Chapter 18

QUADRATIC EQUATIONS

QUADRATIC EQUATION

An equation of the form $ax^2 + bx + c = 0$, where a, b and c are real numbers and $a \neq 0$, is called a ***quadratic equation*** *in the variable x.*

For example :

$x^2 - 3x + 2 = 0$ and $6x^2 + x - 1 = 0$ are quadratic equations in the variable x.

A number α is a **root** (or **solution**) of the quadratic equation $ax^2 + bx + c = 0$ if it satisfies the equation *i.e.* if $a\alpha^2 + b\alpha + c = 0$.

For example :

When we substitute $x = 2$ in the quadratic equation $x^2 - 3x + 2 = 0$, we get $2^2 - 3 \times 2 + 2 = 0$ *i.e.* $4 - 6 + 2 = 0$ *i.e.* $0 = 0$, which is true. Therefore, 2 is a root of the quadratic equation $x^2 - 3x + 2 = 0$.

When we substitute $x = 3$ in the quadratic equation $x^2 - 3x + 2 = 0$, we get $3^2 - 3 \times 3 + 2 = 0$ *i.e.* $9 - 9 + 2 = 0$ *i.e.* $2 = 0$, which is wrong.

Therefore, 3 is not a root of the quadratic equation $x^2 - 3x + 2 = 0$.

SOLVING QUADRATIC EQUATIONS

by factorisation

Factorisation can be used to solve quadratic equations.

The equation $x^2 - 3x + 2 = 0$ can be written as $(x - 1)(x - 2) = 0$.

This equation can be solved by using a property of real numbers called ***zero-product rule***.

Zero-product rule

If a and b are two numbers or expressions and if $ab = 0$, then either $a = 0$ or $b = 0$ or both $a = 0$ and $b = 0$

Using this rule, the solutions of the equation $(x - 1)(x - 2) = 0$ can be obtained by putting each factor equal to zero and then solving for x. Thus, we get

$$x - 1 = 0 \text{ or } x - 2 = 0$$

$$\Rightarrow \quad x = 1 \text{ or } x = 2.$$

Hence, the solutions of the equation $x^2 - 3x + 2 = 0$ are 1 and 2.

Method to Solve a Quadratic Equation by Factorisation

Proceed as under :

* *Clear all fractions and write the equation in the form $ax^2 + bx + c = 0$.*
* *Factorise the left hand side into product of two linear factors.*
* *Put each linear factor equal to zero and solve the resulting linear equations.*

Remark

The solutions (roots) may be checked by substituting in the original equation.

Example 1. Solve : $x^2 - 2x - 15 = 0$.

Solution. Given $x^2 - 2x - 15 = 0$

$\Rightarrow \quad x^2 - 5x + 3x - 15 = 0$

$\Rightarrow \quad x(x - 5) + 3(x - 5) = 0$

$\Rightarrow \quad (x - 5)(x + 3) = 0$

$\Rightarrow \quad x - 5 = 0$ or $x + 3 = 0$

$\Rightarrow \quad x = 5$ or $x = -3$.

Hence, the roots of the given equation are 5, –3.

Example 2. Solve : $6x^2 + x - 1 = 0$.

Solution. Given $6x^2 + x - 1 = 0$

$\Rightarrow \quad 6x^2 + 3x - 2x - 1 = 0$

$\Rightarrow \quad 3x(2x + 1) - 1(2x + 1) = 0$

$\Rightarrow \quad (2x + 1)(3x - 1) = 0$

$\Rightarrow \quad 2x + 1 = 0$ or $3x - 1 = 0$

$\Rightarrow \quad x = -\frac{1}{2}$ or $x = \frac{1}{3}$.

Hence, the roots of the given equation are $-\frac{1}{2}, \frac{1}{3}$.

Example 3. Solve : $9x^2 + 6x + 1 = 0$.

Solution. Given $9x^2 + 6x + 1 = 0$

$\Rightarrow \quad 9x^2 + 3x + 3x + 1 = 0$

$\Rightarrow \quad 3x(3x + 1) + 1(3x + 1) = 0$

$\Rightarrow \quad (3x + 1)(3x + 1) = 0$

$\Rightarrow \quad 3x + 1 = 0$ or $3x + 1 = 0$

$\Rightarrow \quad x = -\frac{1}{3}$ or $x = -\frac{1}{3}$.

Hence, the roots of the given equation are $-\frac{1}{3}, -\frac{1}{3}$.

Note that the given quadratic equation has equal roots.

Example 4. Solve : $2x - \frac{1}{x} = 1$.

Solution. To clear the fractions, multiply both sides of the given equation by x. We get

$2x^2 - 1 = x$

$\Rightarrow \quad 2x^2 - x - 1 = 0$

$\Rightarrow \quad 2x^2 - 2x + x - 1 = 0$

$\Rightarrow$ $2x(x-1)+1(x-1)=0$

$\Rightarrow$ $(2x+1)(x-1)=0$

$\Rightarrow$ $2x+1=0$ or $x-1=0$

$\Rightarrow$ $x=-\frac{1}{2}$ or $x=1$.

Hence, the roots of the given equation are $-\frac{1}{2}$, 1.

Example 5. Solve : $\frac{x+3}{x-1}=\frac{2x+1}{3x-5}$.

Solution. Given $\frac{x+3}{x-1}=\frac{2x+1}{3x-5}$

$\Rightarrow$ $(x+3)(3x-5)=(x-1)(2x+1)$ (By cross-multiplication)

$\Rightarrow$ $3x^2-5x+9x-15=2x^2+x-2x-1$

$\Rightarrow$ $3x^2+4x-15=2x^2-x-1$

$\Rightarrow$ $3x^2-2x^2+4x+x-15+1=0$

$\Rightarrow$ $x^2+5x-14=0$

$\Rightarrow$ $x^2+7x-2x-14=0$

$\Rightarrow$ $x(x+7)-2(x+7)=0$

$\Rightarrow$ $(x+7)(x-2)=0$

$\Rightarrow$ $x+7=0$ or $x-2=0$

$\Rightarrow$ $x=-7$ or $x=2$.

Hence, the roots of the given equation are –7, 2.

Example 6. Solve $\frac{x}{x-1}+\frac{x-1}{x}=2\frac{1}{2}$.

Solution. Given $\frac{x}{x-1}+\frac{x-1}{x}=\frac{5}{2}$

To clear the fractions, multiply both sides of the given equation by L.C.M. of denominators *i.e.* by $2x(x-1)$. We get

$2x\times x+2(x-1)(x-1)=5x(x-1)$

$\Rightarrow$ $2x^2+2x^2-4x+2=5x^2-5x$

$\Rightarrow$ $4x^2-4x+2=5x^2-5x$

$\Rightarrow$ $4x^2-5x^2-4x+5x+2=0$

$\Rightarrow$ $-x^2+x+2=0$

$\Rightarrow$ $x^2-x-2=0$

$\Rightarrow$ $x^2-2x+x-2=0$

$\Rightarrow$ $x(x-2)+1(x-2)=0$

$\Rightarrow$ $(x+1)(x-2)=0$

$\Rightarrow$ $x+1=0$ or $x-2=0$

$\Rightarrow$ $x=-1$ or $x=2$.

Hence, the roots of the given equation are –1, 2.

Exercise 18

Solve the following (1 to 12) equations :

1. (i) $x^2 - 11x + 30 = 0$ (ii) $4x^2 - 25 = 0$
2. (i) $2x^2 - 5x = 0$ (ii) $x^2 - 2x = 48$
3. (i) $6 + x = x^2$ (ii) $2x^2 - 3x + 1 = 0$
4. (i) $3x^2 = 2x + 8$ (ii) $4x^2 + 15 = 16x$
5. (i) $x(2x + 5) = 25$ (ii) $(x + 3)(x - 3) = 40$
6. (i) $(2x + 3)(x - 4) = 6$ (ii) $(3x + 1)(2x + 3) = 3$
7. (i) $4x^2 + 4x + 1 = 0$ (ii) $(x - 4)^2 + 5^2 = 13^2$
8. (i) $21x^2 = 4(2x + 1)$ (ii) $\frac{2}{3}x^2 - \frac{1}{3}x - 1 = 0$
9. (i) $6x + 29 = \frac{5}{x}$ (ii) $x + \frac{1}{x} = 2\frac{1}{2}$
10. (i) $3x - \frac{8}{x} = 2$ (ii) $\frac{x}{3} + \frac{9}{x} = 4$
11. (i) $\frac{x-1}{x+1} = \frac{2x-5}{3x-7}$ (ii) $\frac{1}{x+2} + \frac{1}{x} = \frac{3}{4}$
12. (i) $\frac{8}{x+3} - \frac{3}{2-x} = 2$ (ii) $\frac{x}{x+1} + \frac{x+1}{x} = 2\frac{1}{6}$

Summary

- An equation of the form $ax^2 + bx + c = 0$, where a, b and c are real numbers and $a \neq 0$, is called a quadratic equation in the variable x.
- A number α is a root (or solution) of the equation $ax^2 + bx + c = 0$ if it satisfies it *i.e.* if $a\alpha^2 + b\alpha + c = 0$.
- A quadratic equation can be solved by factorisation and then using zero-product rule.
- To solve a quadratic equation, proceed as under :
  - (i) Clear all fractions and write the equation in the form $ax^2 + bx + c = 0$.
  - (ii) Factorise left hand side into product of two linear factors.
  - (iii) Put each linear factor equal to zero and solve the resulting linear equations.

Check Your Progress

Solve the following (1 to 3) equations :

1. (i) $x(2x + 5) = 3$ (ii) $3x^2 - 4x - 4 = 0$
2. (i) $4x^2 - 2x + \frac{1}{4} = 0$ (ii) $2x^2 + 7x + 6 = 0$
3. (i) $\frac{x-1}{x-2} + \frac{x-3}{x-4} = 3\frac{1}{3}$ (ii) $\frac{6}{x} - \frac{2}{x-1} = \frac{1}{x-2}$