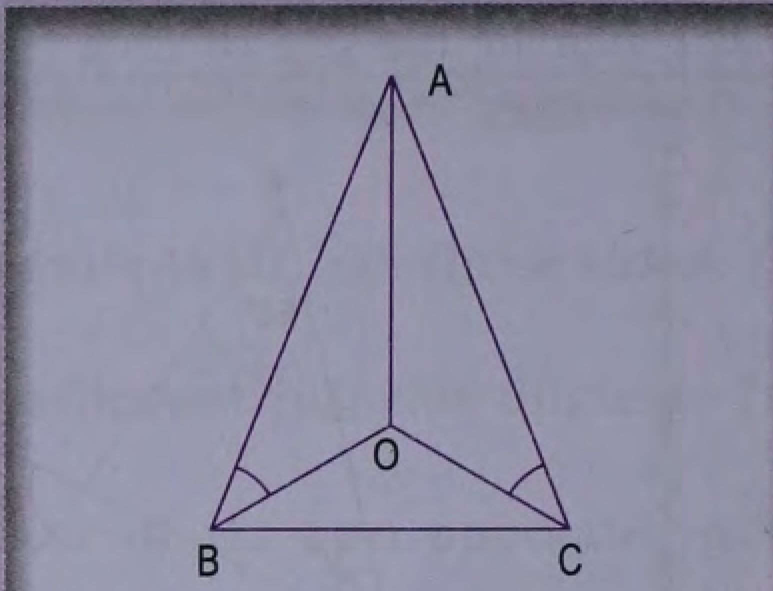
8. In Δ ABC, AB = 8 cm and BC = 6 cm. Find the greatest length and least length that CA can have.



9. In the adjoining figure, AD bisects ∠A. Arrange AB, BD and DC in the ascending order of their lengths.

10. In the adjoining figure, OA bisects ∠A and ∠ABO = ∠OCA Prove that OB = OC.

[Hint. Δ OAB ≅ Δ OAC.]



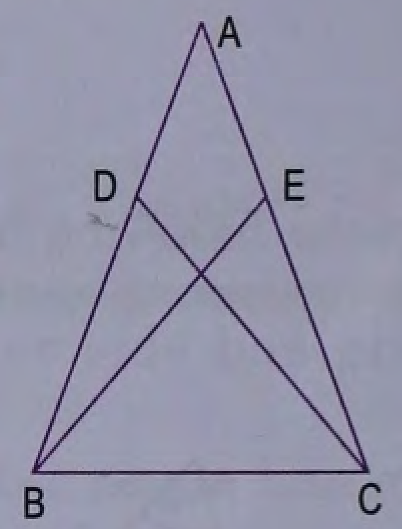
11. In the adjoining figure, AB = AC and AD = AE. Prove that BE = CD.

[Hint. AB = AC ⇒ AD + DB = AE + EC

⇒ DB = EC, ∵ AD = AE

Also AB = AC ⇒ ∠ABC = ∠ACB]

Prove that Δ BCE ≅ Δ CBD.



12. In the adjoining figure, AB = AC and AD = AE. Prove that

(*i*) ∠ADB = ∠AEC

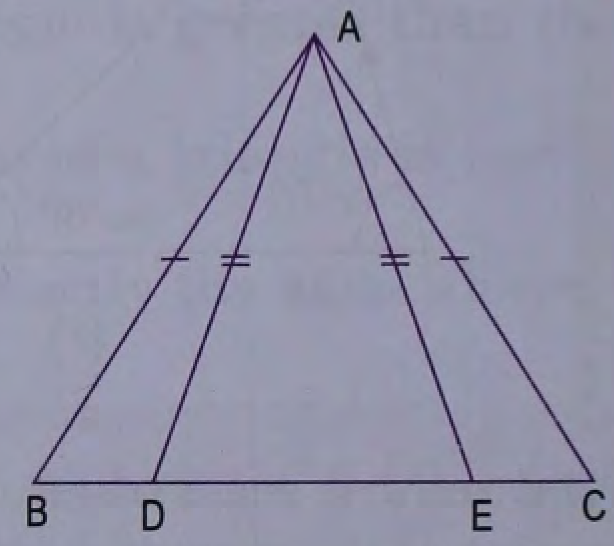
(*ii*) ΔABD ≅ ΔACE

(*iii*) BE = DC.

[Hint. (*i*) AD = AE ⇒ ∠ADE = ∠AED

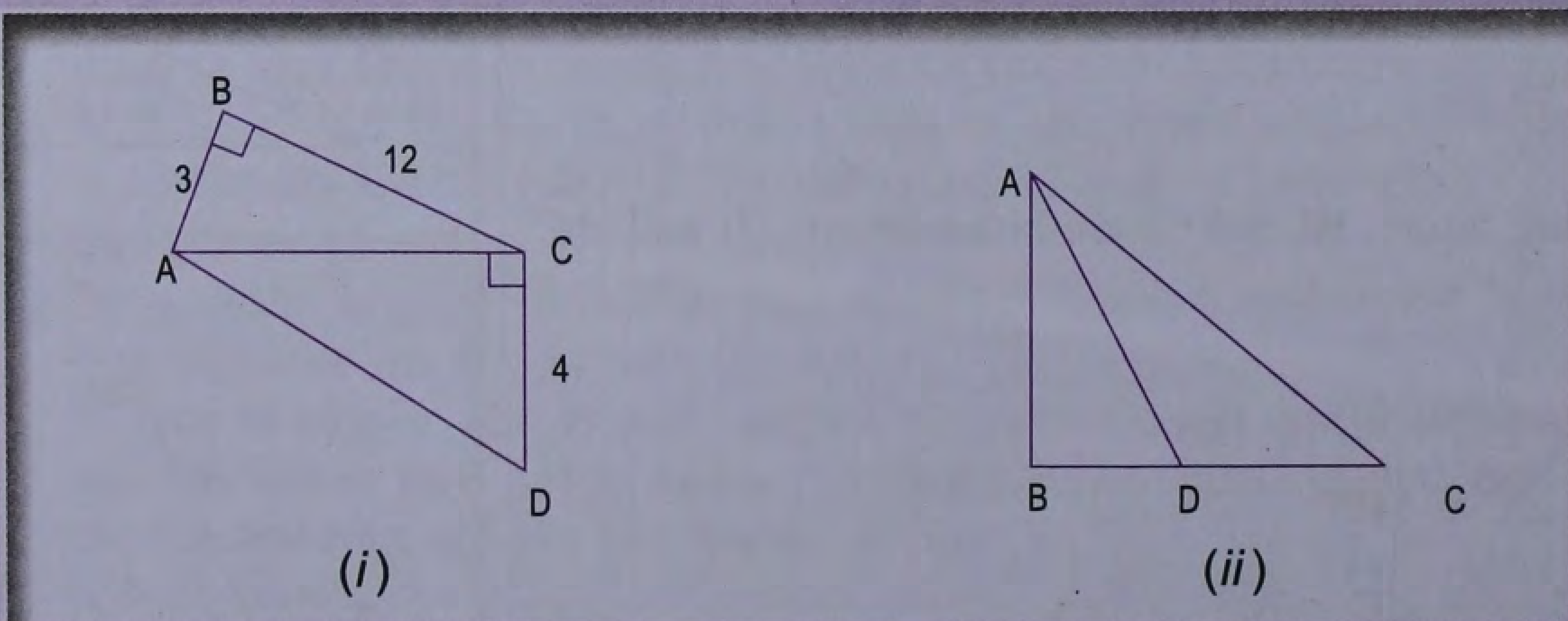
Also ∠ADB + ∠ADE = 180°

= ∠AED + ∠AEC.]



13. (*a*) In the figure (*i*) given below, all measurements are in centimetres. Find the length of AD.

(*b*) In the figure (*ii*) given below, AB = 6 cm, BD = 3 cm and AD = BC. If ∠ABC = 90°, find the length of AC.

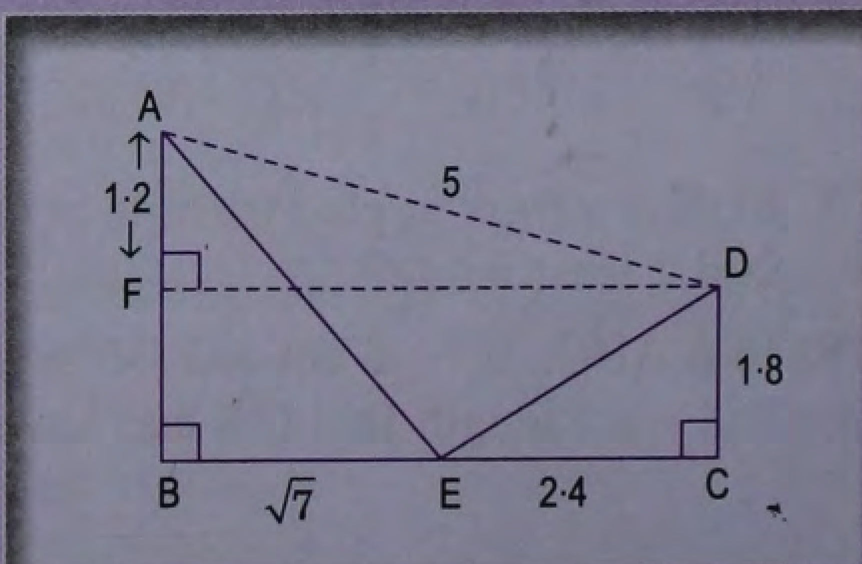


14. In the adjoining figure, all measurements are in centimetres. Find

(*i*) ED

(*ii*) AE.

Hence, prove that $AE^2 + ED^2 = AD^2$. Give the measure of ∠AED.



15. One diagonal of a rhombus is 30 cm long. If one of its side is 17 cm, find the length of the other diagonal.