Chapter 21

FUNDAMENTAL GEOMETRICAL CONCEPTS

In this chapter, we shall have a brief revision of geometrical concepts, angles, their types and properties of angles associated with parallel lines. Stress is being laid on their applications and numerical problems.

GEOMETRICAL CONCEPTS

Point

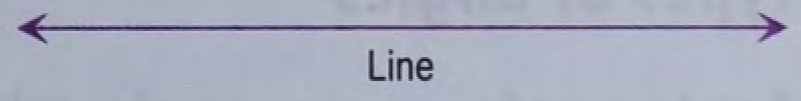
A small dot marked by a sharp pencil on a sheet of paper or a prick made by a fine needle on a paper are examples of a point.

A point determines a location in space. It has no length, breadth or thickness.

Line

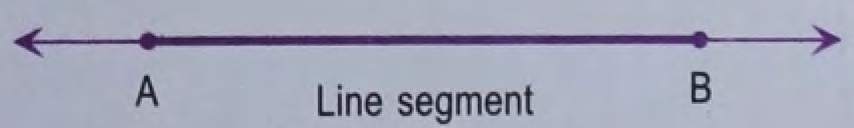
A line has length only. It has no breadth or thickness.

The basic concept of a line is its *straightness* and it extends indefinitely in both directions. The two arrowheads in the opposite directions indicate that the length of a line is unlimited *i.e.* it has no definite length.



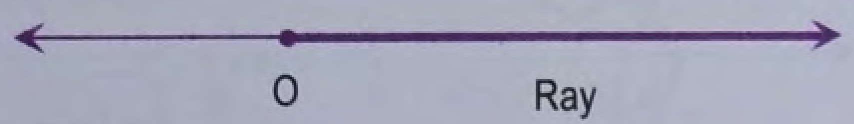
A line has no end points and it consists of an infinite (uncountable) number of points.

Line segment



A line segment is a portion of a line. It has two end points and a definite length.

Ray



A ray is a part of a line that extends indefinitely in one direction from a point on the line, say O. O is the *initial point* of the ray.

A ray has only one end point and it has unlimited length.

Plane

A plane has length and breadth. It has no thickness.

The basic concept of a plane is its *flatness* and it extends indefinitely in all directions. Thus, a plane is a flat surface which extends indefinitely in all directions. The length and breadth of a plane are unlimited *i.e.* a plane has no definite length and no definite breadth.

An unlimited number of lines can be drawn in a plane through a given point in a plane.

Exactly one line passes through two different given points in a plane and it lies wholly in that plane.

Two different lines in a plane either intersect at exactly one point or are parallel.

Space

Space is the set of all points in the universe. So, a point or a line or a plane is a subset of space. In fact, everything we look at is a part of space.

Axiom

The statements which are obviously true are called *axioms* (or *postulates*) and are accepted without proof. In fact, these form the basis of the subject.

Theorem

A *theorem* is a statement of mathematical truth which has to be proved from the facts already proved or assumed.

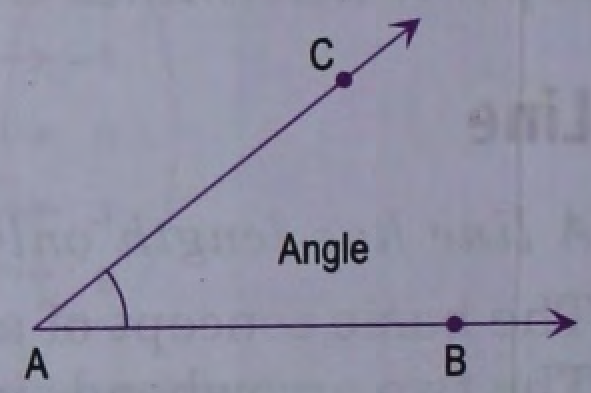
Corollary

A statement whose truth can easily be derived from a theorem is called its *corollary*.

ANGLE

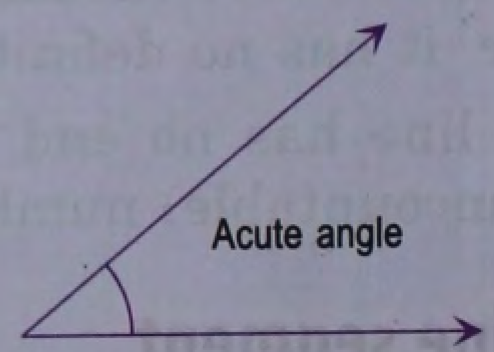
An angle is the figure formed by two rays (or line segments) with same initial (or common) point.

The initial (or common) point is its vertex and the two rays (or line segments) are its arms or sides.

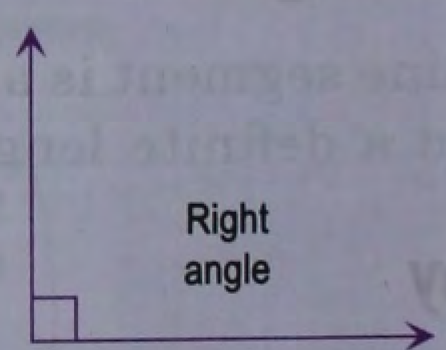


Types of angles

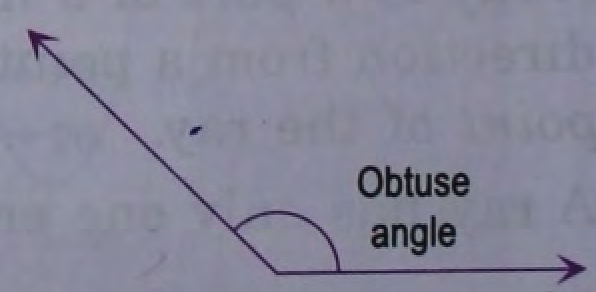
Acute angle — an angle whose measure lies between 0° and 90°.



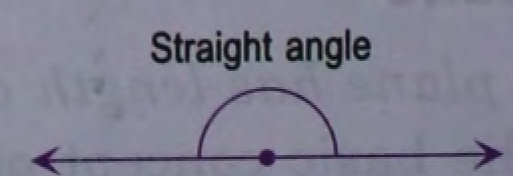
Right angle — an angle whose measure is 90°. It is a quarter turn.



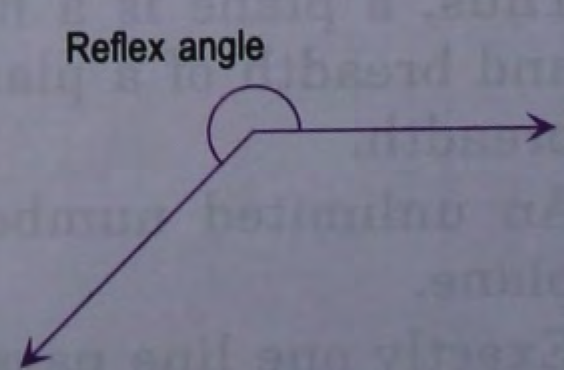
Obtuse angle — an angle whose measure lies between 90° and 180°.



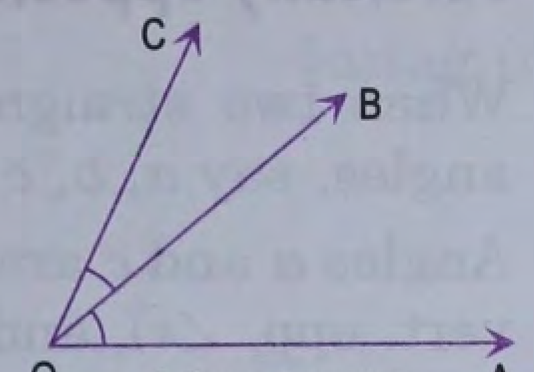
Straight angle — an angle whose measure is 180°. It is a half turn.



Reflex angle — an angle whose measure lies between 180° and 360°.



Adjacent angles

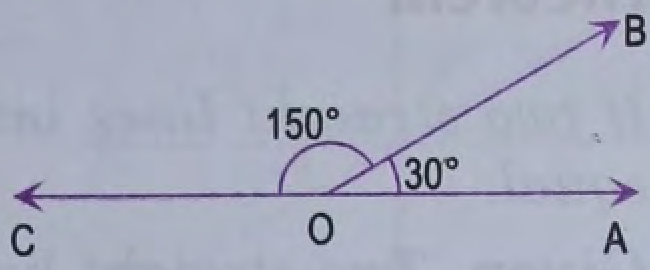


Adjacent angles

Two angles are called *adjacent angles* if they have a common vertex, a common arm and their other arms lie on either side of the common arm.

In the adjoining figure, ∠AOB and ∠BOC are adjacent angles.

If the sum of any two adjacent angles is 180° then their exterior arms are in the same straight line.

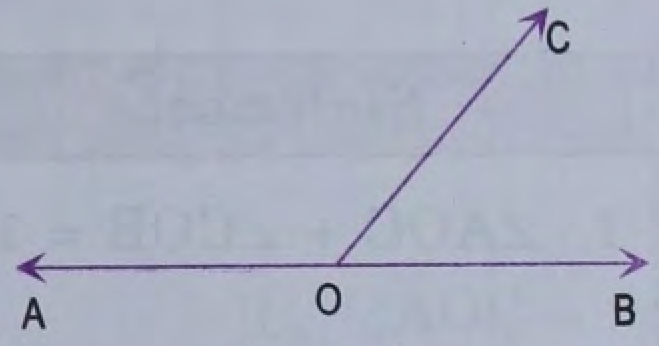


In the adjoining figure,

∠AOB + ∠BOC = 30° + 150° = 180°.

So COA is a straight line.

Linear pair



Linear pair

Two adjacent angles whose exterior arms are in a straight line are said to form a *linear pair i.e.* two adjacent angles are said to form a *linear pair* if the sum of their measures is 180°.

Here, AOB is a straight line, so ∠ AOB = 180°.

Therefore, ∠ AOC and ∠ COB form a linear pair.

Complementary angles

Two angles are called *complementary angles* if the sum of their measures is 90°. Each angle is called the complement of the other.

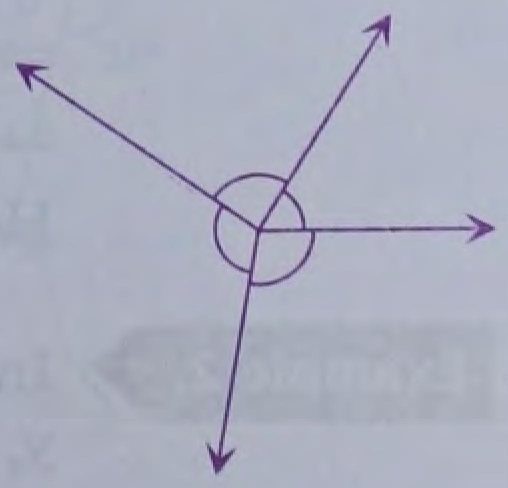
Complementary angles need not be adjacent angles.

Supplementary angles

Two angles are called *supplementary angles* if the sum of their measures is 180°. Each angle is called the supplement of the other.

Supplementary angles need not be adjacent angles.

Angles at a point

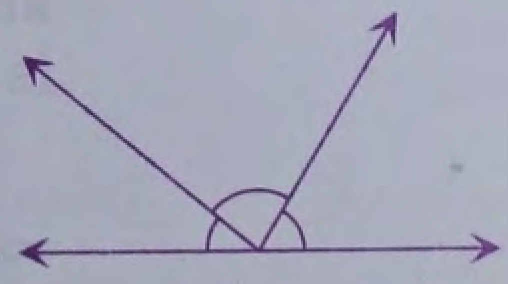


In the adjoining figure, the four angles together make one complete rotation, so they add upto 360°.

This is true no matter how many angles are formed at a point. Thus :

Sum of angles at a point = 360°

Angles on one side of a straight line

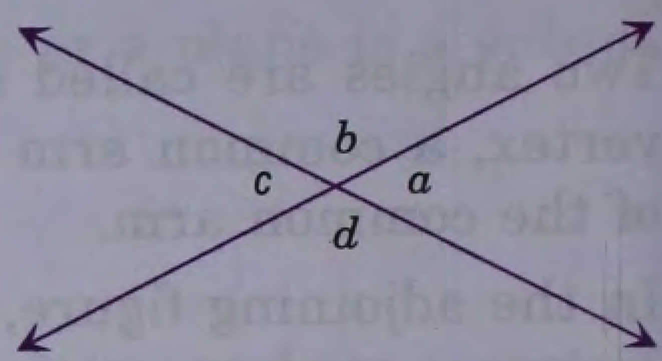


In the adjoining figure, the three angles together make a straight line, so they add upto 180°.

This is true no matter how many angles make up the straight line. Thus :

Sum of angles at a point on one side of a straight line = 180°

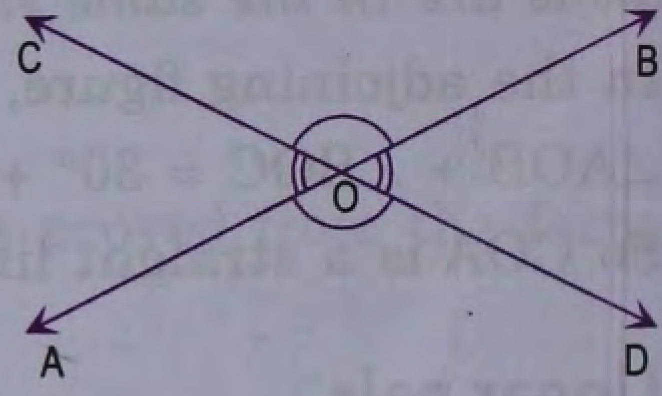
Vertically opposite angles



When two straight lines intersect each other, they form four angles, say a, b, c and d.

Angles a and c are called *vertically opposite angles* (abbreviated vert. opp. ∠s), and so are angles b and d.

Theorem



If two straight lines intersect, the vertically opposite angles are equal.

Given. Two straight lines AB and CD intersect at the point O.

To prove. ∠AOC = ∠BOD and ∠COB = ∠AOD

Proof.

Statements	*Reasons*
1. ∠AOC + ∠COB = 180°	1. AOB is a straight line, sum of angles at a point on one side of a line = 180°
2. ∠COB + ∠BOD = 180°	2. COD is a straight line
3. ∠AOC + ∠COB = ∠COB + ∠BOD	3. From 1 and 2
4. ∠AOC = ∠BOD Similarly ∠COB = ∠AOD † **Q.E.D.**	4. ∠COB is common to both sides

Example 1. An angle is one-fourth of its complement. Find the size of the angles.

Solution. Let the angle be $x°$, then its complement is $90° - x°$.

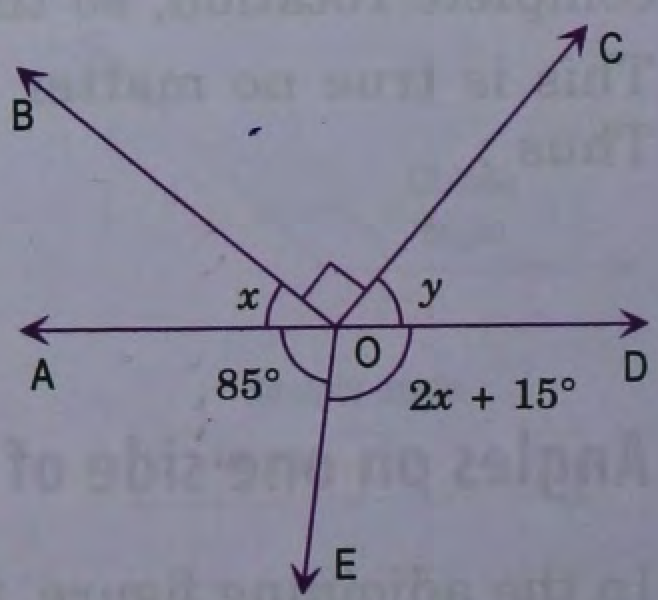
According to given information, $x° = \frac{1}{4}(90° - x°)$

$\Rightarrow \quad 4x° = 90° - x° \quad \Rightarrow \quad 5x° = 90°$

$\Rightarrow \quad x° = 18°$ and $90° - x° = 90° - 18° = 72°$

Hence, the required angles are 18° and 72°.

Example 2. In the adjoining figure, find the values of x and y, given that AOD is a straight line.



Solution.

State which geometrical fact is used to calculate angle

Since AOD is a straight line and the sum of angles at a point on one side of a straight line is 180°,

$85° + 2x + 15° = 180°$

$\Rightarrow \quad 2x = 180° - 85° - 15° = 80°$

$\Rightarrow \quad x = 40°.$ ...(*i*)

† The letters Q.E.D. (quod erat demonstrandum) written at the end, indicate that the proof has been completed.

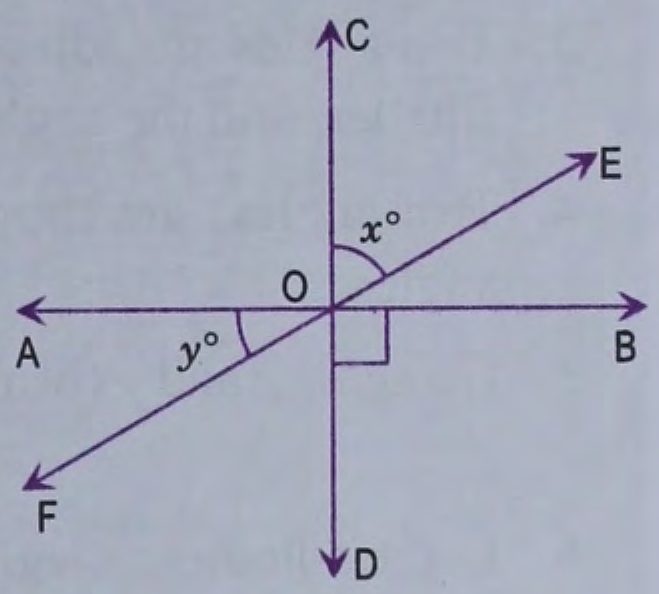
Also $x + 90° + y = 180°$

$\Rightarrow$ $40° + 90° + y = 180°$ [using (*i*)]

$\Rightarrow$ $y = 180° - 40° - 90° = 50°$

Hence, $x = 40°$ and $y = 50°$.

Example 3. In the adjoining figure, the three straight lines AB, CD and EF all pass through the point O. If $\angle BOD = 90°$ and $x : y = 2 : 1$, find $\angle BOE$ and $\angle FOD$.



Solution. Given $x : y = 2 : 1$ $\Rightarrow$ $\frac{x}{y} = \frac{2}{1}$

$\Rightarrow$ $x = 2y$ ...(*i*)

$\angle AOC = \angle BOD$ (vert. opp. $\angle s$)

$\Rightarrow$ $\angle AOC = 90°$ ($\because \angle BOD = 90°$ given)

Since FOE is a straight line, sum of angles at a point on one side of a straight line is 180°.

$\angle FOA + \angle AOC + \angle COE = 180°$

$\Rightarrow$ $y° + 90° + x° = 180°$ ($\because \angle AOC = 90°$)

$\Rightarrow$ $y° + 2y° = 180° - 90°$ [using (*i*)]

$\Rightarrow$ $3y° = 90° \Rightarrow y = 30$

$\therefore$ $x = 2 \times 30 = 60$ [using (*i*)]

$\angle BOE = y°$ (vert. opp. $\angle s$)

$\Rightarrow$ $\angle BOE = 30°$ ($\because y = 30$)

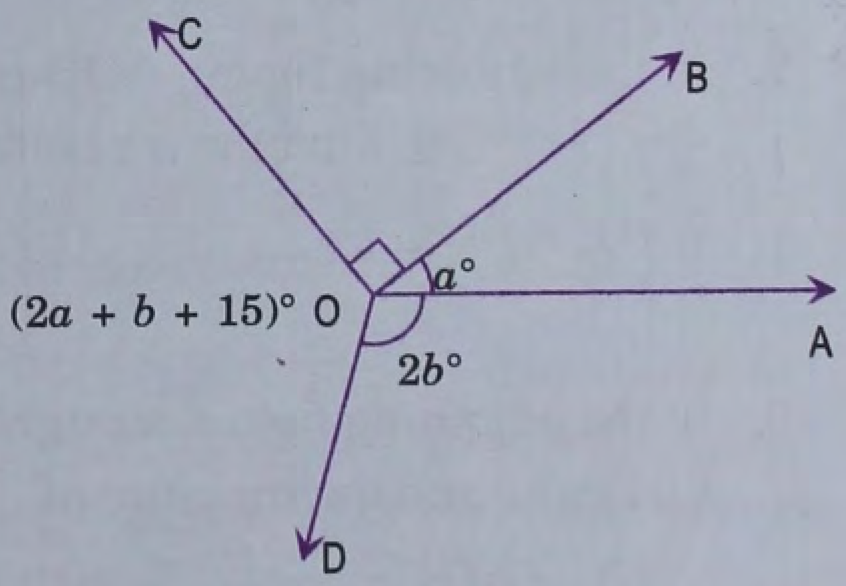
$\angle FOD = x°$ (vert. opp. $\angle s$)

$\Rightarrow$ $\angle FOD = 60°$. ($\because x = 60$)

Example 4. In the given figure, $2b - a = 65$ and $\angle BOC = 90°$. Find the measures of

(*i*) $\angle AOB$ (*ii*) $\angle AOD$

(*iii*) $\angle COD$.



Solution. Since the sum of angles at a point $= 360°$,

$\therefore$ $a° + 90° + (2a + b + 15)° + 2b° = 360°$

$\Rightarrow$ $3a + 3b = 360 - 90 - 15 = 255$

$\Rightarrow$ $a + b = 85$ ...(*i*)

Also $2b - a = 65$ ...(*ii*) (given)

Adding (*i*) and (*ii*), we get

$3b = 150 \Rightarrow b = 50$

Substituting this value of b in (*i*), we get

$a + 50 = 85 \Rightarrow a = 35$

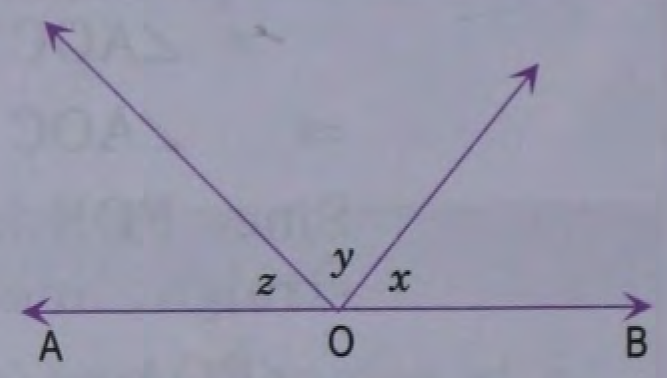
(*i*) $\angle AOB = a° = 35°$

(*ii*) $\angle AOD = 2b° = 2 \times 50° = 100°$

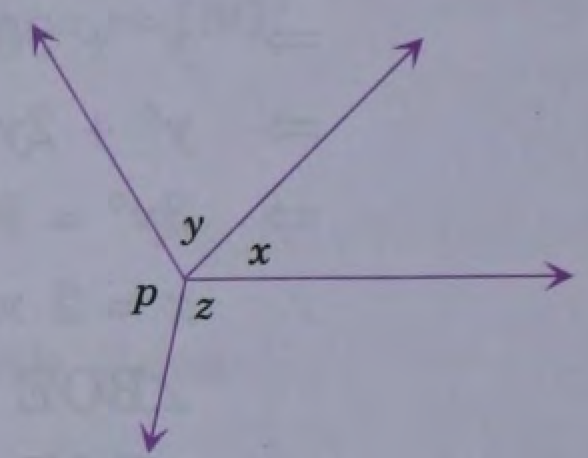
(*iii*) $\angle COD = (2a + b + 15)° = (2 \times 35 + 50 + 15)° = 135°$.

Exercise 21.1

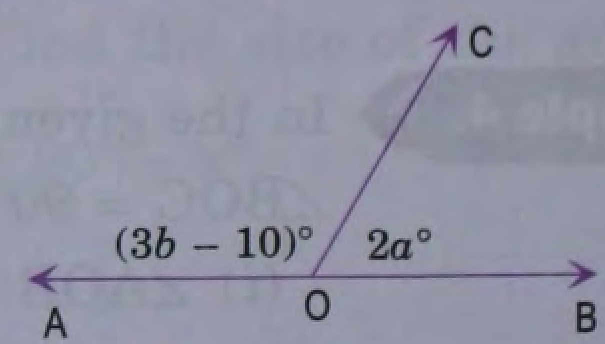
1. Two complementary angles are in the ratio 7 : 11. Find the angles.
2. An angle is 30° more than one-half of its complement. Find the angle in degrees.
3. Two angles are adjacent and form an angle of 135°. If the larger is 15° more than three times the smaller, find the angles.
4. Two angles are supplementary and the larger is 20° less than three times the smaller. Find the angles.
5. If angles x and y form a linear pair and $2x - 3y = 35°$, find the values of x and y.



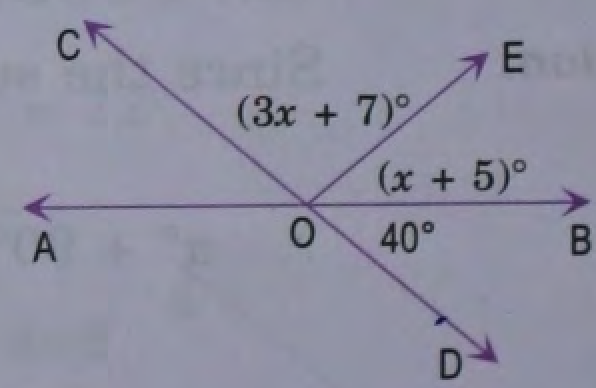
6. In the adjoining diagram, AOB is a straight line.
   (*i*) If $x = 55°$, $z = 66°$, find y.
   (*ii*) If $x = 35°$, $y = 2x$, find z.
   (*iii*) If $x : y : z = 3 : 5 : 7$, find x.



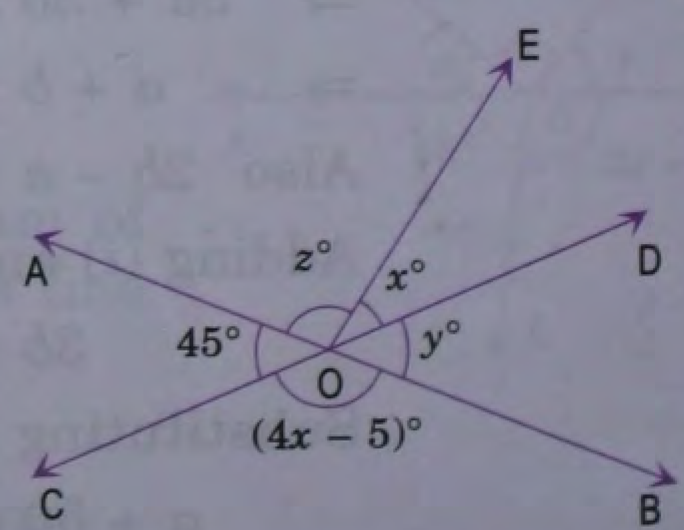
7. In the adjoining figure, $x : y : z : p = 3 : 5 : 6 : 10$. Find the values of x, y, z and p.
8. If the measures of two adjacent angles on a straight line are $(2x + 5)°$ and $3x°$, find the angles.



9. In the adjoining figure, AOB is a straight line and $4a - 3b = 20$. Find the measures of
   (*i*) ∠BOC (*ii*) ∠AOC.



10. In the adjoining figure, straight lines AB and CD intersect at O. Find x and the measure of
    (*i*) ∠AOC (*ii*) ∠AOD
    (*iii*) ∠COE.



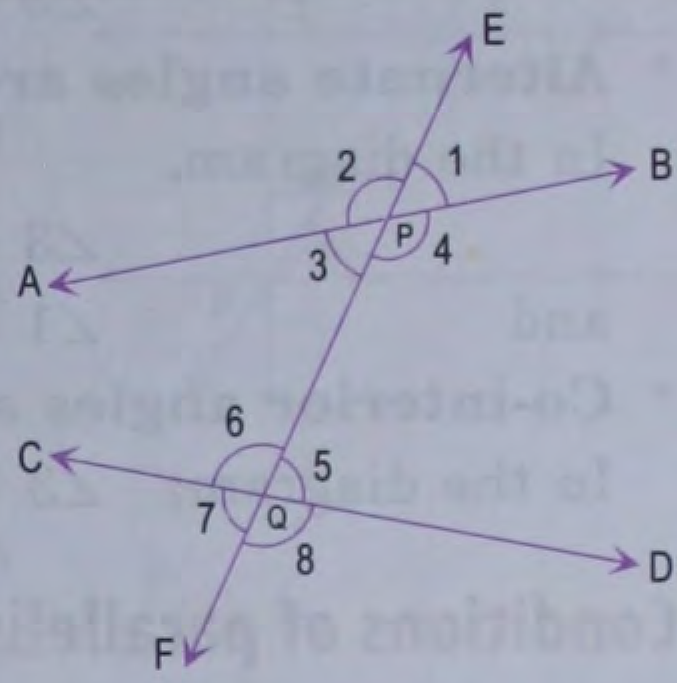
11. In the adjoining figure, lines AB and CD intersect each other at the point O. If ∠AOC = 45°, find the values of x, y and z.
12. Prove that the bisectors of two adjacent supplementary angles are at right angles.
13. Angles x and y are adjacent. State whether their exterior arms are in a straight line when
    (*i*) $x = 112°$, $y = 68°$ (*ii*) $x = 56°$, $y = 114°$.
    [Hint. Find sum of angles.**]**

TRANSVERSAL

A straight line that intersects two (or more) straight lines in a plane at different points is called a *transversal*.

In the adjoining figure, line EF intersects two lines AB and CD at distinct points P and Q. So line EF is a transversal.

Notice that four angles are formed at P and four angles are formed at Q. For convenience, we label them by the numerals 1 to 8 (as shown in figure).



Interior angles

The angles that contain line segment PQ as one of their arms are called *interior angles*. Here angles 3, 4, 5 and 6 are interior angles.

Exterior angles

The angles that do not contain line segment PQ as one of their arms are called *exterior angles*. Here angles 1, 2, 7 and 8 are exterior angles.

Some of these eight angles can be paired and have special names.

Corresponding angles

The four pairs $\angle 1, \angle 5$; $\angle 2, \angle 6$; $\angle 3, \angle 7$ and $\angle 4, \angle 8$ are called pairs of *corresponding angles* (abbreviated corres. $\angle s$).

Alternate angles

The two pairs $\angle 3, \angle 5$ and $\angle 4, \angle 6$ are called *alternate interior angles* or simply the *alternate angles* (abbreviated alt. $\angle s$).

The two pairs $\angle 1, \angle 7$ and $\angle 2, \angle 8$ are called *alternate exterior angles* (abbreviated alt. ext. $\angle s$).

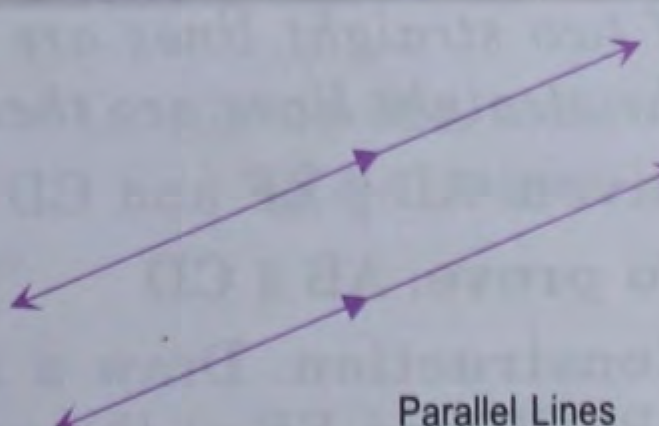
Co-interior angles

The two pairs $\angle 3, \angle 6$ and $\angle 4, \angle 5$ are called *co-interior* or *consecutive interior* or *conjoined* or *allied angles* (abbreviated co-int. $\angle s$).

PARALLEL LINES

Two different lines drawn in a plane are *parallel* if they do not meet.

The distance between the parallel lines remains the same.

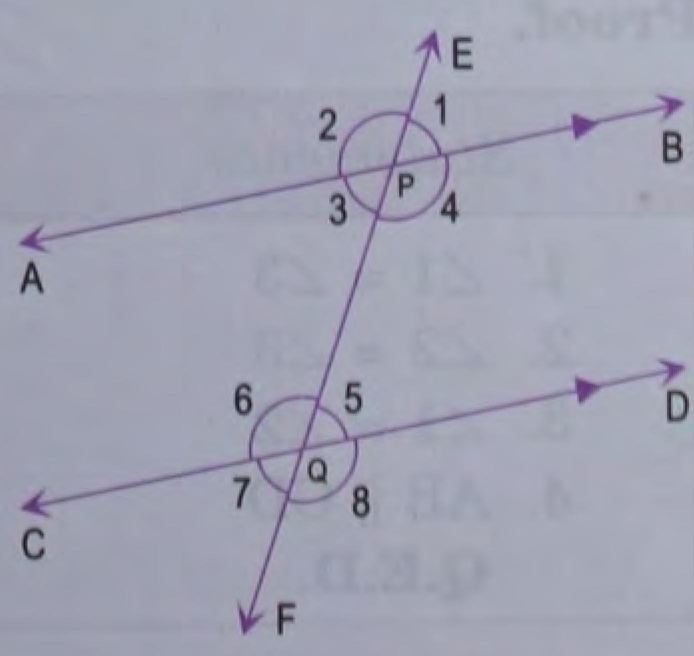


Parallel Lines

Properties of angles associated with parallel lines

Let a transversal EF intersect two parallel lines AB and CD at the points P and Q respectively.

Eight angles are formed — four at P and four at Q. For convenience, label these angles by the numerals 1 to 8. Then :



* **Corresponding angles are equal.**

In the diagram, $\angle 1 = \angle 5$, $\angle 2 = \angle 6$,
$\angle 3 = \angle 7$ and $\angle 4 = \angle 8$

* **Alternate angles are equal.**
In the diagram,
$\angle 3 = \angle 5$, $\angle 4 = \angle 6$ (alternate interior angles)
and $\angle 1 = \angle 7$, $\angle 2 = \angle 8$ (alternate exterior angles)

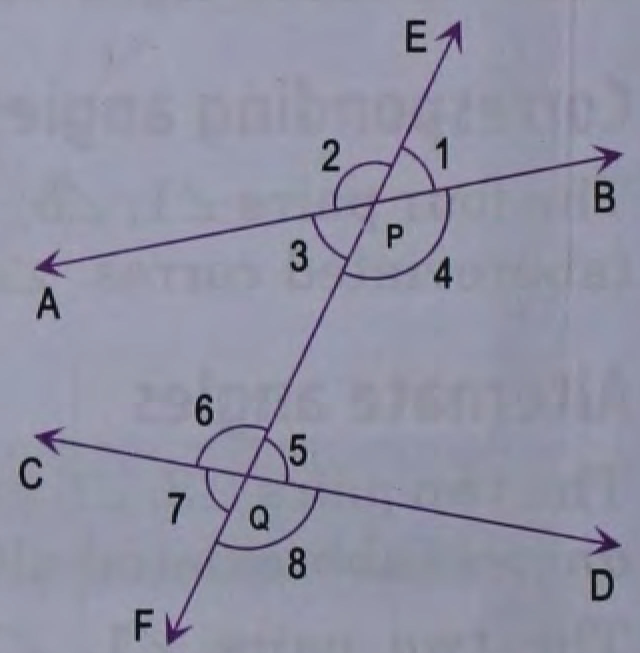
* **Co-interior angles are supplementary angles.**
In the diagram, $\angle 3 + \angle 6 = 180°$, $\angle 4 + \angle 5 = 180°$

Conditions of parallelism

The converse of the above results is also true.

- *If two lines are cut by a transversal such that a pair of corresponding angles is equal, then the lines are parallel.*
- *If two lines are cut by a transversal such that a pair of alternate angles is equal, then the lines are parallel.*
- *If two lines are cut by a transversal such that a pair of co-interior angles is supplementary, then the lines are parallel.*

In the adjoining figure, a transversal EF intersects two **non-parallel lines** AB and CD at the points P and Q respectively. Eight angles are formed – four at P and four at Q. For convenience, label these angles by the numerals 1 to 8.

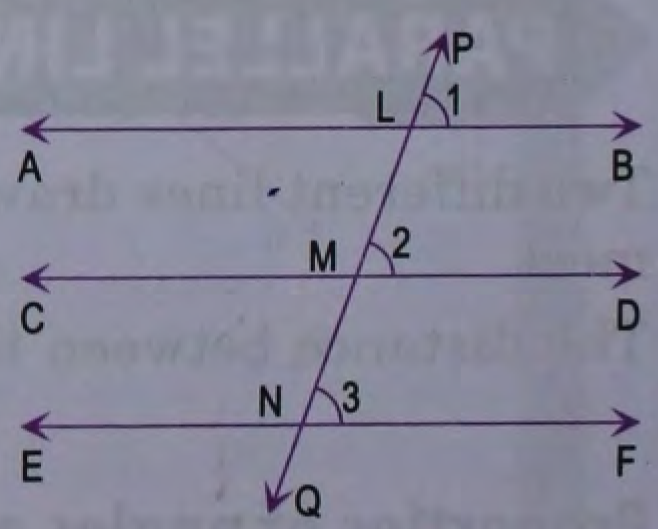


As the lines AB and CD are *not parallel*, we have :

- *Corresponding angles are not equal.*
$\angle 1 \neq \angle 5$, $\angle 2 \neq \angle 6$,
$\angle 3 \neq \angle 7$ and $\angle 4 \neq \angle 8$.
- *Alternate angles are not equal.*
$\angle 3 \neq \angle 5$, $\angle 4 \neq \angle 6$, (alternate interior angles)
$\angle 1 \neq \angle 7$ and $\angle 2 \neq \angle 8$. (alternate exterior angles)
- *Co-interior angles are not supplementary angle.*
$\angle 3 + \angle 6 \neq 180°$, $\angle 4 + \angle 5 \neq 180°$.

Theorem

If two straight lines are parallel to a third straight line, then the straight lines are themselves parallel.



Given. AB || EF and CD || EF

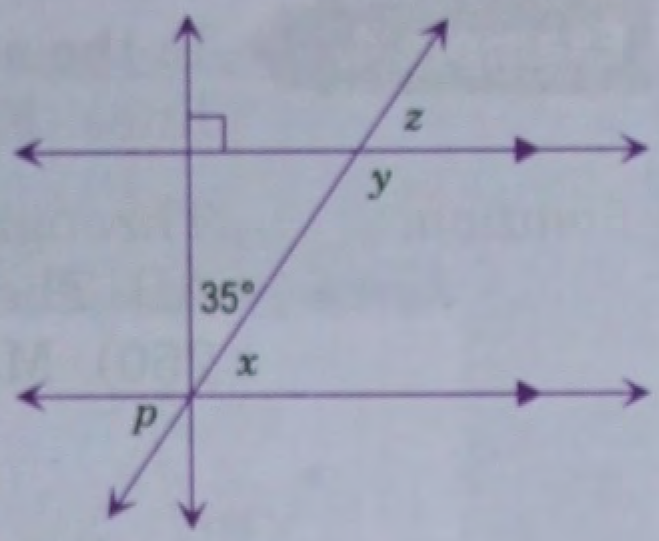
To prove. AB || CD

Construction. Draw a transversal PQ to cut the given lines AB, CD and EF at the points L, M and N respectively.

Proof.

Statements	*Reasons*
1. $\angle 1 = \angle 3$	1. AB || EF, property of corres. $\angle s$
2. $\angle 2 = \angle 3$	2. CD || EF, property of corres. $\angle s$
3. $\angle 1 = \angle 2$	3. From 1 and 2
4. AB || CD **Q.E.D.**	4. Condition of parallelism

Example 1. From the adjoining diagram, find the values of x, y, z and p.



Solution.

$x + 35° = 90°$ (corres. ∠s)

$\Rightarrow x = 90° - 35° = 55°$

$x + y = 180°$ (co-int. ∠s)

$\Rightarrow 55° + y = 180°$

$\Rightarrow y = 180° - 55° = 125°$

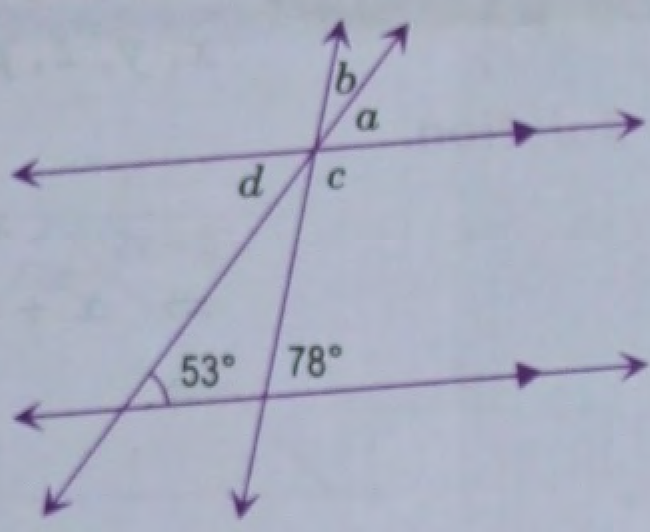
$z = x$ (corres. ∠s)

$\Rightarrow z = 55°$ ($\because x = 55°$)

$p = x$ (vert. opp. ∠s)

$\Rightarrow p = 55°.$ ($\because x = 55°$)

Example 2. From the adjoining diagram, find the values of a, b, c and d.



Solution.

$a = 53°$ (corres. ∠s)

$78° + c = 180°$ (co-int. ∠s)

$\Rightarrow c = 180° - 78° = 102°$

$b + a = 78°$ (corres. ∠s)

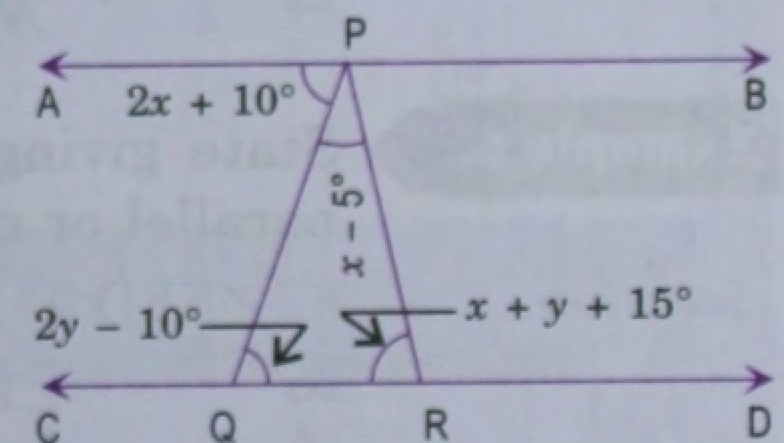
$\Rightarrow b + 53° = 78°$ ($\because a = 53°$)

$\Rightarrow b = 78° - 53° = 25°$

$d = a$ (vert. opp. ∠s)

$\Rightarrow d = 53°.$ ($\because a = 53°$)

Example 3. In the adjoining figure, lines AB and CD are parallel. Find the values of x and y.



Solution.

$2x + 10° = 2y - 10°$ (alt. ∠s, AB ∥ CD)

$\Rightarrow 2x - 2y + 20° = 0$

$\Rightarrow x - y + 10° = 0$ ...(*i*)

$\angle RPB = x + y + 15°$ (alt. ∠s, AB ∥ CD)

$\angle APQ + \angle QPR + \angle RPB = 180°$ (angle on the same side of a straight line)

$\Rightarrow (2x + 10°) + (x - 5°) + (x + y + 15°) = 180°$

$\Rightarrow 4x + y + 20° - 180° = 0$

$\Rightarrow 4x + y - 160° = 0$ ...(*ii*)

On adding (*i*) and (*ii*), we get

$5x - 150° = 0 \Rightarrow x = 30°$

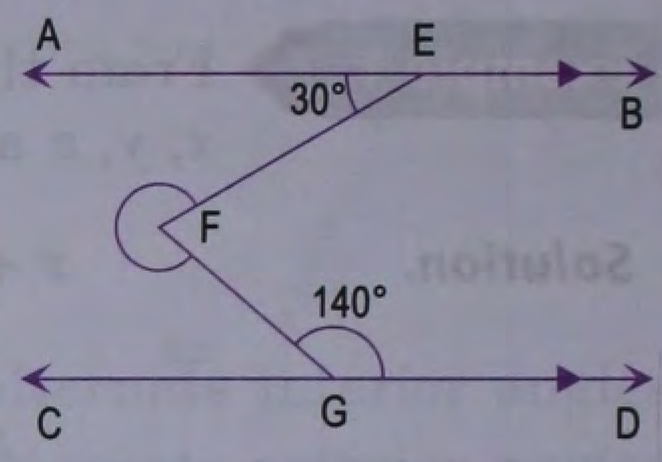
Putting $x = 30°$ in (*i*), we get

$30° - y + 10° = 0$

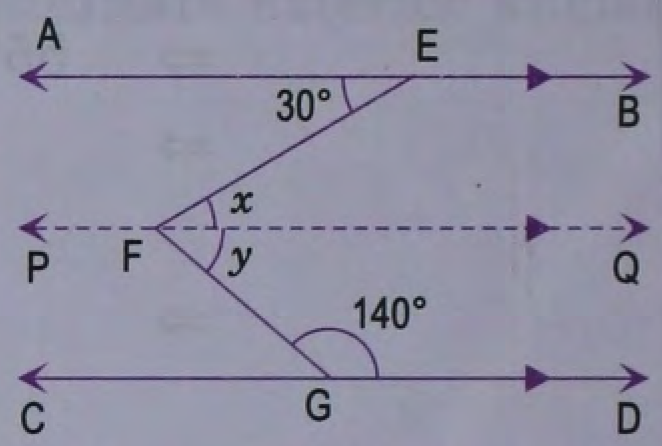
$\Rightarrow 40° - y = 0 \Rightarrow y = 40°$

Hence, $x = 30°$ and $y = 40°$.

Example 4. In the adjoining diagram, AB and CD are parallel lines. Find the size of the reflex angle EFG.



Solution. Through F draw a straight line PQ parallel to AB. Then, PQ is parallel to CD (theorem on page 250). Mark the angles as shown.



$x = 30°$ (alt. $\angle s$)

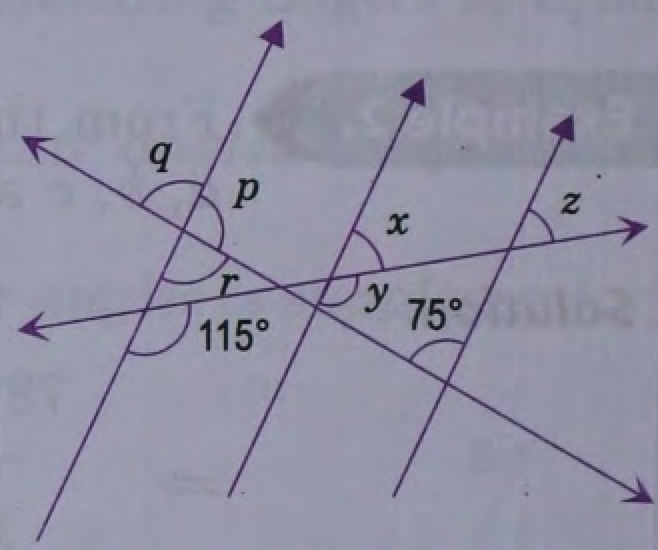
$y + 140° = 180°$ (co-int. $\angle s$)

$\Rightarrow$ $y = 180° - 140° = 40°$

$\therefore$ $\angle EFG = x + y = 30° + 40° = 70°$

$\therefore$ Reflex $\angle EFG = 360° - 70° = 290°$.

Example 5. From the adjoining diagram, find the values of x, y, z, p, q and r.



Solution. $y = 115°$ (corres. $\angle s$)

$x + y = 180°$ (linear pair)

$\Rightarrow$ $x + 115° = 180°$

$\Rightarrow$ $x = 180° - 115° = 65°$

$z = x$ (corres. $\angle s$)

$\Rightarrow$ $z = 65°$ ($\because$ $x = 65°$)

$r = 75°$ (alt. $\angle s$)

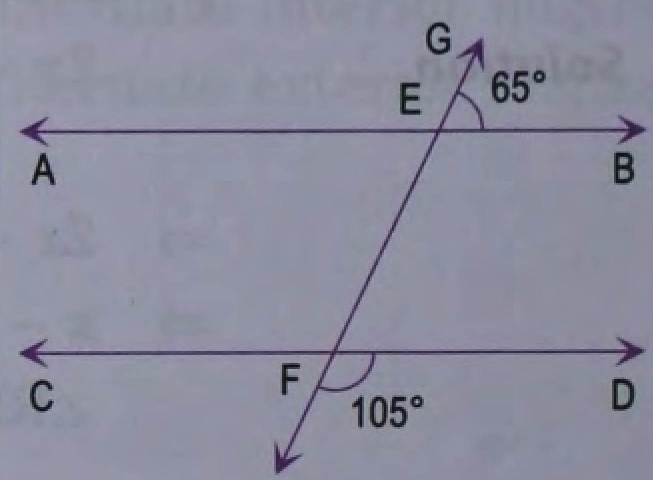
$p + r = 180°$ (linear pair)

$\Rightarrow$ $p + 75° = 180° \Rightarrow p = 180° - 75° = 105°$

$q = r$ (vert. opp. $\angle s$)

$\Rightarrow$ $q = 75°$. ($\because$ $r = 75°$)

Example 6. State giving reasons whether AB and CD are parallel or not.



Solution. $\angle EFD + 105° = 180°$ (linear pair)

$\Rightarrow$ $\angle EFD = 180° - 105° = 75°$

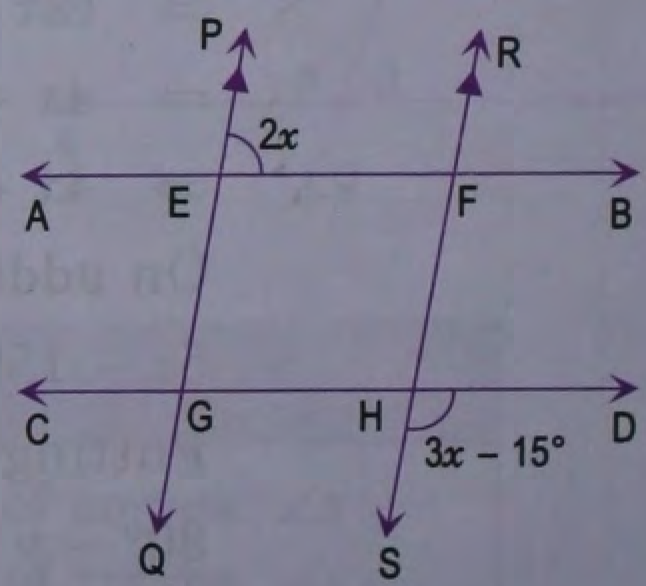
$\angle GEB = 65°$ (given)

$\Rightarrow$ $\angle GEB \neq \angle EFD$

$\Rightarrow$ corresponding angles are not equal.

Hence, AB and CD are not parallel lines.

Example 7. In the adjoining diagram, lines PQ and RS are parallel. For what value of x will the lines AB and CD be parallel?



Solution. $\angle EFH = 2x$ (alt. $\angle s$)

$\angle HFB + \angle EFH = 180°$ (linear pair)

$\Rightarrow$ $\angle HFB + 2x = 180°$

$\Rightarrow$ $\angle HFB = 180° - 2x$

Now, the lines AB and CD will be parallel if

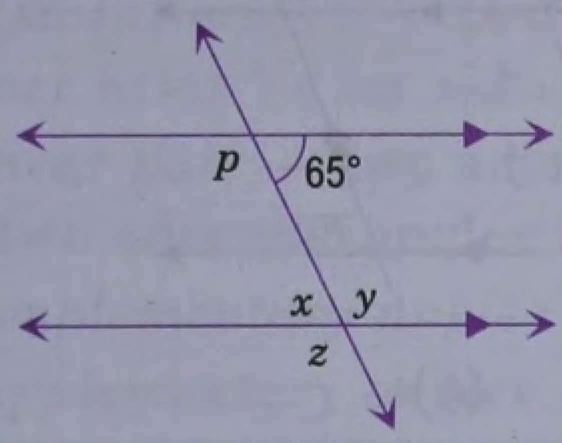
$\angle HFB = \angle SHD$ (corres. $\angle s$)

$\Rightarrow$ $180° - 2x = 3x - 15°$

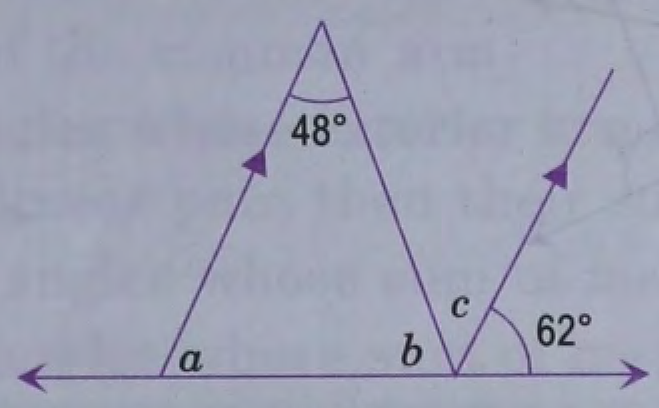
$\Rightarrow$ $5x = 195°$ $\Rightarrow$ $x = 39°$.

Exercise 21.2

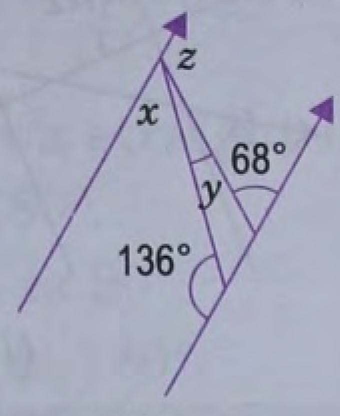
1. Calculate the size of each lettered angle in the following figures :



(i)

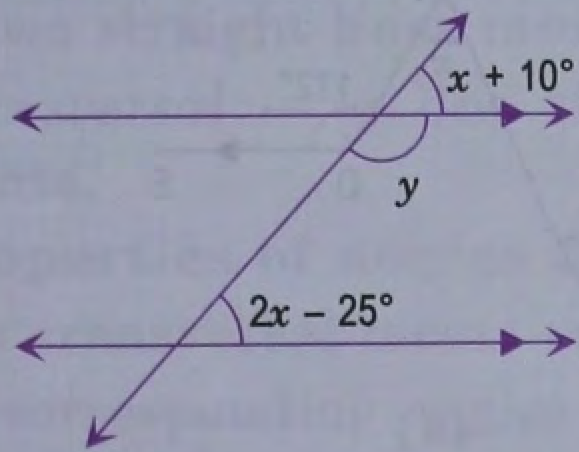


(ii)

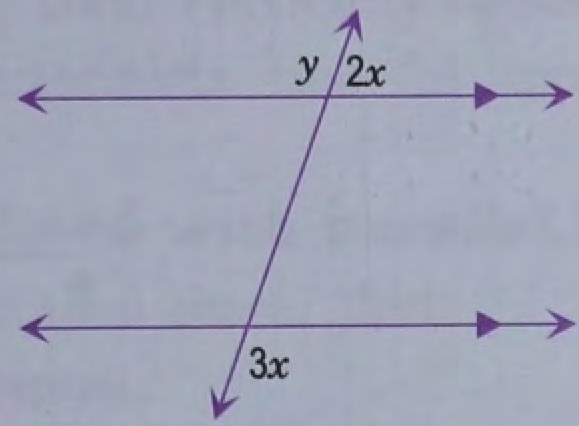


(iii)

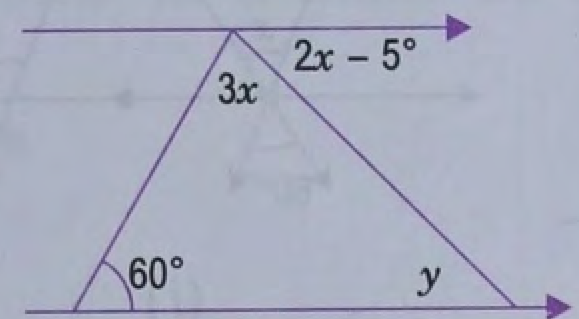
2. Find the values of x and y from the following figures :



(i)

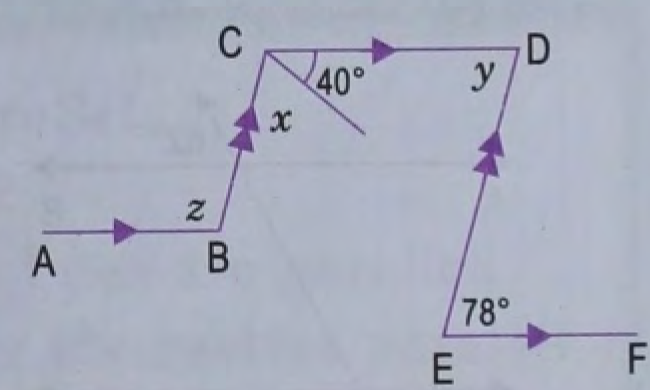


(ii)

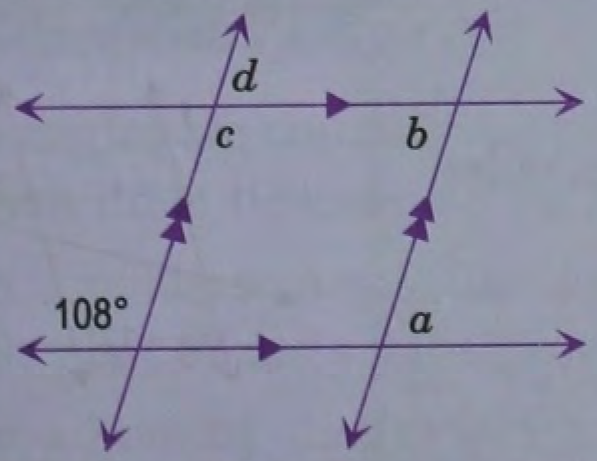


(iii)

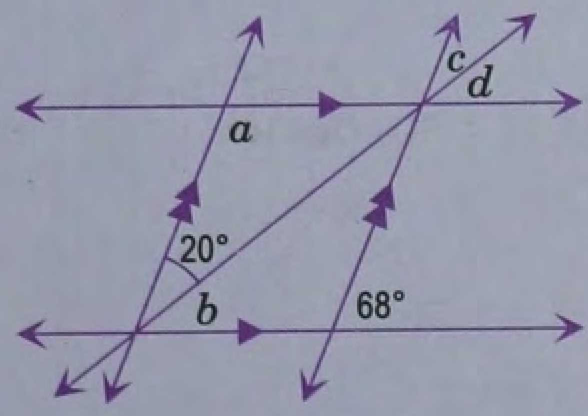
3. In the adjoining figure, AB || CD || EF and BC || ED. Find the values of x, y and z.



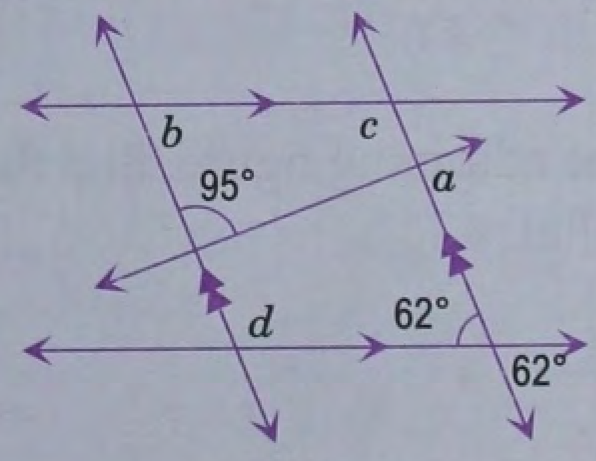
4. Calculate the size of each lettered angle in the following figures :



(i)

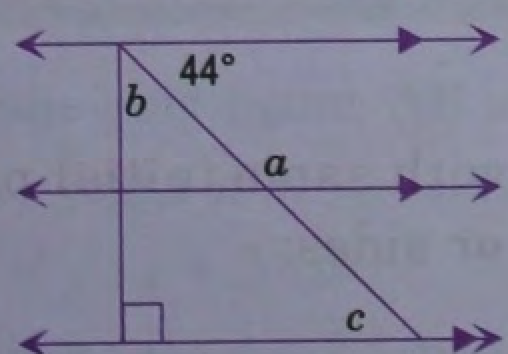


(ii)

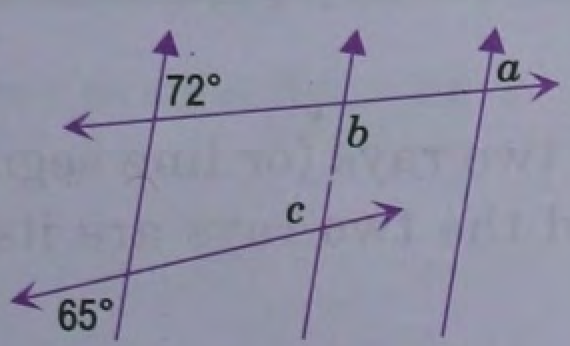


(iii)

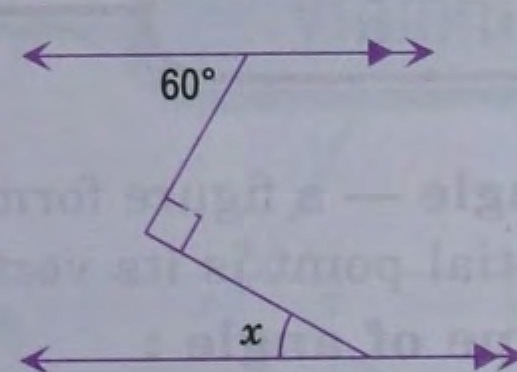
5. Calculate the size of each lettered angle in the following figures :



(i)

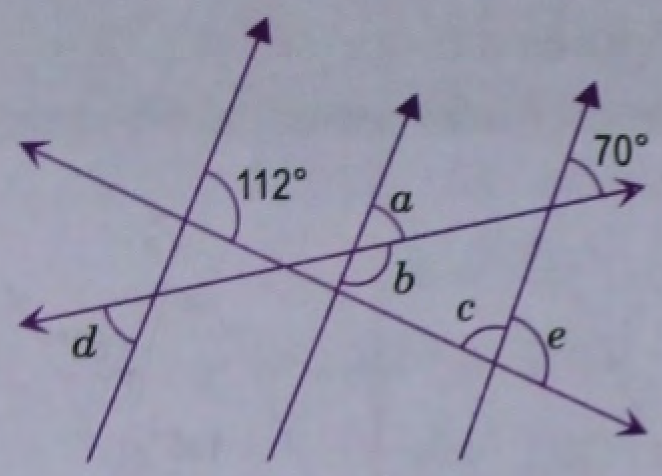


(ii)

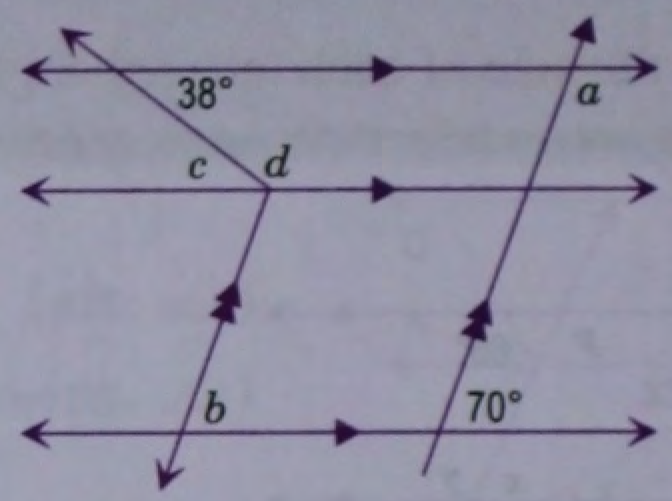


(iii)

6. Find the measure of each lettered angle in the following figures :

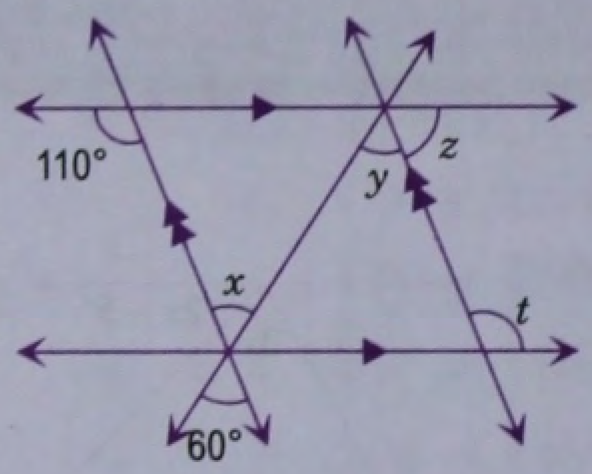


(*i*)

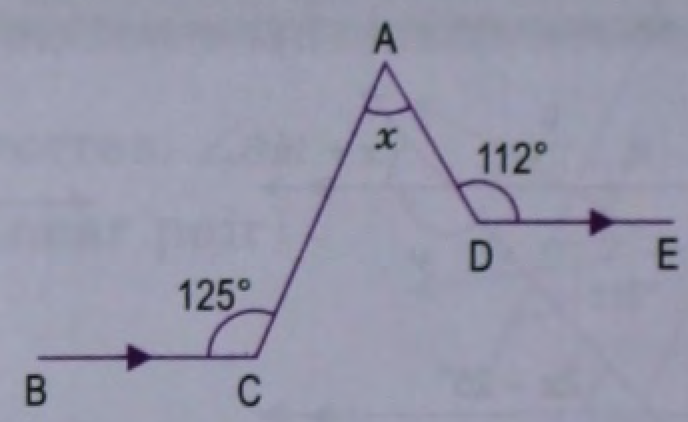


(*ii*)

7. Find the measure of each lettered angle in the following figures :



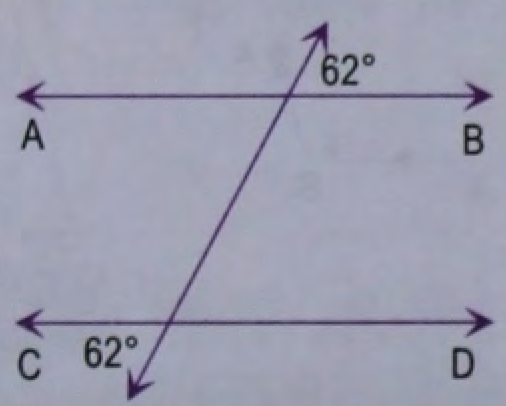
(*i*)



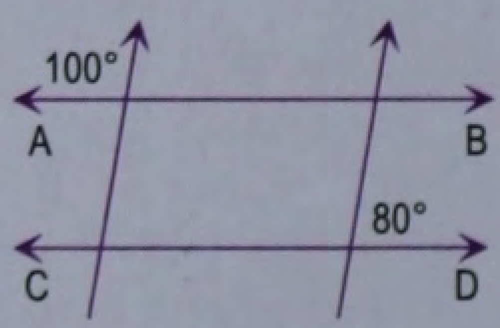
(*ii*)

[Hint. (*ii*) Through A, draw a straight line parallel to BC.**]**

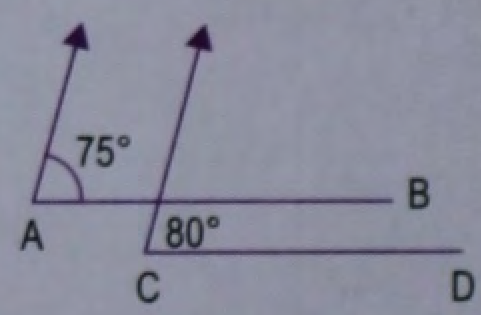
8. State, giving reason, whether AB and CD are parallel or not :



(*i*)

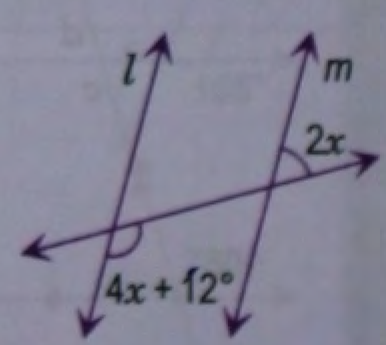


(*ii*)



(*iii*)

9. In the adjoining figure, find the value of x so that the lines l and m may be parallel.



Summary

- **Angle —** a figure formed by two rays (or line segments) with same initial point. The initial point is its vertex and the two rays are its arms or sides.
- **Type of angle :**

  Acute angle — an angle whose measure lies between 0° and 90°

  Right angle — an angle whose measure is 90°

Obtuse angle — an angle whose measure lies between 90° and 180°

Straight angle — an angle whose measure is 180°

Reflex angle — an angle whose measure lies between 180° and 360°

- **Types of angles :**

  Adjacent angles — two angles that have a common vertex, a common arm and their other arms lie on either side of the common arm.

  Linear pair — two adjacent angles whose exterior arms are in a straight line. Thus, if two adjacent angles form a linear pair, then their sum is 180°.

  Complementary angles — two angles whose sum of measures is 90°.

  Supplementary angles — two angles whose sum of measures is 180°.
- Sum of angles at a point = 360°
- Sum of angles at a point on one side of a straight line = 180°
- If the sum of two adjacent angles is 180°, then their exterior arms are in a straight line.
- If two straight lines intersect, then vertically opposite angles are equal.
- *Transversal* — a line that intersects two (or more) lines in a plane at distinct points.
- **Properties of angles associated with parallel lines**

  If a transversal meets two parallel lines, then
  - corresponding angles are equal.
  - alternate angles are equal.
  - co-interior angles are supplementary angles.
- **Conditions of parallelism**

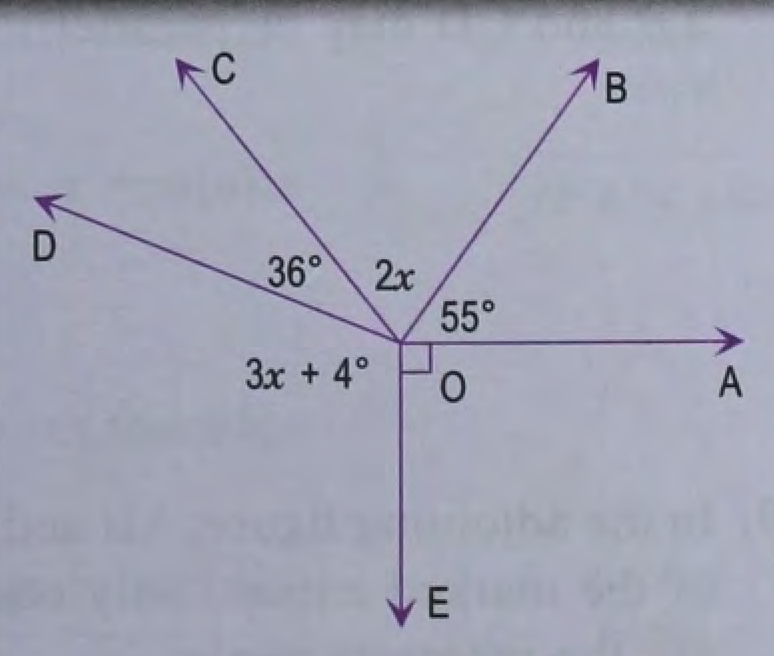
  If two lines are cut by a transversal such that a pair of
  - corresponding angles are equal, then the lines are parallel.
  - alternate angles are equal, then the lines are parallel.
  - co-interior angles are supplementary angles, then the lines are parallel.
- If two lines are parallel to a third line, then the lines are themselves parallel.

Check Your Progress

1. If two angles are complementary angles and one angle is 10° less than three times the other, find the angles.
2. If two supplementary angles are in the ratio 2 : 7, find the complement of the smaller angle.
3. In the adjoining diagram, find the value of x and hence complete the following:

   (*i*) ∠AOC = ...

   (*ii*) ∠DOE = ...



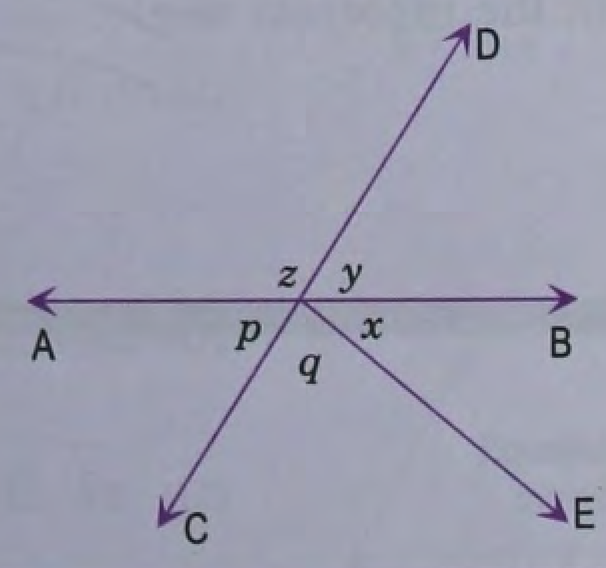
4. In the adjoining figure, AB and CD are straight lines. Find

   (*i*) x, y and z if $x = p$ and $q = 80°$

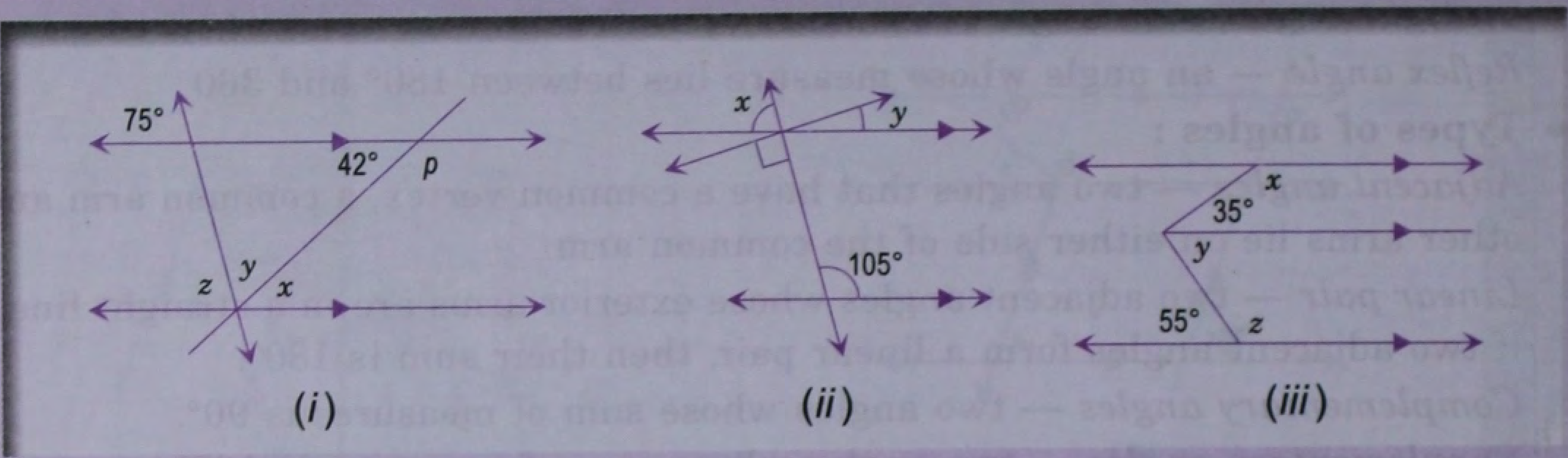
   (*ii*) p if $y : z = 2 : 3$

   (*iii*) z if $p : q : x = 2 : 3 : 1$

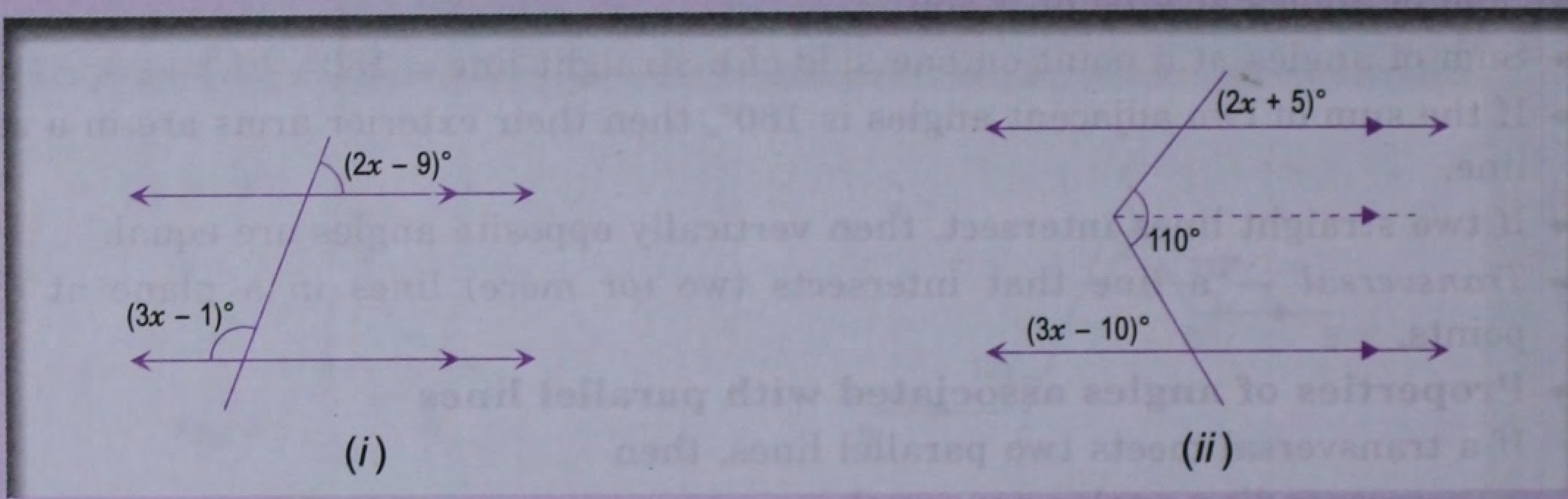
   (*iv*) x if $y = 40°$ and $q = 2p + 10°$.



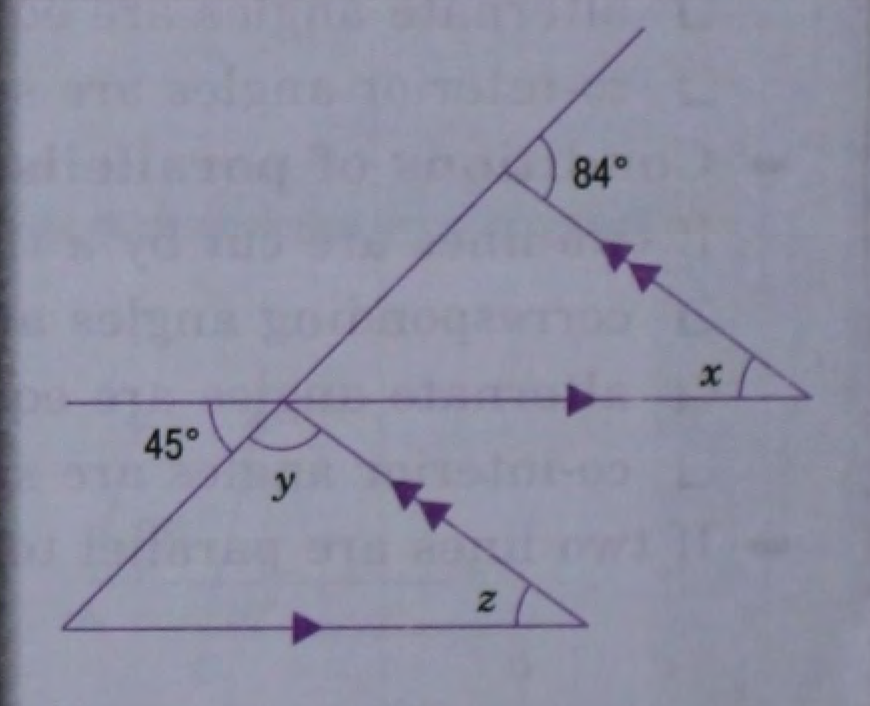
5. Find the measure of each lettered angle in the following figures :



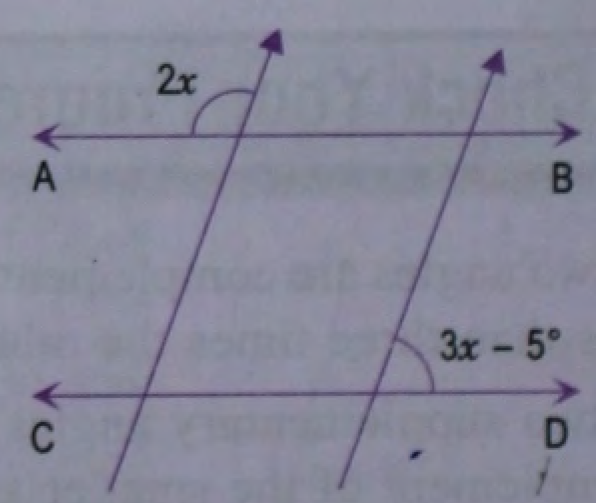
6. Find the value of x from the following sketches :



7. From the adjoining figure, find the measure of each lettered angle.



8. In the adjoining figure, find the value of x so that the lines AB and CD may be parallel.



9. In the adjoining figure, AB and CD are parallel lines. Out of the marked angles, only one angle is incorrect. Point out the incorrect angle.