Chapter 3

NUMBER SYSTEMS

You are already familiar with the system of natural numbers, whole numbers, integers and the four fundamental operations of arithmetic on them — addition, subtraction, multiplication and division. In this chapter, we shall review the same and extend our study to the system of rational numbers, irrational numbers and real numbers.

NATURAL NUMBERS

The counting numbers 1, 2, 3, 4, ... are called ***natural numbers***. The set of natural numbers is denoted by **N**. Thus,

$$\mathbf{N} = \{1, 2, 3, 4, ...\}.$$

WHOLE NUMBERS

The number 0 together with the natural numbers *i.e.* the numbers 0, 1, 2, 3, 4,... are called ***whole numbers***. The set of whole numbers is denoted by **W**. Thus,

$$\mathbf{W} = \{0, 1, 2, 3, 4, ...\} \quad i.e. \quad \mathbf{W} = \mathbf{N} \cup \{0\}.$$

We already know the four fundamental operations of addition, subtraction, multiplication and division on the whole numbers. In particular,

if a, b ≠ 0 are any two whole numbers, then there exist unique whole numbers q and r such that

$$a = b \times q + r \text{ where } 0 \le r < b$$

i.e.   *dividend = divisor × quotient + remainder.*

This is called **division algorithm** or **division rule**.

INTEGERS

We know that the set of whole numbers is ***closed*** with respect to the operations of addition and multiplication. However, the set of whole numbers is ***not closed*** with respect to the operation of subtraction.

For example, when we subtract 57 from 39 we do not get a whole number *i.e.* 39 – 57 is not a whole number. So, the system of whole numbers is inadequate for subtraction. Thus, to provide an answer to all the problems of subtraction, we have to enlarge the whole number system. We introduce another kind of numbers called negative integers

i.e. $-1, -2, -3, -4, ...$

The set of whole numbers together with negative integers is called *set of* ***integers***. It is denoted by **I** or **Z**. Thus,

$$\mathbf{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}.$$

All positive and all negative integers including zero are called ***directed numbers***.

The set of integers is a set of directed numbers.

You already know the four fundamental operations of addition, subtraction, multiplication and division on the set of integers. To simplify expressions involving integers, you need to know the **rules of calculations**. The order in which several operations must be done can be remembered with the help of the word **'BODMAS'**.

B — ***Brackets***. First carry out the operation inside brackets.

O — ***Of***. Change 'of' to '×' and work it out.

D — ***Division***. After 'of', carry out division.

M — ***Multiplication***. After division, carry out multiplication.

A — ***Addition***. After multiplication, carry out addition.

S — ***Subtraction***. Finally carry out subtraction.

According to the rule of BODMAS, calculations must be done in order of the letters in this word.

Brackets

The brackets are the grouping symbols. The different kinds of brackets are:

(*i*) [ ] are known as rectangular brackets or big brackets.

(*ii*) { } are known as braces or curly brackets.

(*iii*) ( ) are known as parenthesis or common brackets.

(*iv*) — is known as line or bar bracket or vinculum.

The order of removing brackets is :

(*i*) line bracket (*ii*) common brackets

(*iii*) curly brackets and lastly (*iv*) rectangular brackets.

Example 1. Write whole numbers between 5 to 85 using the digits 8, 0, 6. Repetition of digits is

(*i*) not allowed (*ii*) allowed.

Solution. The whole numbers between 5 to 85 consist of one digit or two digits.

The one digit whole numbers formed by the given digits are 0, 6, 8.

From the given digits, the possible ways of choosing two digits are

0, 6; 0, 8; 6, 8

Remember that the digit 0 can not be put at ten's place because that would make the number only one-digited.

(*i*) When the repetition of digits is not allowed, two-digited whole number formed by the given digits are

60, 80, 68, 86.

∴ The whole numbers between 5 to 85 formed by the given digits are

6, 8, 60, 68, 80.

(*ii*) When the repetition of digits is allowed, two-digited whole numbers formed by the given digits are

60, 66, 80, 88, 68, 86.

∴ The whole numbers between 5 to 85 formed by the given digits are

6, 8, 60, 66, 68, 80.

Example 2. Write (*i*) the smallest (*ii*) the greatest seven-digit whole number having four different digits.

Solution. (*i*) The smallest four different digits are 0, 1, 2, 3.

The smallest seven-digit whole number having four different digits

= 1000023

(*ii*) The greatest four different digits are 9, 8, 7, 6

The greatest seven-digit number having four different digits

= 9999876

Example 3. Write the greatest and the smallest five-digit numbers using the digits 0, 2, 5, 7 with the condition that one digit is repeated twice.

Solution. The digits to be used are 0, 2, 5, 7 and one digit is to be used twice.

Greatest number =

7	7	5	2	0

Smallest number =

2	0	0	5	7

Example 4. Write the greatest and the smallest six-digit numbers using four different digits with the condition that 5 occurs at thousand's place.

Solution. Greatest number =

9	9	5	9	8	7

Smallest number =

1	0	5	0	0	2

Example 5. Find the largest four-digit natural number which is exactly divisible by 439.

Solution. The largest four-digit natural number = 9999.

We divide 9999 by 439 and find the remainder.

$$\begin{array}{r|r|l} 439 & 9999 & 22 \\ & 878 & \\ \hline & 1219 & \\ & 878 & \\ \hline & 341 & \end{array}$$

∴ The least number which should be subtracted from 9999 so that the remaining number is exactly divisible by 439 is 341.

Hence, the required number

= 9999 – 341 = 9658.

Example 6. Find the smallest five-digit whole number which is exactly divisible by 657.

Solution. The smallest five-digit whole number = 10000.

We divide 10000 by 657 and find the remainder.

$$\begin{array}{r|r|l} 657 & 10000 & 15 \\ & 657 & \\ \hline & 3430 & \\ & 3285 & \\ \hline & 145 & \end{array}$$

∴ The least number which should be added to 10000 so that the sum is exactly divisible by 657 = 657 – 145 = 512.

Hence, the smallest five-digit whole number which is exactly divisible by 657 = 10000 + 512 = 10512.

Example 7. Simplify the following :

(*i*) $13 - [3 \text{ of } (-2) - 18 \div (5 - \overline{11-4})]$

(*ii*) $(-3)^3 - 6 \div (-3) + 7 \text{ of } (-5)$.

Solution. (*i*) $13 - [3 \text{ of } (-2) - 18 \div (5 - \overline{11 - 4})]$

$= 13 - [3 \times (-2) - 18 \div (5 - 7)]$

$= 13 - [-6 - 18 \div (-2)] = 13 - [-6 - \frac{18}{-2}] = 13 - [-6 - (-9)]$

$= 13 - [-6 + 9] = 13 - 3 = 10.$

(*ii*) $(-3)^3 - 6 \div (-3) + 7 \text{ of } (-5)$

$= (-3) \times (-3) \times (-3) - \frac{6}{-3} + 7 \times (-5) = -27 - (-2) + (-35)$

$= -27 + 2 - 35 = -60.$

Exercise 3.1

1. Write the smallest (*i*) natural number (*ii*) whole number. Can you write the largest whole number?
2. Write all possible three-digit whole numbers using the digits 5, 0, 8; repetition of digits not allowed. Also find their sum.
3. Write all possible two-digit whole numbers using the digits 3, 0, 7; repetition of digits allowed.
4. Write all natural numbers between 7 to 95 using the digits 8, 0, 9. The repetition of digits is
   (*i*) not allowed (*ii*) allowed.
5. Write down the smallest and the greatest natural numbers of five digits that can be formed by using the digits 0, 1, 3, 7, 8; using each digit only once.
6. Write the greatest and the smallest natural number of five digits; repetition of digits not allowed.
7. Write (*i*) the smallest (*ii*) the greatest natural number of six digits having three different digits.
8. Write the greatest and the smallest five-digit numbers by using one digit twice from the digits:
   (*i*) 1, 3, 0, 8 (*ii*) 3, 5, 7, 0.
9. Write the greatest and the smallest six-digit numbers using four different digits with the conditions as given below :
   (*i*) digit 2 occurs at hundred's place (*ii*) digit 6 occurs at thousand's place.
10. Find the largest four digit number which is exactly divisible by 459.
11. Find the smallest four digit number which is exactly divisible by 76.
12. Write two consecutive integers between the integers –7 and –3.
13. State whether each of the following statements is true or false :
    (*i*) There is no largest integer and no smallest integer.
    (*ii*) Every integer has a successor as well as predecessor in integers.
    (*iii*) 0 has no predecessor in whole numbers.
    (*iv*) –1 is the predecessor of –2 in integers.
14. Simplify the following :
    (*i*) $4 \times 6 + 18 \div (-3)$ (*ii*) $7 - [5 - 21 \div \{5 - (-2)\}]$
    (*iii*) $(-4) \text{ of } (-6) - 8 \{-5 + (-9) \div (-3)\}$ (*iv*) $15 - [11 - \{5 - 3 (9 - \overline{3 - 6})\}]$.

FACTORS AND MULTIPLES

A natural number which divides a given natural number exactly is called a ***factor*** *(or* ***divisor****) of that given number, and the given number is called a* ***multiple*** *of that number.*

In other words, if $a, b, c \in \mathbf{N}$ such that $c = ab$ then each of a and b is called a **factor** (or **divisor**) of c and c is called a **multiple** of each of a and b.

For example :

As $45 = 1 \times 45$, $45 = 3 \times 15$ and $45 = 5 \times 9$,
1, 3, 5, 9, 15 and 45 are all factors of 45
and 45 is a multiple of each of 1, 3, 5, 9, 15 and 45.

Types of whole numbers

Even number — *a whole number which is divisible by 2 is called an* ***even number****.*

Even numbers are : 0, 2, 4, 6, 8, 10,...

Odd number — *a whole number which is not divisible by 2 is called an* ***odd number****.*

Odd numbers are : 1, 3, 5, 7, 9, 11,...

Prime number — *a natural number greater than 1 is called a* ***prime number*** *if it has no factors other than 1 and itself i.e. if its only factors are 1 and the number itself.*

The prime numbers from 2 to 100 are
2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97.

Composite number — *a natural number is called a* ***composite number*** *if it has atleast one more factor other than 1 and the number itself.*

Some composite numbers are 4, 6, 8, 9, 10, 12, 14, 15,...

Remarks

- 1 is neither prime nor composite.
- Every natural number (except 1) is either prime or composite.
- 2 is the only even prime number.
- There are infinitely many prime numbers.

Prime factorisation — *if a natural number is expressed as the product of prime numbers, then the factorisation of the number is called its* ***prime*** *(or* ***complete****)* ***factorisation****.*

A prime factorisation of a natural number can be expressed in the exponential form.

For example :

(*i*) $48 = 2 \times 2 \times 2 \times 2 \times 3 = 2^4 \times 3$

(*ii*) $420 = 2 \times 2 \times 3 \times 5 \times 7 = 2^2 \times 3 \times 5 \times 7$

Example 1. Write the prime factorisation of 4725 in the exponential form.

Solution.

3	4725
3	1575
3	525
5	175
5	35
	7

$\therefore \quad 4725 = 3 \times 3 \times 3 \times 5 \times 5 \times 7$
$= 3^3 \times 5^2 \times 7.$

Least Common Multiple

Least common multiple *(abbreviated* ***L.C.M.****) of two natural numbers is the smallest natural number which is a multiple of both the numbers.*

In other words, L.C.M. is the smallest element of the set of common multiples of given numbers.

Example 2. Find the L.C.M. of 72, 240 and 196.

Solution. *Prime factorisation method*

$$72 = 2 \times 2 \times 2 \times 3 \times 3 = 2^3 \times 3^2$$
$$240 = 2 \times 2 \times 2 \times 2 \times 3 \times 5 = 2^4 \times 3 \times 5$$
$$196 = 2 \times 2 \times 7 \times 7 = 2^2 \times 7^2$$

L.C.M. of the given numbers = product of all the prime factors of each of the given number with greatest index of common prime factors

$$= 2^4 \times 3^2 \times 5 \times 7^2 = 16 \times 9 \times 5 \times 49 = 35280.$$

Division method

2	72,	240,	196
2	36,	120,	98
2	18,	60,	49
3	9,	30,	49
	3	10,	49

L.C.M. of the given numbers
= product of divisors and the remaining numbers
$= 2 \times 2 \times 2 \times 3 \times 3 \times 10 \times 49$
$= 72 \times 10 \times 49 = 35280.$

Example 3. Find the greatest number of four digits which is exactly divisible by each of 16, 24, 35 and 42.

Solution. First, we find the L.C.M. of 16, 24, 35 and 42.

2	16,	24,	35,	42
2	8,	12,	35,	21
2	4,	6,	35,	21
3	2,	3,	35,	21
7	2,	1	35,	7
	2,	1,	5,	1

∴ L.C.M. of given numbers $= 2 \times 2 \times 2 \times 3 \times 7 \times 2 \times 5$
$= 1680$

Greatest number of four digits = 9999

According to the given condition, we need a greatest number of four digits which is exactly divisible by 1680.

We divide 9999 by 1680 and find the remainder.

∴ The smallest number which should be subtracted from 9999 so that the remaining number is exactly divisible by 1680 is 1599.

$$\begin{array}{r|l} 1680 & 9999 \quad (5 \\ & \underline{8400} \\ & 1599 \end{array}$$

Hence, the required number = 9999 – 1599 = 8400.

Example 4. Find the least number of five digits which is exactly divisible by each of 8, 18, 24, 30 and 54.

Solution. First, we find the L.C.M. of 8, 18, 24, 30 and 54.

2	8,	18,	24,	30,	54
2	4,	9,	12,	15,	27
2	2,	9,	6,	15,	27
3	1,	9,	3,	15,	27
3	1,	3,	1,	5,	9
	1,	1,	1,	5,	3

∴ L.C.M. of given numbers
$= 2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 3$
$= 1080$

Least number of five digits = 10000

According to the given condition, we need a smallest number of five digits which is exactly divisible by 1080.

We divide 10000 by 1080 and find the remainder.

$$\begin{array}{r|l} 1080 & 10000 \; (9 \\ & \underline{9720} \\ & \;\;280 \end{array}$$

∴ The least number which should be added to 10000 so that the sum is exactly divisible by 1080

= 1080 – 280 = 800.

Hence, the required number = 10000 + 800 = 10800.

Highest Common Factor

Highest common factor *(abbreviated* ***H.C.F.****) of two natural numbers is the largest common factor (or divisor) of the given natural numbers.*

In other words, H.C.F. is the greatest element of the set of common factors of the given numbers.

H.C.F. is also called **greatest common divisor** (abbreviated **G.C.D.**)

Example 5. Find the H.C.F. of 72, 126 and 270.

Solution. *Prime factorisation method*

$72 = 2 \times 2 \times 2 \times 3 \times 3 = 2^3 \times 3^2$

$126 = 2 \times 3 \times 3 \times 7 = 2^1 \times 3^2 \times 7^1$

$270 = 2 \times 3 \times 3 \times 3 \times 5 = 2^1 \times 3^3 \times 5^1$

H.C.F. of the given numbers = the product of common prime factors with least index

$= 2^1 \times 3^2 = 2 \times 3 \times 3 = 18.$

Division method

Find H.C.F. of 72 and 126

$$\begin{array}{l} 72)\,126\,(1 \\ \quad\;\; \underline{72} \\ \quad\;\; 54)\,72\,(1 \\ \quad\quad\quad \underline{54} \\ \quad\quad\quad 18)\,54\,(3 \\ \quad\quad\quad\quad\;\; \underline{54} \\ \quad\quad\quad\quad\;\;\; 0 \end{array}$$

∴ H.C.F. of 72 and 126 = 18

Now find H.C.F. of 18 and 270

$$\begin{array}{l} 18)\,270\,(15 \\ \quad\;\; \underline{18} \\ \quad\;\; 90 \\ \quad\;\; \underline{90} \\ \quad\;\;\; 0 \end{array}$$

∴ H.C.F. of 18 and 270 = 18

Hence, H.C.F. of the given numbers = 18.

Example 6. Find the greatest number that will divide the numbers 620, 1010 and 1265 leaving the remainders 8, 2 and 5 respectively.

Solution. When 620 is divided by that number, 8 is left as a remainder. So 620 – 8 *i.e.* 612 is exactly divisible by that number.

Similarly, 1010 – 2 and 1265 – 5 *i.e.* 1008 and 1260 are divisible by that number. Thus, the required number is the H.C.F. of the numbers 612, 1008 and 1260.

To find their H.C.F., we use prime factorisation method.

$$612 = 2 \times 2 \times 3 \times 3 \times 17 = 2^2 \times 3^2 \times 17^1$$
$$1008 = 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 7 = 2^4 \times 3^2 \times 7^1$$
$$1260 = 2 \times 2 \times 3 \times 3 \times 5 \times 7 = 2^2 \times 3^2 \times 5^1 \times 7^1$$

$\therefore$ H.C.F. = the product of common prime factors with least index

$= 2^2 \times 3^2 = 2 \times 2 \times 3 \times 3 = 36.$

Hence, the required number = 36.

Co-prime numbers — *two natural numbers are called* ***co-prime numbers*** *if they have no common factor other than 1.*

In other words, two natural numbers are co-prime numbers if their H.C.F. is 1.

Some examples of co-prime numbers are : 4, 9; 8, 21; 27, 50.

Relation between L.C.M. and H.C.F. of two natural numbers :

The product of L.C.M. and H.C.F. of two natural numbers = the product of the numbers.

Remarks

- In particular, if two natural numbers are co-prime number, then their L.C.M. = the product of the numbers.
- H.C.F. of two natural numbers always divides their L.C.M.

Example 7. The L.C.M. of two numbers is 6 times their H.C.F. The sum of the H.C.F. and the L.C.M. of these numbers is 210. If one of the numbers is 60, find the other number.

Solution. Let the H.C.F. of the given numbers be x.

Given that L.C.M. is 6 times their H.C.F.

$\Rightarrow$ L.C.M. of the numbers = $6x$

Further, it is given that H.C.F. + L.C.M. = 210

$\Rightarrow x + 6x = 210 \Rightarrow 7x = 210 \Rightarrow x = 30$

$\therefore$ H.C.F. of given numbers = 30 and their L.C.M. = $6 \times 30 = 180$

We know that the product of H.C.F. and L.C.M. of two numbers = the product of the numbers

Here one number = 60 (given),

$\therefore$ the other number $= \frac{30 \times 180}{60} = 90.$

Exercise 3.2

1. Write the prime factorisation of the greatest three-digit number.
2. Write the prime factorisation of the following numbers in exponential form:
   (*i*) 13860 (*ii*) 27830 (*iii*) 21952.
3. Find the smallest number which must be added to 9373 so that it becomes divisible by 4.

4. Find the L.C.M. of :

(*i*) 24, 60 and 112 (*ii*) 70, 84, 336 and 1260.

5. Find the least number which on adding 7 is exactly divisible by 15, 35 and 48.

6. Find the least number which when divided by 16, 28, 40 and 77 leaves 5 as remainder in each case.

7. Find the greatest number of four digits which is exactly divisible by each of 12, 18, 40 and 45.

8. Find the least number of five digits which is exactly divisible by each of 32, 36, 60, 90 and 144.

9. Find the H.C.F. of :

(*i*) 72, 126, 168 (*ii*) 96, 528, 2160, 3520.

10. Find the greatest number that will divide 400, 435 and 541 leaving 9, 10 and 14 as remainders respectively.

11. Which of the following pairs of numbers are co-prime number?

(*i*) 15, 98 (*ii*) 198, 429 (*iii*) 847, 2160.

12. Three drums of capacities 165 litres, 195 litres and 240 litres are to be filled from a tank by using a mug, an integral number of times in each case. What is the largest capacity of the mug?

13. Four ribbons measuring 14 m, 18 m, 22 m and 26 m respectively are to be cut into pieces of equal length.

(*i*) Find the least number of pieces that can be obtained.

(*ii*) What is the length of each piece?

14. If the product of two numbers is 84942 and their H.C.F. is 33, find their L.C.M.

15. The product of H.C.F. and L.C.M. of two numbers is 9072. If one of the numbers is 72, find the other number.

16. The L.C.M. of two numbers is 28 times their H.C.F. The difference of L.C.M. and H.C.F. of these numbers is 810. If one of the numbers is 120, find the other number.

FRACTION

A ***fraction*** *is a number which represents a part of whole. The whole may be a single object or a group of objects.*

A fraction is written as $\frac{p}{q}$ where p and q are whole numbers and $q \neq 0$. Numbers such as $\frac{1}{2}, \frac{2}{3}, \frac{4}{5}, \frac{11}{7}$ are all fractions. In the fraction $\frac{p}{q}$ where p and q are whole numbers and $q \neq 0$, the horizontal line is called the *division line*. The number below the division line *i.e.* q is called *denominator* and it tells us into how many equal part a whole is divided. The number above the division line *i.e.* p is called *numerator* and it tells us how many equal parts are taken.

TYPES OF FRACTIONS

Proper fraction — *a fraction whose numerator is greater than zero but less than its denominator is called a* ***proper fraction****.*

For example, $\frac{3}{5}, \frac{4}{7}, \frac{43}{128}$ are all proper fractions.

Improper fraction — *a fraction whose numerator is equal to or greater than its denominator is called an* ***improper fraction***.

For example, $\frac{12}{5}, \frac{11}{7}, \frac{217}{89}$ are all improper fractions.

Notice that every natural number can be written as a fraction.

For example, $7 = \frac{7}{1}$, $39 = \frac{39}{1}$.

So every integer is an improper fraction.

Mixed fraction (or mixed number) — *a number that consists of two parts—a natural number and a proper fraction is called a* ***mixed fraction*** *or* ***mixed number***.

For example, $3\frac{5}{7}, 8\frac{9}{11}$ are mixed fractions.

In fact, $3\frac{5}{7} = 3 + \frac{5}{7}$ and $8\frac{9}{11} = 8 + \frac{9}{11}$.

Every mixed fraction can be written as an improper fraction and every improper fraction can be written as a mixed fraction.

For example, $3\frac{5}{7} = 3 + \frac{5}{7} = \frac{3 \times 7 + 5}{7} = \frac{26}{7}$ and

$$\frac{51}{11} = \frac{11 \times 4 + 7}{11} = 4 + \frac{7}{11} = 4\frac{7}{11}.$$

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11) 51 (4
    44
    --
     7
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Negative fractions — As we have negative integers, in a similar way we have negative fractions.

For example, $-\frac{4}{7}, -\frac{11}{5}, -4\frac{3}{13}$ are all negative fractions.

The fractions $-\frac{4}{7}, -\frac{3}{11}, -\frac{13}{89}$ are proper negative fractions.

The fractions $-\frac{13}{7}, -\frac{21}{5}$ are improper negative fractions.

The fractions $-7\frac{2}{5}, -11\frac{5}{13}$ are mixed negative fractions.

Note that $-7\frac{2}{5} = -\left(7 + \frac{2}{5}\right) = -\frac{35+2}{5} = -\frac{37}{7}$ and

$$-\frac{21}{5} = -\left(\frac{21}{5}\right) = -\left(4 + \frac{1}{5}\right) = -4\frac{1}{5}$$

Thus, a mixed negative fraction can be changed to an improper negative fraction and an improper negative fraction can be changed to a mixed negative fraction.

Simple fraction — *a fraction in which both numerator and denominator are whole numbers is called a* ***simple fraction***.

For example, $\frac{7}{11}, \frac{27}{5}, -\frac{31}{7}$ are simple fractions.

Complex fraction — *a fraction in which either numerator or denominator or both are fractions is called a* ***complex fraction***.

For example, $\frac{\frac{3}{5}}{7}, \frac{5}{\frac{7}{11}}, \frac{\frac{17}{3}}{5\frac{1}{7}}, -\frac{\frac{7}{17}}{\frac{43}{5}}$ are complex fractions.

Decimal fraction — *a fraction whose denominator is 10, 100, 1000,... i.e. whose denominator is of the form 10^n, $n \in$ **N**, is called a* ***decimal fraction***.

For example, $\frac{3}{10}, \frac{327}{100}, -\frac{21}{1000}$ are decimal fractions.

Vulgar fraction — *a fraction whose denominator is a number other than 10, 100, 1000,... i.e. whose denominator is not of the form 10^n, $n \in$ **N**, is called a* ***vulgar fraction***.

For example, $\frac{3}{7}, -\frac{5}{214}, \frac{37}{18}, \frac{671}{150}$ are vulgar fractions.

Like fractions — *two (or more) fractions having same denominator are called* ***like fractions***.

For example, $\frac{3}{11}, \frac{15}{11}, -\frac{9}{11}, \frac{45}{11}$ are like fractions.

Unlike fractions — *two (or more) fractions having different denominators are called* ***unlike fractions***.

For example, $\frac{4}{5}, \frac{9}{16}, -\frac{15}{11}$ are unlike fractions

Equivalent fractions — *two (or more) fractions are called* ***equivalent fractions*** *if they have same value.*

Simplest form of a fraction — *if the numerator and denominator of a fraction have no common factor (except 1), then the fraction is said to be in its* ***simplest (irreducible) form*** *or in* ***lowest terms***.

Basic properties of fractions

The value of a fraction does not change if the numerator and the denominator of a fraction are :

(*i*) multiplied by the same (non-zero) number

(*ii*) divided by the same (non-zero) number.

Example 1. Express

(*i*) $\frac{107}{11}$ as a mixed fraction (*ii*) $-4\frac{7}{13}$ as an improper fraction.

Solution. (*i*) $\frac{107}{11} = \frac{9 \times 11 + 8}{11} = 9 + \frac{8}{11} = 9\frac{8}{11}$.

$$11 \overline{)107}(9 \quad 99 \quad 8$$

(*ii*) $-4\frac{7}{13} = -\left(4 + \frac{7}{13}\right) = -\frac{52+7}{13} = -\frac{59}{13}$.

Example 2. Convert $\frac{7}{25}, \frac{9}{10}, \frac{19}{30}$ into equivalent like fractions with least denominator.

Solution. L.C.M. of 25, 10 and 30

$= 2 \times 5 \times 5 \times 3 = 150$

2	25,	10,	30
5	25,	5,	15
	5,	1,	3

$\frac{7}{25} = \frac{7 \times 6}{25 \times 6} = \frac{42}{150}$,

$\frac{9}{10} = \frac{9 \times 15}{10 \times 15} = \frac{135}{150}$, $\frac{19}{30} = \frac{19 \times 5}{30 \times 5} = \frac{95}{150}$

Hence, the given fractions are equivalent to $\frac{42}{150}, \frac{135}{150}, \frac{95}{150}$ respectively.

Example 3. Arrange the fractions $\frac{13}{18}, \frac{8}{15}, \frac{17}{24}, \frac{7}{12}$ in descending order.

Solution. L.C.M. of 18, 15, 24 and 12

$= 2 \times 2 \times 3 \times 3 \times 5 \times 2$

$= 360$

2	18,	15,	24,	12
2	9,	15,	12,	6
3	9,	15,	6,	3
	3,	5,	2,	1

Write the given fractions as equivalent like fractions :

$\frac{13}{18} = \frac{13 \times 20}{18 \times 20} = \frac{260}{360}$, $\frac{8}{15} = \frac{8 \times 24}{15 \times 24} = \frac{192}{360}$,

$\frac{17}{24} = \frac{17 \times 15}{24 \times 15} = \frac{255}{360}$, $\frac{7}{12} = \frac{7 \times 30}{12 \times 30} = \frac{210}{360}$

Descending means greater to smaller

As 260 > 255 > 210 > 192,

$\frac{260}{360} > \frac{255}{360} > \frac{210}{360} > \frac{192}{360} \Rightarrow \frac{13}{18} > \frac{17}{24} > \frac{7}{12} > \frac{8}{15}$

Hence, the given fractions in descending order are $\frac{13}{18}, \frac{17}{24}, \frac{7}{12}, \frac{8}{15}$.

To insert a fraction between two given fractions

If $\frac{a}{b}$ and $\frac{c}{d}$ are any two fractions, then the fraction $\frac{a+c}{b+d}$ lies between the given fractions $\frac{a}{b}$ and $\frac{c}{d}$.

For example, let $\frac{1}{2}$ and $\frac{3}{5}$ be two given fractions then by the above rule, the fraction $\frac{1+3}{2+5}$ *i.e.* $\frac{4}{7}$ lies between the fractions $\frac{1}{2}$ and $\frac{3}{5}$.

If $\frac{a}{b} < \frac{c}{d}$, then $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.

Moreover, $\frac{a+c}{b+d}$ is not the only fraction between $\frac{a}{b}$ and $\frac{c}{d}$. In fact, there are infinitely many fractions between two given fractions.

Example 4. Insert three fractions between $\frac{2}{5}$ and $\frac{4}{7}$.

Solution. By the above rule, the fraction $\frac{2+4}{5+7}$ *i.e.* $\frac{6}{12}$ *i.e.* $\frac{1}{2}$ lies between the fractions $\frac{2}{5}$ and $\frac{4}{7}$.

Again the fraction $\frac{2+1}{5+2}$ *i.e.* $\frac{3}{7}$ lies between $\frac{2}{5}$ and $\frac{1}{2}$.

Further, the fraction $\frac{1+4}{2+7}$ *i.e.* $\frac{5}{9}$ lies between $\frac{1}{2}$ and $\frac{4}{7}$.

Hence, the fractions $\frac{3}{7}, \frac{1}{2}$ and $\frac{5}{9}$ lie between the given fractions $\frac{2}{5}$ and $\frac{4}{7}$.

Simplification of expressions involving fractions

You already know the four fundamental operations of addition, subtraction, multiplication and division on fractions. To simplify expressions involving fractions, use rule of **BODMAS**.

Note. Multiplication sign is often dropped before a bracket and between brackets.

Example 5. Simplify the following expressions :

(*i*) $\frac{2}{3}$ of $\left(5\frac{1}{6} - 4\frac{3}{8}\right)$ (*ii*) $\dfrac{3\frac{1}{3} \text{ of } \frac{1}{5} + \frac{1}{2}}{\frac{3}{4} - \frac{4}{5}}$.

Solution. (*i*) $\frac{2}{3}$ of $\left(5\frac{1}{6}-4\frac{3}{8}\right) = \frac{2}{3}$ of $\left(\frac{31}{6}-\frac{35}{8}\right)$ **Use rule of BODMAS**

$= \frac{2}{3}$ of $\frac{31\times 4 - 35\times 3}{24} = \frac{2}{3}$ of $\frac{124-105}{24}$

$= \frac{2}{3}$ of $\frac{19}{24} = \frac{\not{2}^{1}}{3} \times \frac{19}{\not{24}_{12}} = \frac{19}{36}$.

(*ii*) $\dfrac{3\frac{1}{3} \text{ of } \frac{1}{5} + \frac{1}{2}}{\frac{3}{4}-\frac{4}{5}} = \dfrac{\frac{\not{10}^{2}}{3}\times\frac{1}{\not{5}_1}+\frac{1}{2}}{\frac{3}{4}-\frac{4}{5}}$ **Use rule of BODMAS**

$= \dfrac{\frac{2}{3}+\frac{1}{2}}{\frac{15-16}{20}} = \dfrac{\frac{4+3}{6}}{-\frac{1}{20}} = \dfrac{\frac{7}{6}}{-\frac{1}{20}} = \frac{7}{\not{6}_3}\times\left(-\frac{\not{20}^{10}}{1}\right) = -\frac{70}{3} = -23\frac{1}{3}$.

Example 6. Simplify the following expression :

$\left(\frac{5}{6}+\frac{3}{4}\right)$ of $\frac{9}{10} \div \frac{3}{16} - \frac{1}{2}\times\frac{3}{4}\times\left(-\frac{5}{12}\right) + 3\frac{1}{2}$. **Use rule of BODMAS**

Solution. $\left(\frac{5}{6}+\frac{3}{4}\right)$ of $\frac{9}{10} \div \frac{3}{16} - \frac{1}{2}\times\frac{3}{4}\times\left(-\frac{5}{12}\right) + 3\frac{1}{2}$

$= \left(\frac{5\times 2+3\times 3}{12}\right)$ of $\frac{9}{10} \div \frac{3}{16} - \frac{1\times \not{3}^{1}\times(-5)}{2\times 4\times \not{12}_4} + \frac{7}{2}$

$= \frac{19}{12}\times\frac{9}{10}\div\frac{3}{16} - \frac{1\times 1\times(-5)}{2\times 4\times 4} + \frac{7}{2}$

$= \frac{19}{\not{12}_{\not{4}_1}}\times\frac{\not{9}^{\not{3}^1}}{\not{10}_5}\times\frac{\not{16}^{\not{8}^2}}{\not{3}_1} - \left(\frac{-5}{32}\right) + \frac{7}{2}$

$= \frac{19\times 1\times 2}{1\times 5\times 1} + \frac{5}{32} + \frac{7}{2} = \frac{38}{5} + \frac{5}{32} + \frac{7}{2} = \frac{38\times 32 + 5\times 5 + 7\times 80}{160}$

$= \frac{1216+25+560}{160} = \frac{1801}{160} = 11\frac{41}{160}$.

Use of fractions

Example 7. Mr. Mukerjee's monthly salary is ₹ 16000. He spends $\frac{1}{4}$ of his salary on food. Out of the remaining, he spends $\frac{3}{10}$ on house rent and $\frac{5}{24}$ on the education of children. Find how much money is still left with him.

Solution. Money spent on food $= \frac{1}{4}$ of ₹ 16000 = ₹ $\left(\frac{1}{4}\times 16000\right)$ = ₹ 4000

Remaining money with him = ₹ 16000 – ₹ 4000 = ₹ 12000

Money spent on house rent $= \frac{3}{10}$ of ₹ 12000 = ₹ $\left(\frac{3}{10}\times 12000\right)$ = ₹ 3600

Money spent on the education of children

$= \frac{5}{24}$ of ₹ 12000 = ₹ $\left(\frac{5}{24}\times 12000\right)$ = ₹ 2500

∴ Total money spent = ₹ 4000 + ₹ 3600 + ₹ 2500 = ₹ 10100

∴ Money left with him = ₹ 16000 – ₹ 10100 = ₹ 5900.

Example 8. If $\frac{3}{5}$ of a number exceeds its $\frac{2}{7}$ by 44, find the number.

Solution. Let the required number be x.

According to the given condition, $\frac{3}{5}$ of $x - \frac{2}{7}$ of $x = 44$

$\Rightarrow \quad \frac{3}{5}x - \frac{2}{7}x = 44$ (Multiply both sides by 35)

$\Rightarrow \quad 21x - 10x = 35 \times 44$

$\Rightarrow \quad 11x = 35 \times 44 \Rightarrow x = \frac{35 \times \cancel{44}}{\cancel{11}} = 140$

Hence, the required number is 140.

Example 9. The highest score in a cricket test match in an innings was $\frac{3}{11}$ of the total and the next highest was $\frac{3}{10}$ of the remaining. If the difference between two scores was 24 runs, what was the total score?

Solution. Let the total score of the innings be x runs.

Then the highest score of the innings = $\frac{3}{11}$ of $x = \frac{3}{11}x$

$\therefore$ The remaining score = $x - \frac{3}{11}x = \frac{11x - 3x}{11} = \frac{8x}{11}$

Next highest score of the innings = $\frac{3}{10}$ of $\frac{8x}{11}$

$= \frac{3}{\cancel{10}_5} \times \frac{\cancel{8x}^4}{11} = \frac{12x}{55}$

According to given condition, $\frac{3x}{11} - \frac{12x}{55} = 24$ (Multiply both sides by 55)

$\Rightarrow \quad 15x - 12x = 55 \times 24$

$\Rightarrow \quad 3x = 55 \times 24 \quad \Rightarrow \quad x = \frac{55 \times \cancel{24}^8}{\cancel{3}_1} = 440$

Hence, the total score of the innings was 440 runs.

Exercise 3.3

1. Express the following as mixed fractions :
   (i) $\frac{23}{7}$ (ii) $\frac{112}{15}$ (iii) $-\frac{1234}{105}$.
2. Express the following as improper fractions :
   (i) $4\frac{3}{7}$ (ii) $13\frac{21}{47}$ (iii) $-5\frac{5}{8}$.
3. Reduce the following fractions to their lowest terms :
   (i) $\frac{77}{88}$ (ii) $\frac{375}{625}$ (iii) $-\frac{72}{336}$ (iv) $-\frac{276}{115}$.
4. Convert the following fractions into equivalent like fractions with least denominator :
   (i) $\frac{13}{16}, \frac{17}{24}, \frac{23}{32}$ (ii) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}$.

5. Arrange the given fractions in ascending order :

(*i*) $\frac{3}{4}, \frac{9}{16}, \frac{5}{12}$ (*ii*) $\frac{2}{3}, \frac{5}{7}, \frac{9}{13}, \frac{5}{6}$.

6. Arrange the given fractions in descending order :

(*i*) $\frac{13}{18}, \frac{11}{15}, \frac{37}{45}$ (*ii*) $\frac{17}{24}, \frac{7}{12}, \frac{13}{18}, \frac{8}{15}$.

7. (*i*) Insert one fraction between $\frac{7}{12}$ and $\frac{3}{4}$.

(*ii*) Insert two fractions between $\frac{3}{5}$ and $\frac{7}{8}$.

(*iii*) Insert three fractions between $\frac{4}{5}$ and $\frac{3}{7}$.

8. Simplify the following expressions :

(*i*) $\left(5\frac{1}{18}-2\frac{5}{6}\right)$ of $2\frac{3}{5} \times \frac{3}{4}$ (*ii*) $2-1\frac{1}{4} \div \frac{1}{5}$ of $6\frac{1}{4}$.

9. Simplify the expression : $30 \div \left[3\frac{1}{2}-\overline{4\frac{1}{4}-2\frac{1}{3}}\right]$.

10. (*i*) What fraction of $6\frac{2}{3}$ is $11\frac{5}{7}$?

(*ii*) Express 45 mm as a fraction of 21 m.

11. One-fifth of a rod is 20 cm. What is the length of the rod?

12. Ankita did $\frac{1}{4}$ of a certain work and Anshul did $\frac{1}{3}$ of it. What fraction of the work is left?

13. There were 60 students in a class. On a rainy day, $\frac{1}{5}$th of them were absent and $\frac{7}{12}$th of those present brought raincoats. How many of them did not bring raincoat?

14. At a cricket match, $\frac{2}{7}$th of the spectators were in a covered stand while 15000 were in open. Find the total number of spectators.

15. Robert spent $\frac{3}{8}$ of his monthly income on food and $\frac{3}{10}$ of the remaining on house rent. What fraction of the income is left with him? If the money left is ₹3570, what is his monthly income?

16. In a class of 56 students, the number of boys is $\frac{2}{5}$th of the number of girls. Find the number of boys and girls.

17. A man donated $\frac{1}{10}$ of his money to a school, $\frac{1}{6}$th of the remaining to a church and the remaining money he distributed equally among his three children. If each child gets ₹ 50000, how much money did the man originally have?

18. If $\frac{1}{4}$ of a number is added to $\frac{1}{3}$ of that number, the result is 15 greater than half of that number. Find the number.

19. A student was asked to multiply a given number by $\frac{4}{5}$. By mistake, he divided the given number by $\frac{4}{5}$. His answer was 36 more than the correct answer. What was the given number?

[Hint. Let the number be x, then $\frac{5}{4}x-\frac{4}{5}x=36$.]

DECIMAL NUMBERS

Look at the number 7023·602. The small dot which appears in between the digits 3 and 6 is called **decimal point**. Such a number is called a **decimal number**; 7023 is called its ***integral part*** and ·602 is called its ***decimal part***. The position of each digit in the given number determines the value of that digit.

The first digit '3' on the left of decimal point is the unit's digit and it represents 3 units. The second digit '2' on the left of decimal point is the ten's digit and it represents two tens *i.e.* twenty. The next digit '0' is the hundred's digit and it represents zero hundred, and the next digit '7' represents seven thousand. The first digit '6' on the right of decimal point is the tenth's digit and it represents six tenths. The second digit '0' on the right of decimal point is the hundredth's digit and it represents zero hundredth. The next digit '2' is the thousandth's digit and it represents two thousandths.

Thus, the place values of the digits that are on the left of decimal point in order are 1, 10, 100, 1000 and so on; and the place values of the digits that are on the right of decimal point are $\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}$ and so on.

$$\therefore \quad 7023{\cdot}602 = 7 \times 1000 + 0 \times 100 + 2 \times 10 + 3 + 6 \times \frac{1}{10} + 0 \times \frac{1}{100} + 2 \times \frac{1}{1000}.$$

It is known as ***expanded form***.

In a decimal number, the number of digits that follow the decimal point is called the number of *decimal places*. Thus, 7023·602 has 3 decimal places.

- **Decimal fractions can be written as decimal numbers.**

  Count the number of zeros in the denominator and then count the same number of digits in the numerator starting from the unit's digit and moving to the left, and then place the decimal point.

For example :

$$\frac{3}{10} = 0{\cdot}3,\ \frac{51}{100} = 0{\cdot}51,\ \frac{93}{1000} = 0{\cdot}093,$$

$$\frac{123}{10} = 12{\cdot}3,\ \frac{2503}{100} = 25{\cdot}03,\ \frac{853012}{1000} = 853{\cdot}012$$

- **Decimal numbers can be written as decimal fractions.**

  In the denominator (i.e. below the division line) after 1, write the number of zeros equal to the number of decimal places in the given number.

For example :

$$43{\cdot}5 = \frac{435}{10},\ 72{\cdot}09 = \frac{7209}{100},\ 3{\cdot}107 = \frac{3107}{1000},$$

$$0{\cdot}7 = \frac{7}{10},\ 0{\cdot}08 = \frac{8}{100},\ 0{\cdot}033 = \frac{33}{1000}.$$

Remarks

- Any extra zero (or zeros) written at the end of a decimal number does not change its value. For example, 4·7 is same 4·70 or 4·700.
- An integer can be expressed as a decimal number by writing zero (or zeros) in the decimal part.
  For example, 19 is same as 19·0 or 19·00.

Example 1. Convert the following decimal numbers to fractions :

(*i*) 0·36 (*ii*) 7·25 (*iii*) 31·065

Solution. (*i*) $0{\cdot}36 = \frac{36}{100} = \frac{9}{25}$.

(*ii*) $7{\cdot}25 = 7 + \frac{25}{100} = 7 + \frac{1}{4} = 7\frac{1}{4}$.

(*iii*) $31{\cdot}065 = 31 + \frac{65}{1000} = 31 + \frac{13}{200} = 31\frac{13}{200}$.

Example 2. Express the following fractions as decimals :

(*i*) $\frac{3}{25}$ (*ii*) $\frac{13}{125}$ (*iii*) $\frac{5}{8}$ (*iv*) $7\frac{3}{40}$

Solution. (*i*) $\frac{3}{25} = \frac{3 \times 4}{25 \times 4} = \frac{12}{100} = 0{\cdot}12.$

(*ii*) $\frac{13}{125} = \frac{13 \times 8}{125 \times 8} = \frac{104}{1000} = 0{\cdot}104.$

(*iii*) $\frac{5}{8} = \frac{5 \times 125}{8 \times 125} = \frac{625}{1000} = 0{\cdot}625$

(*iv*) $7\frac{3}{40} = 7 + \frac{3}{40} = 7 + \frac{3 \times 25}{40 \times 25} = 7 + \frac{75}{1000} = 7{\cdot}075.$

Convert each fraction into an equivalent decimal fraction

Simplification of expressions involving decimal numbers

You already know the four fundamental operations of addition, subtraction, multiplication and division on decimal numbers. To simplify expressions involving decimal numbers, use rule of **BODMAS**.

Example 3. Simplify the following :

(*i*) 2·31 × 0·019 (*ii*) 1·3 ÷ 0·08 (*iii*) $\frac{0{\cdot}03 \times 0{\cdot}9}{0{\cdot}0036}$

Solution. (*i*) 2·31 × 0·019

```
      2 3 1
  ×     1 9
  ---------
    2 0 7 9
    2 3 1 ×
  ---------
    4 3 8 9
  0 4 3 8 9
```

Steps

(*i*) Ignore the decimal points and multiply the numbers.

(*ii*) Count the number of decimal places in each number, their sum = 2 + 3 = 5.

(*iii*) Using this sum, start at the right of the product 4389 and count off 5 places. Note that in the product there are only 4 digits, so write it as 04389 to make 5 places.

∴ 2·31 × 0·019 = 0·04389.

(*ii*) $1{\cdot}3 \div 0{\cdot}08 = \frac{1{\cdot}3}{0{\cdot}08}$

$= \frac{1{\cdot}30}{0{\cdot}08} = \frac{130}{8}$

$= \frac{65}{4} = 16\frac{1}{4}$

$= 16 + \frac{1}{4} = 16 + \frac{1 \times 25}{4 \times 25} = 16 + \frac{25}{100} = 16{\cdot}25.$

To convert divisor into a natural number, shift decimal point to the right by two places

(*iii*) First we find 0·03 × 0·9

Since 3 × 9 = 27 and the sum of decimal places in the given numbers = 2 + 1 = 3,

$\therefore\ 0{\cdot}03 \times 0{\cdot}9 = 0{\cdot}027$

Shift the decimal point to the right by four places

$$\therefore\ \frac{0{\cdot}03 \times 0{\cdot}9}{0{\cdot}0036} = \frac{0{\cdot}027}{0{\cdot}0036} = \frac{0{\cdot}0270}{0{\cdot}0036}$$

$$= \frac{270}{36} = \frac{15}{2} = 7\frac{1}{2}$$

$$= 7 + \frac{1}{2} = 7 + \frac{1 \times 5}{2 \times 5} = 7 + \frac{5}{10} = 7{\cdot}5.$$

Example 4. Simplify the following :

(*i*) $(3{\cdot}7 - 2{\cdot}931 + 0{\cdot}12)$ of $0{\cdot}37$ (*ii*) $\dfrac{9{\cdot}75 \times 4{\cdot}8}{13} + \dfrac{1{\cdot}95}{15}$

Solution. (*i*) $3{\cdot}7 - 2{\cdot}931 + 0{\cdot}12$

$= 3{\cdot}82 - 2{\cdot}931$

$= 0{\cdot}889$

$$\begin{array}{r} 3{\cdot}70 \\ +\ 0{\cdot}12 \\ \hline 3{\cdot}82 \end{array} \qquad \begin{array}{r} 3{\cdot}820 \\ -\ 2{\cdot}931 \\ \hline 0{\cdot}889 \end{array} \qquad \begin{array}{r} 889 \\ \times\ 37 \\ \hline 6223 \\ 2667\times \\ \hline 32893 \end{array}$$

$\therefore\ (3{\cdot}7 - 2{\cdot}931 + 0{\cdot}12)$ of $0{\cdot}37$

$= 0{\cdot}889 \times 0{\cdot}37$

$= 0{\cdot}32893$

(*ii*) $9{\cdot}75 \times 4{\cdot}8 = 46{\cdot}800$

$= 46{\cdot}8$

$$\begin{array}{r} 975 \\ \times\ 48 \\ \hline 7800 \\ 3900\times \\ \hline 46800 \end{array}$$

$$\therefore\ \frac{9{\cdot}75 \times 4{\cdot}8}{13} + \frac{1{\cdot}95}{15} = \frac{46{\cdot}8}{13} + \frac{1{\cdot}95}{15}$$

$= 3{\cdot}6 + 0{\cdot}13$

$= 3{\cdot}73.$

$$13\,\overline{)\,46{\cdot}8\,}\,(3{\cdot}6:\quad 39,\ 78,\ 78,\ 0 \qquad 15\,\overline{)\,1{\cdot}95\,}\,(0{\cdot}13:\quad 15,\ 45,\ 45,\ 0$$

Types of decimal numbers

(*i*) **Terminating decimals**

We know that $\frac{3}{5} = 0{\cdot}6$, $\frac{7}{25} = 0{\cdot}28$, $\frac{5}{2} = 2{\cdot}5$, $\frac{3}{8} = 0{\cdot}375$

In these cases, the dividend is exactly divisible after a few steps. Such decimal numbers are called **terminating decimals**.

(*ii*) **Non-terminating decimals**

There are fractions which give unending decimal values. For example :

$\frac{2}{3} = 0{\cdot}66666...$, $\frac{8}{15} = 0{\cdot}533333...$

$\frac{5}{11} = 0{\cdot}454545...$, $\frac{15}{7} = 2{\cdot}142857142857...$

From these examples, we note that the division is not exact. Such decimal numbers are called **non-terminating decimals**.

Further, in the above divisions, we obtain a certain digit or a certain block of digits which repeat over and over again. Such repeating decimal numbers are called **non-terminating repeating decimals or recurring** (or **periodic**) **decimals**. This fact is represented by putting a dot (or bar) over the repeated digit or digits.

Thus, $\frac{2}{3} = 0.66666... \quad = 0.\dot{6}$ or $0.\overline{6}$

$\frac{8}{15} = 0.533333... \quad = 0.5\dot{3}$ or $0.5\overline{3}$

$\frac{5}{11} = 0.45454545... \quad = 0.\dot{4}\dot{5}$ or $0.\overline{45}$

However, if a recurring decimal contains more than two repeating digits, then put a dot on the first repeating digit and another on the last digit or put a bar over the entire block of repeating digits.

Thus, $\frac{15}{7} = 2.142857142857... = 2.\dot{1}4285\dot{7}$ or $2.\overline{142857}$

There do exist **non-terminating and non-recurring** decimal numbers. We shall deal with such decimal numbers in the next section of this chapter.

Remark

(*i*) The number 0.666666... can be written as $0.66\dot{6}$.

(*ii*) The number 0.533333... can be written as $0.53\dot{3}$.

(*iii*) The number 0.454545... can be written as $0.45\overline{45}$ etc.

Example 5. Convert the following recurring decimals to fractions :

(*i*) $0.\dot{3}$ (*ii*) $0.1\dot{6}$ (*iii*) $0.\overline{27}$

Solution. (*i*) Let $x = 0.\dot{3} = 0.3333....$

Multiplying both sides by 10

$\therefore \quad 10x = 3.3333....$

$- \quad x = .3333....$

$9x = 3$

$\Rightarrow \quad x = \frac{3}{9} = \frac{1}{3}.$

(*ii*) Let $x = 0.1\dot{6} = 0.166666....$

Multiplying both sides by 10

$10x = 1.66666....$

$- \quad x = 0.16666....$

$9x = 1.5$

$\Rightarrow \quad x = \frac{1.5}{9} = \frac{15}{90} = \frac{1}{6}.$

(*iii*) Let $x = 0.\overline{27} = 0.27272727....$

Multiplying both sides by 100

$100x = 27.272727....$

$- \quad x = 0.272727....$

$99x = 27$

$\Rightarrow \quad x = \frac{27}{99} = \frac{3}{11}.$

Approximation

The population of Delhi is 1,82,00,000.

It takes 2·5 years to travel to Mars.

The radius of the Earth is 6400 km.

Each of the above statement contains a number which is not exact. The population of Delhi is not exactly 1,82,00,000 although this figure gives you a good idea of its size. The second and the third statements are about the time and length which are approximate. In each of these statements, the number is correct to a stated accuracy (unit). We say that the number has been **rounded off** or **corrected to** the stated accuracy.

Rounding to whole numbers

How would you round a number like 8·62 to the nearest unit? The number 8·62 lies between 8 and 9. Do you round it up or down? Notice that 8·62 is closer to 9 than to 8, so 8·62 = 9 rounded to the nearest unit.

Look at the following examples and make sure that I have rounded them correctly:

257 = 260 to the nearest ten
253 = 250 to the nearest ten
255 = 260 to the nearest ten
7584 = 7600 to the nearest hundred
7584 = 8000 to the nearest thousand
257 = 300 to the nearest hundred
37·6 = 38 to the nearest unit
37·46 = 37 to the nearest unit
37·512 = 38 to the nearest unit
0·56 = 1 to the nearest unit

Rounding to decimal places

To round a number to a certain number of decimal places, proceed as under:

(i) Work out the number to one more place than you need.

(ii) If the extra digit is 5 or more, add 1 to the number before it.

(iii) If the extra digit is less than 5, leave the number before it as it is.

Look at the following examples and make sure that I have rounded them correctly:

49·3795 = 49·4 correct to 1 decimal place
49·3795 = 49·38 correct to 2 decimal places
7·2485 = 7·249 correct to 3 decimal places
7·2485 = 7·25 correct to 2 decimal places
7·2485 = 7·2 correct to 1 decimal place
0·0535 = 0·1 correct to 1 decimal place
0·0535 = 0·05 correct to 2 decimal places
0·0535 = 0·054 correct to 3 decimal places
2·397 = 2·40 correct to 2 decimal places

Significant figures

Significant figures are the digits used in expressing the given number (quantity) with stated accuracy (unit).

The following rules help to determine the significant figures in a number:

(*i*) *All digits in a whole number are significant.*

For example :

3570 has four significant figures.

504 has three significant figures.

(*ii*) If the decimal number is greater than 1, ignore the decimal point. All the non-zero digits and all the zeros between them and after them (if any) are significant figures.

For example :

5·203 has four significant figures.

34·0104 has six significant figures.

7·20 has three significant figures.

39·050100 has eight significant figures.

(*iii*) If the number lies between 0 and 1 then the initial zeros are not significant.

For example :

0·02034 has four significant figures and first significant figure is 2.

0·000302 has three significant figures and first significant figure is 3.

(*iv*) The final zero(s) of an approximated number when expressed as a decimal number are significant.

For example :

The number 2·397 approximated to two decimal places is 2·40.

Here the number 2·40 has three significant figures.

Rounding to significant figures

To round a decimal number to a certain number of significant figures, proceed in a way similar to that used for decimal places. Look at the following examples and make sure that I have rounded them correctly :

3·10465 = 3·105 correct to four significant figures

1·0716 = 1·1 correct to two significant figures

0·003561 = 0·004 correct to one significant figure

37·3417 = 37·34 correct to four significant figures

128·255 = 128·26 correct to five significant figures

0·002456 = 0·00246 correct to three significant figures

7·998 = 8·00 correct to three significant figures.

Example. Round the following quantities to the accuracy stated :

(*i*) A crowd of sixty seven thousand four hundred and forty eight to the nearest hundred.

(*ii*) ₹8674513 to the nearest thousand.

(*iii*) ₹2·496 to the nearest paisa.

(*iv*) 53·67 m to the nearest 10 cm.

(*v*) 0·00549 kg to the nearest gram.

(*vi*) 7·451 cm to the nearest mm.

(*vii*) ₹13·53 to the nearest 5 paise.

Solution. (*i*) The crowd is of 67448 people.
Obviously, the choice is between 67400 and 67500.
The nearest to 67448 is 67400.
Hence, the crowd of 67448 rounding to the nearest hundred is 67400.

(*ii*) The choice is between ₹8674000 and ₹8675000.
The nearest to ₹8674513 is ₹8675000.
Hence, ₹8674513 rounding to the nearest thousand is ₹8675000.

(*iii*) ₹2·496 = 2·496 × 100 paise = 249·6 paise.
Obviously, the choice is between 249 paise and 250 paise.
The nearest to 249·6 paise is 250 paise.
Hence, ₹2·496 rounding to the nearest paisa is 250 paise *i.e.* ₹2·50.

(*iv*) 53·674 m = 53·67 × 100 cm = 5367 cm.
Obviously, the choice is between 5360 cm and 6370 cm.
The nearest to 5367 cm is 5370 cm.
Hence, 53·67m rounding to the nearest 10 cm is 5370 cm *i.e.* 53·70 m.

(*v*) 0·00549 kg = 0·00549 × 1000 gm = 5·49 gm.
The choice is between 5 gm and 6 gm.
The nearest to 5·49 gm is 5 gm.
Hence, 0·00549 kg rounding to the nearest gram is 5 gm *i.e.* 0·005 kg.

(*vi*) 7·451 cm = 7·451 × 10 mm = 74·51 mm.
The choice is between 74 mm and 75 mm.
The nearest to 74·51 mm is 75 mm.
Hence, 7·451 cm rounding to the nearest mm is 75 mm *i.e.* 7·5 cm.

(*vii*) ₹13·53 = 13·53 × 100 paise = 1353 paise.
Obviously, the choice is between 1350 paise and 1355 paise.
The nearest to 1353 paise is 1355 paise.
Hence, ₹13·53 rounding to the nearest 5 paise is 1355 paise *i.e.* ₹13·55.

Exercise 3.4

1. Convert the following decimal numbers into fractions (in lowest terms) :

   (*i*) 0·625 (*ii*) 3·025 (*iii*) 4·072

2. Write the following fractions as decimal numbers :

   (*i*) $\frac{31}{1000}$ (*ii*) $13\frac{57}{100000}$ (*iii*) $\frac{7}{8}$ (*iv*) $7\frac{5}{16}$

   (*v*) $\frac{17}{6}$ (*vi*) $\frac{9}{110}$ (*vii*) $\frac{13}{44}$ (*viii*) $\frac{4}{7}$

3. Write the following decimal numbers in ascending order :

   (*i*) 0·3, 0·33, 3·03, 0·303 (*ii*) 0·643, 0·634, 0·6034, 0·6304

4. Calculate the following :

(*i*) $89{\cdot}2 - 11{\cdot}04 - 38{\cdot}0093$

(*ii*) $37{\cdot}051 - 48{\cdot}2378 - 0{\cdot}307 + 18{\cdot}07$

(*iii*) $3{\cdot}07 \times 0{\cdot}045 \times 0{\cdot}2$

(*iv*) $(0{\cdot}4)^2 \times 0{\cdot}07 + 6 \div 0{\cdot}2$

(*v*) $\dfrac{(0{\cdot}3)^3 \times 5{\cdot}1}{(0{\cdot}09)^2}$

(*vi*) $\dfrac{0{\cdot}235 \times 11{\cdot}5}{94 \times 2{\cdot}5} + \dfrac{48{\cdot}36}{12}$

5. Simplify the following :

(*i*) $3{\cdot}2 + (3{\cdot}2 - 0{\cdot}4)$ of $0{\cdot}4$

(*ii*) $4{\cdot}5 - [2 - 0{\cdot}125\ \{4 - 6 \div (2{\cdot}7 - 0{\cdot}7)\}]$

(*iii*) $\dfrac{1{\cdot}8 \text{ of } 9{\cdot}75}{0{\cdot}0195} - \dfrac{0{\cdot}75}{0{\cdot}005}$

6. (*i*) Express $1{\cdot}08$ as a fraction of $0{\cdot}39$.

(*ii*) Find the value of $2{\cdot}5$ of $5{\cdot}630$ kg.

7. Express $22 \div 7$ correct to

(*i*) four decimal places

(*ii*) four significant figures.

8. Express $\dfrac{5}{48}$ correct to four significant figures.

9. Evaluate $9{\cdot}8 + 0{\cdot}91 - 6{\cdot}38$ correct to the nearest whole number.

10. Round the following quantities (measurements) to the accuracy stated :

(*i*) ₹ 54566 to the nearest hundred.

(*ii*) ₹ 63745 to the nearest ten.

(*iii*) ₹ 27·998 to the nearest paisa.

(*iv*) 4·357 m to the nearest cm.

(*v*) 63·89 m to the nearest 10 cm.

(*vi*) 0·0764 kg to the nearest gm.

(*vii*) 3·47 m to the nearest 5 cm.

REAL NUMBERS

Module 4 (2 classes)

Rational numbers

Any number that can be expressed in the form $\frac{m}{n}$, where m and n are integers and $n \neq 0$ is called a ***rational number****.*

The set of rational numbers is denoted by **Q**. Thus,

$$\mathbf{Q} = \left\{\frac{m}{n}; m, n \in \mathbf{Z} \text{ and } n \neq 0\right\}.$$

Earlier in this chapter, we learnt that every number of the form $\frac{m}{n}$ where m and n are integers and $n \neq 0$ can be expressed as either a terminating decimal or a recurring decimal. Hence, *every terminating decimal or a recurring decimal is a* ***rational number****.*

Remarks

- Every integer (positive, negative or zero) can be written in the form $\frac{m}{n}$ where $n = 1$. For example,

$$5 = \frac{5}{1}, -3 = \frac{-3}{1}, 0 = \frac{0}{1}$$

Hence, every integer is a rational number and the set **Z** of integers is a proper subset of the set Q of rational numbers *i.e.* $\mathbf{Z} \subset \mathbf{Q}$.

- Since the division by zero is not allowed, $\frac{1}{0}$ is not a rational number *i.e.* the reciprocal of zero is not allowed.
- When we write a rational number in the form $\frac{m}{n}$, $m, n \in \mathbf{Z}$ and $n \neq 0$, usually we take $n > 0$, while m may be positive, negative or zero.

Further, if m and n have no common factor except 1, the number is in lowest terms.

- Every rational number can be expressed in the lowest terms.

Properties of rational numbers

Let **Q** be the set of rational numbers and a, b be any members of **Q**, then the following results hold :

- $a, b \in \mathbf{Q} \Rightarrow a + b \in \mathbf{Q}$

For example, $\frac{1}{3} + \frac{2}{7} = \frac{7+6}{21} = \frac{13}{21}$, which is a rational number.

- $a, b \in \mathbf{Q} \Rightarrow a - b \in \mathbf{Q}$

For example, $\frac{1}{3} - \frac{2}{7} = \frac{7+6}{21} = \frac{1}{21}$, which is a rational number.

- $a, b \in \mathbf{Q} \Rightarrow ab \in \mathbf{Q}$

For example, $\frac{1}{3} \times \frac{2}{7} = \frac{2}{21}$, which is a rational number.

- $a, b \in \mathbf{Q}, b \neq 0 \Rightarrow \frac{a}{b} \in \mathbf{Q}$

For example, $\frac{1}{3} \div \frac{2}{7} = \frac{1}{3} \times \frac{7}{2} = \frac{7}{6}$, which is a rational number.

Thus, the set of rational numbers is closed under all the four fundamental operations of arithmetic.

If a and b are any two different rational numbers, then $\frac{a+b}{2}$ is a rational number and it lies between them *i.e.* if $a < b$ then $a < \frac{a+b}{2} < b$. Continuing this process, we find that there are **infinitely many** rational numbers between two different rational numbers. For example,

$\dfrac{\frac{1}{3}+\frac{2}{7}}{2} = \frac{13}{42}$ is a rational number which lies between $\frac{1}{3}$ and $\frac{2}{7}$.

Example 1. Insert a rational number between $\frac{4}{5}$ and $\frac{5}{9}$ and arrange in ascending order.

Solution. The L.C.M. of 5 and 9 is 45.

$$\therefore \frac{4}{5} = \frac{4\times9}{5\times9} = \frac{36}{45}, \frac{5}{9} = \frac{5\times5}{9\times5} = \frac{25}{45}.$$

Since $25 < 36, \frac{25}{45} < \frac{36}{45} \Rightarrow \frac{5}{9} < \frac{4}{5}$.

A rational number between $\frac{5}{9}$ and $\frac{4}{5} = \frac{\frac{5}{9}+\frac{4}{5}}{2} = \frac{\frac{25+36}{45}}{2} = \frac{61}{45} \times \frac{1}{2} = \frac{61}{90}$

and the numbers in ascending order are $\frac{5}{9}, \frac{61}{90}, \frac{4}{5}$.

Example 2. Insert three rational numbers between 3 and 3·5.

Solution. A rational number between 3 and $3{\cdot}5 = \frac{3+3{\cdot}5}{2} = \frac{6{\cdot}5}{2} = 3{\cdot}25$

A rational number between 3 and $3{\cdot}25 = \frac{3+3{\cdot}25}{2} = \frac{6{\cdot}25}{2} = 3{\cdot}125$

A rational number between 3 and $3{\cdot}125 = \frac{3+3{\cdot}125}{2} = \frac{6{\cdot}125}{2} = 3{\cdot}0625$

We note that $3 < 3{\cdot}0625 < 3{\cdot}125 < 3{\cdot}25 < 3{\cdot}5$, therefore, three rational numbers between 3 and 3·5 are 3·0625, 3·125, 3·25.

Irrational numbers

The number whose square is 2 is written as $\sqrt{2}$. In fact, $\sqrt{2}$ is not a rational number *i.e.* $\sqrt{2}$ cannot be expressed as $\frac{m}{n}$ where $m, n \in \mathbf{Z}$ and $n \neq 0$. It follows that $\sqrt{2}$ is neither a terminating decimal nor a repeating decimal. Such numbers are called **irrational numbers**.

The numbers $\sqrt{2}, \sqrt{3}, 5\sqrt{6}, -\sqrt{5}, \frac{2}{\sqrt{7}}, 3+\sqrt{11}, 2-\sqrt{7}, \sqrt{3}+\sqrt{5}, \sqrt[3]{2}$ etc. are *irrational numbers*.

In fact, all non-terminating and non-recurring decimal numbers are irrational numbers.

Note. The number π is irrational and its approximate value is $\frac{22}{7}$.

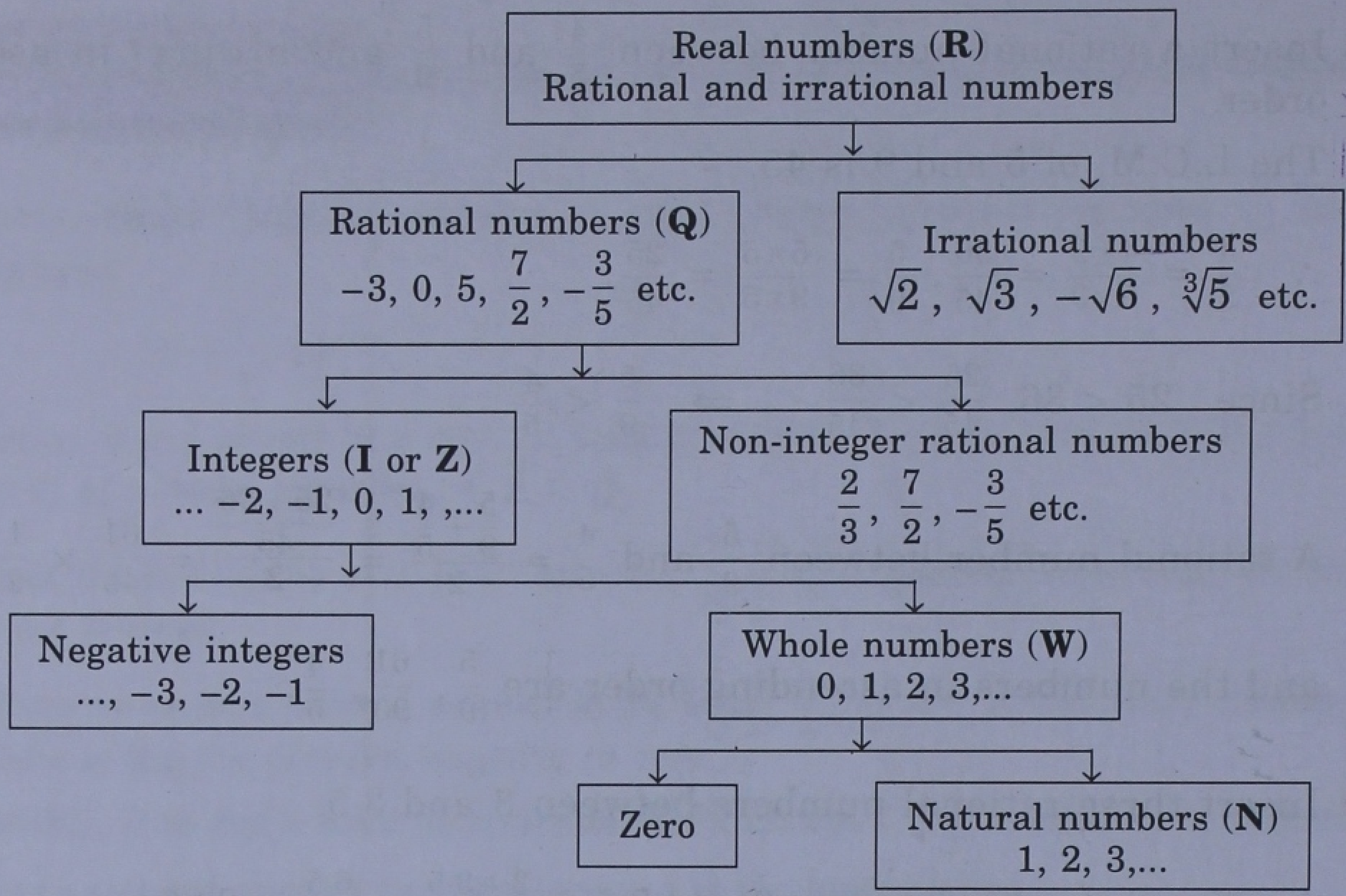
Real numbers

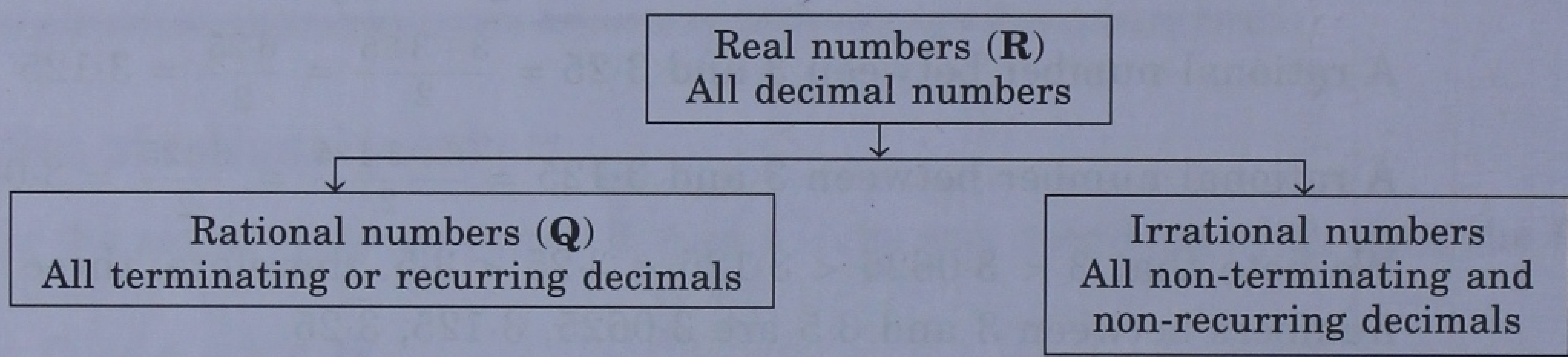
All rational numbers and all irrational numbers are ***real numbers***. The set of real numbers is denoted by **R**. Thus,

$$\mathbf{R} = \{x : x \text{ is rational or irrational}\}.$$

Note that $\mathbf{N} \subset \mathbf{W} \subset \mathbf{Z} \subset \mathbf{Q} \subset \mathbf{R}$.

All decimal numbers (terminating, recurring, or non-terminating and non-recurring) are ***real numbers***.

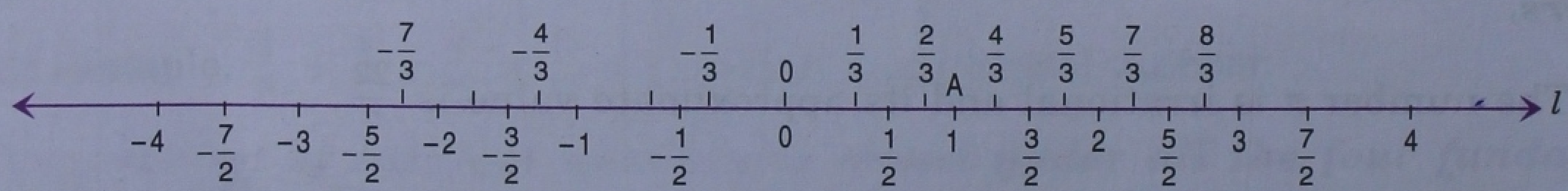




Representation of real numbers on a line

Let l be a straight line which extends endlessly on both sides. Mark the positive direction to the right by an arrowhead. Take a point O on l and label it 0 (zero). Next choose another point, say A, on l towards the right of 0 and label it 1 (one). Thus, the points O and A represent the number 0 and 1 respectively. The length of the segment OA represents unit length. Now mark points, on l to the right of A at unit length's interval and label these 2, 3,.... Similarly, mark points on l to the left of O and label them, −1, −2, −3,....

Next, we consider the representation of fractions on the line l. Take one-half of the unit length and mark points on l on both sides of O, these points will represent the numbers $\frac{1}{2}, \frac{2}{2}, \frac{3}{2}$ and $-\frac{1}{2}, -\frac{2}{2}, -\frac{3}{2}$..., shown in the figure.



Similarly, take one-third of the unit length and mark points on l on both sides of O, these points will represent the numbers $\frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{4}{3}$,... and $-\frac{1}{3}, -\frac{2}{3}, -\frac{3}{3}, -\frac{4}{3}$,.... Thus, every rational number has been represented by one and only one point on the line l.

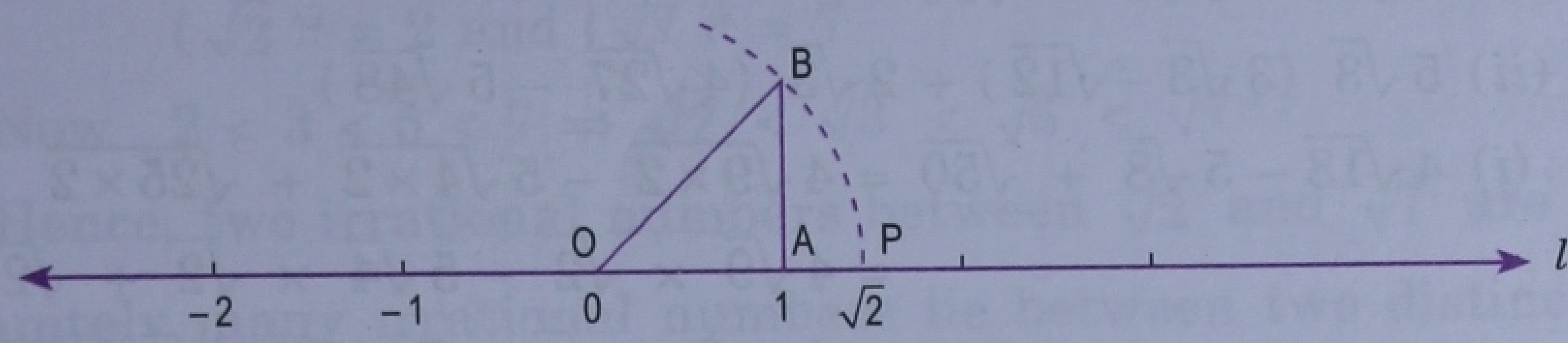
Next, to represent irrational numbers on the line l, we recall a result from geometry, known as ***Pythagoras theorem*** :

> ***The square of the hypotenuse of a right angled triangle is equal to the sum of the squares of the other two sides.***

Construct a right angled triangle OAB at A such that OA = AB = 1, then by Pythagoras theorem,

$$OB^2 = OA^2 + AB^2 \Rightarrow OB^2 = 1^2 + 1^2 = 2 \Rightarrow OB = \sqrt{2}.$$

Now mark a point, say P, on l on the right of O such that OP = OB = $\sqrt{2}$, the point P represents the irrational number $\sqrt{2}$ (shown in the figure), and so on.



Hence, corresponding to every real number (rational or irrational) there exists one and only one point on the line l and conversely corresponding to every point on the line l there exists one and only one real number.

The line l is called the ***number line****.*

Some facts about irrational numbers

- $\sqrt{5} + \sqrt{3} \neq \sqrt{5+3}$ *i.e.* $\sqrt{5} + \sqrt{3} \neq \sqrt{8}$, $\sqrt{5} - \sqrt{3} \neq \sqrt{5-3}$ *i.e* $\sqrt{5} - \sqrt{3} \neq \sqrt{2}$ etc.
- $\sqrt{3} + \sqrt{3} \neq \sqrt{6}$ but $\sqrt{3} + \sqrt{3} = 2\sqrt{3}$,

  $5\sqrt{2} - 3\sqrt{2} = (5-3)\sqrt{2} = 2\sqrt{2}$ etc.
- If a and b are positive rational numbers, then

$$\sqrt{ab} = \sqrt{a} \times \sqrt{b} \text{ and } \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}.$$

For example : $\sqrt{6} = \sqrt{2\times3} = \sqrt{2} \times \sqrt{3}$,

$$\sqrt{45} = \sqrt{9\times5} = \sqrt{9} \times \sqrt{5} = 3 \times \sqrt{5} = 3\sqrt{5},$$

$$\sqrt{\frac{5}{9}} = \frac{\sqrt{5}}{\sqrt{9}} = \frac{\sqrt{5}}{3}, \quad \frac{\sqrt{12}}{\sqrt{3}} = \sqrt{\frac{12}{3}} = \sqrt{4} = 2,$$

$$\sqrt{45} - \sqrt{5} = \sqrt{9\times5} - \sqrt{5} = \sqrt{9} \times \sqrt{5} - \sqrt{5} = 3\sqrt{5} - \sqrt{5}$$

$$= (3-1)\sqrt{5} = 2\sqrt{5}.$$

Rationalisation

If the product of two irrational numbers is a rational number, then each number is called the ***rationalising factor*** *of the other number.*

For example :

(*i*) $\sqrt{5} \times \sqrt{5} = 5$, therefore $\sqrt{5}$ is a rationalising factor of $\sqrt{5}$.

(*ii*) $(4 + \sqrt{3})(4 - \sqrt{3}) = 4^2 - (\sqrt{3})^2 = 16 - 3 = 13$, therefore $4 + \sqrt{3}$ and $4 - \sqrt{3}$ are rationalising factors of each other.

(*iii*) $(\sqrt{5} - \sqrt{7})(\sqrt{5} + \sqrt{7}) = (\sqrt{5})^2 - (\sqrt{7})^2 = 5 - 7 = -2$, therefore $\sqrt{5} - \sqrt{7}$ and $\sqrt{5} + \sqrt{7}$ are rationalising factors of each other.

The process of multiplying an irrational number by its rationalising factor is called ***rationalisation****.*

Rule to rationalise the denominator of a fraction :

Multiply and divide the numerator and denominator of the given fraction by the rationalising factor of its denominator and simplify it.

Example 3. Evaluate (simplify) the following :

(*i*) $4\sqrt{18} - 5\sqrt{8} + \sqrt{50}$

(*ii*) $5\sqrt{3}\ (3\sqrt{3} - \sqrt{12}) + 2\sqrt{3}\ (4\sqrt{27} - 5\sqrt{48})$

Solution. (*i*) $4\sqrt{18} - 5\sqrt{8} + \sqrt{50} = 4\sqrt{9\times 2} - 5\sqrt{4\times 2} + \sqrt{25\times 2}$

$= 4\sqrt{9} \times \sqrt{2} - 5\sqrt{4} \times \sqrt{2} + \sqrt{25} \times \sqrt{2}$

$= 4 \times 3 \times \sqrt{2} - 5 \times 2 \times \sqrt{2} + 5\sqrt{2}$

$= 12\sqrt{2} - 10\sqrt{2} + 5\sqrt{2}$

$= (12 - 10 + 5) \times \sqrt{2} = 7\sqrt{2}$.

(*ii*) $5\sqrt{3}\ (3\sqrt{3} - \sqrt{12}) + 2\sqrt{3}\ (4\sqrt{27} - 5\sqrt{48})$

$= 5\sqrt{3}\ (3\sqrt{3} - \sqrt{4\times 3}) + 2\sqrt{3}\ (4\sqrt{9\times 3} - 5\sqrt{16\times 3})$

$= 5\sqrt{3}\ (3\sqrt{3} - \sqrt{4} \times \sqrt{3}) + 2\sqrt{3}\ (4 \times \sqrt{9} \times \sqrt{3} - 5 \times \sqrt{16} \times \sqrt{3})$

$= 5\sqrt{3}\ (3\sqrt{3} - 2\sqrt{3}) + 2\sqrt{3}\ (4 \times 3\sqrt{3} - 5 \times 4\sqrt{3})$

$= 5\sqrt{3}\ (3 - 2)\sqrt{3} + 2\sqrt{3}\ (12 - 20)\sqrt{3}$

$= 5 \times 1 \times \sqrt{3} \times \sqrt{3} + 2 \times (-8) \times \sqrt{3} \times \sqrt{3} = 5 \times 3 - 16 \times 3$

$= 15 - 48 = -33$.

Example 4. Rationalise the denominator of the following :

(*i*) $\frac{3}{4\sqrt{5}}$ (*ii*) $\frac{3}{4-\sqrt{7}}$ (*iii*) $\frac{30}{3\sqrt{2}-2\sqrt{3}}$

Solution. (*i*) $\frac{3}{4\sqrt{5}} = \frac{3}{4\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}} = \frac{3\sqrt{5}}{4\times 5} = \frac{3\sqrt{5}}{20}$.

(*ii*) $\frac{3}{4-\sqrt{7}} = \frac{3}{4-\sqrt{7}} \times \frac{4+\sqrt{7}}{4+\sqrt{7}} = \frac{3(4+\sqrt{7})}{4^2 - (\sqrt{7})^2}$

$= \frac{3(4+\sqrt{7})}{16-7} = \frac{3(4+\sqrt{7})}{9} = \frac{4+\sqrt{7}}{3}$.

(*iii*) $\frac{30}{3\sqrt{2}-2\sqrt{3}} = \frac{30}{3\sqrt{2}-2\sqrt{3}} \times \frac{3\sqrt{2}+2\sqrt{3}}{3\sqrt{2}+2\sqrt{3}} = \frac{30(3\sqrt{2}+2\sqrt{3})}{(3\sqrt{2})^2-(2\sqrt{3})^2}$

$= \frac{30(3\sqrt{2}+2\sqrt{3})}{9\times 2-4\times 3} = \frac{30(3\sqrt{2}+2\sqrt{3})}{18-12} = \frac{30(3\sqrt{2}+2\sqrt{3})}{6}$

$= 5(3\sqrt{2} + 2\sqrt{3})$.

Example 5. Write in ascending order : $3\sqrt{2}$, $2\sqrt{8}$, $\sqrt{50}$, 4, $4\sqrt{3}$.

Solution. Write all the numbers as square roots under one radical.

$3\sqrt{2} = \sqrt{9} \times \sqrt{2} = \sqrt{18}$, $2\sqrt{8} = \sqrt{4} \times \sqrt{8} = \sqrt{32}$, $\sqrt{50} = \sqrt{50}$,

$4 = \sqrt{16}$, $4\sqrt{3} = \sqrt{16} \times \sqrt{3} = \sqrt{48}$

Since $16 < 18 < 32 < 48 < 50$

$\Rightarrow \sqrt{16} < \sqrt{18} < \sqrt{32} < \sqrt{48} < \sqrt{50}$

$\Rightarrow 4 < 3\sqrt{2} < 2\sqrt{8} < 4\sqrt{3} < \sqrt{50}$.

Hence, the given numbers in ascending order are
$4, 3\sqrt{2}, 2\sqrt{8}, 4\sqrt{3}, \sqrt{50}$.

Example 6. Insert two irrational numbers between $\sqrt{2}$ and $\sqrt{7}$.

Solution. Consider the squares of $\sqrt{2}$ and $\sqrt{7}$.

$$(\sqrt{2})^2 = 2 \text{ and } (\sqrt{7})^2 = 7$$

Now $2 < 3 < 5 < 7 \Rightarrow \sqrt{2} < \sqrt{3} < \sqrt{5} < \sqrt{7}$

Hence, two irrational numbers between $\sqrt{2}$ and $\sqrt{7}$ are $\sqrt{3}$ and $\sqrt{5}$.

Note. Since infinitely many irrational numbers lie between two distinct real numbers, $\sqrt{3}$ and $\sqrt{5}$ are not the only irrational numbers between $\sqrt{2}$ and $\sqrt{7}$.

Exercise 3.5

1. Insert one rational number between $\frac{5}{7}$ and $\frac{4}{9}$, and arrange in ascending order.
2. Insert a rational number between $\frac{2}{9}$ and $\frac{3}{8}$, and arrange in descending order.
3. Insert three rational numbers between 5 and 5·6.
4. Evaluate the following :

   (*i*) $5\sqrt{2} - \sqrt{8}$ (*ii*) $4\sqrt{3} + \sqrt{27}$ (*iii*) $\sqrt{48} - \sqrt{12}$

   (*iv*) $5\sqrt{3} \times \sqrt{8}$ (*v*) $\frac{\sqrt{15}}{\sqrt{3}}$ (*vi*) $\sqrt{3}(\sqrt{12} + \sqrt{27})$
5. Rationalise the denominator of the following :

   (*i*) $\frac{2\sqrt{2}}{\sqrt{5}}$ (*ii*) $\frac{1}{3-\sqrt{2}}$ (*iii*) $\frac{4}{2+\sqrt{3}}$

   (*iv*) $\frac{3}{5-\sqrt{13}}$ (*v*) $\frac{17}{3\sqrt{2}+1}$ (*vi*) $\frac{1}{\sqrt{3}-\sqrt{2}}$

   (*vii*) $\frac{8\sqrt{2}}{\sqrt{5}-\sqrt{3}}$ (*viii*) $\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}$ (*ix*) $\frac{\sqrt{3}}{\sqrt{6}+\sqrt{3}}$
6. Simplify the following :

   (*i*) $\frac{\sqrt{15}-2}{\sqrt{15}+2} + \frac{\sqrt{15}+2}{\sqrt{15}-2}$ (*ii*) $\frac{4+\sqrt{13}}{4-\sqrt{13}} - \frac{4-\sqrt{13}}{4+\sqrt{13}}$
7. Write the following numbers in ascending order :

   (*i*) $3\sqrt{2}, 2\sqrt{3}, \sqrt{15}, 4$ (*ii*) $4\sqrt{5}, 5\sqrt{3}, 10, 3\sqrt{7}, 6\sqrt{2}$
8. Insert an irrational number between $\sqrt{5}$ and $\sqrt{8}$.
9. Insert two irrational numbers between 2 and 3.
10. Insert three irrational numbers between $\sqrt{12}$ and 4.

Module 5 (2 classes)

SQUARES AND SQUARE ROOTS

Square of a number

When a number is multiplied by itself, the product so obtained is called the ***square*** *of that number.*

For example :

(*i*) $5 \times 5 = 25$ *i.e.* $5^2 = 25$, so 25 is the square of 5

(*ii*) $\frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$ *i.e.* $\left(\frac{2}{3}\right)^2 = \frac{4}{9}$, so $\frac{4}{9}$ is the square of $\frac{2}{3}$

(*iii*) $0.4 \times 0.4 = 0.16$ *i.e.* $(0.4)^2 = 0.16$, so 0.16 is the square of 0.4

Note. In this section, we shall consider only positive numbers.

Some facts about squares

- The square of an odd natural number is an odd natural number.

For example :

(*i*) $7 \times 7 = 49$, 7 is odd and 49 is also odd.

(*ii*) $15 \times 15 = 225$, 15 is odd and 225 is also odd.

- The square of an even natural number is an even natural number.

For example :

(*i*) $6 \times 6 = 36$, 6 is even and 36 is also even.

(*ii*) $14 \times 14 = 196$, 14 is even and 196 is also even.

- The square of a proper fraction is smaller than the fraction.

For example :

(*i*) $\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$, $\frac{1}{9} < \frac{1}{3}$ (*ii*) $\frac{3}{4} \times \frac{3}{4} = \frac{9}{16}$, $\frac{9}{16} < \frac{3}{4}$.

- The square of a decimal number less than one is smaller than the decimal number.

For example :

(*i*) $0.6 \times 0.6 = 0.36$, $0.36 < 0.6$

(*ii*) $0.21 \times 0.21 = 0.0441$, $0.0441 < 0.21$

Square root of a number

*The **square root** of a number is that number which when multiplied by itself gives the original number.*

For example :

(*i*) As $5 \times 5 = 25$, square root of 25 is 5.

(*ii*) As $\frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$, square root of $\frac{4}{9}$ is $\frac{2}{3}$.

(*iii*) $0.4 \times 0.4 = 0.16$, square root of $0.16 = 0.4$.

The square root of a is denoted by $\sqrt{a}$.

Thus, $\sqrt{25} = 5$, $\sqrt{\frac{4}{9}} = \frac{2}{3}$ and $\sqrt{0.16} = 0.4$.

Square root of a number by prime factorisation

Method to find the square root of a number by resolving it into prime factors:

(*i*) *Express the given number as the product of primes.*

(*ii*) *Make groups in pairs of the same prime.*

(*iii*) *Take one factor from each pair of primes. Multiply them together, the product so obtained is the required square root of the given number.*

Remarks

- Instead of writing the prime factors in pairs, we can write them in the exponential notation. Then for finding the square root of the number, take half of each index value.
- Square root of a fraction = $\frac{\text{square root of its numerator}}{\text{square root of its denominator}}$.

Example 1. Find the square root of the following numbers :

(*i*) 7056 (*ii*) $10\frac{86}{121}$ (*iii*) 42·25

Solution. (*i*) $7056 = 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 7 \times 7$

$= 2^2 \times 2^2 \times 3^2 \times 7^2$

$\therefore \sqrt{7056} = 2 \times 2 \times 3 \times 7$

$= 84.$

2	7056
2	3528
2	1764
2	882
3	441
3	147
7	49
	7

(*ii*) $10\frac{86}{121} = \frac{1296}{121}$

Converting mixed fraction to improper fraction

$= \frac{2\times2\times2\times2\times3\times3\times3\times3}{11\times11}$

$= \frac{2^2\times2^2\times3^2\times3^2}{(11)^2}$

$\therefore \sqrt{10\frac{86}{121}} = \frac{2\times2\times3\times3}{11} = \frac{36}{11} = 3\frac{3}{11}.$

(*iii*) $42{\cdot}25 = \frac{4225}{100}$

Converting decimal to fraction

$= \frac{169}{4}$

Expressing in lowest terms

$= \frac{13\times13}{2\times2} = \frac{13^2}{2^2}$

$\therefore \sqrt{42{\cdot}25} = \frac{13}{2} = \frac{13\times5}{2\times5} = \frac{65}{10} = 6{\cdot}5.$

Example 2. In an auditorium, the number of rows is equal to the number of chairs in each row. If the capacity of the auditorium is 1764, find the number of chairs in each row.

Solution. Let the number of rows be x.

Then the number of chairs in each row = x

$\therefore$ The total number of chairs in the auditorium = $x \times x = x^2$

But the capacity of the auditorium = 1764

$\therefore x^2 = 1764$

$= 2 \times 2 \times 3 \times 3 \times 7 \times 7$

$= 2^2 \times 3^2 \times 7^2$

$\Rightarrow x = 2 \times 3 \times 7 = 42$

Hence, the number of chairs in each row = 42.

2	1764
2	882
3	441
3	147
7	49
	7

Square root of a number by division

Rule to find the square root of a number by division method :

In case of numbers where the factorisation is not easy or of big numbers or of numbers whose square root is not exact, we find the square root of such numbers by **division method**. This method is explained below with the help of a few examples.

Example 3. Find the square root of the following numbers :

(*i*) 55696 (*ii*) 288369

Solution. (*i*) **Steps**

	2 3 6
2	5 56 96
	4
43	1 56
	1 29
466	27 96
	27 96
	0

(Double)

1. Place a bar (or arrow) over every pair of digits from right to left (←) *i.e.* starting from unit's digit. If the number of digits is odd, then the left most digit too will have a bar. Each pair of digits and then remaining one digit (if any) on the extreme left is called **period**.
2. Take the first pair of digits or the single digit as the case may be. In this case, it is the digit 5. Find the greatest number whose square is 5 or less than 5. Such a number is 2. Write 2 on the top in the quotient and also in the divisor. Subtract 2^2 *i.e.* 4 from 5. The remainder is 1.
3. Bring down the pair of digits under the next bar (*i.e.* 56 in this case) to the right of the remainder. So the new dividend is 156.
4. Double the quotient (*i.e.* 2 in this case) to get 4 and enter it with a blank on its right at the place of new divisor.
5. Find the largest possible digit to fill the blank which will also become the new digit in the quotient, such that when the new divisor is multiplied by the new digit in the quotient the product is less than or equal to the dividend.

   In this case $43 \times 3 = 129$, so we choose the new digit as 3. Place 129 under 156. Subtract and get the remainder 27.
6. Bring down the pair of digits under the next bar (*i.e.* 96 in this case) to the right of the remainder. So the new dividend is 2796
7. Double the quotient (*i.e.* 23 in this case) to get 46 and enter it with a blank on its right at the place of new divisor.
8. Find the largest possible digit to fill the blank which will also become the new digit in the quotient, such that when the new divisor is multiplied by this new digit in the quotient the product is less than or equal to the dividend.

   In this case $466 \times 6 = 2796$. So we choose the new digit as 6. Place 2796 under 2796. Subtract and get the remainder 0.

   $\therefore \sqrt{55696} = 236.$

(*ii*) **Steps**

1. Place a bar over every pair of digits from right to left (←).
2. Take the first pair of digits. In this case, it is 28. Find the greatest number whose square is 28 or less than 28. Such a number is 5. Write 5 on the top in the quotient and also in the divisor. Subtract 5^2 *i.e.* 25 from 28. The remainder is 3.

	5 3 7
5	28 83 69
	25
103	3 83
	3 09
1067	74 69
	74 69
	0

3. Bring down the pair of digits under the next bar (*i.e.* 83 in this case) to the right of the remainder. So new dividend is 383.
4. Double the quotient (*i.e.* 5 in this case) to get 10 and enter it with a blank at the place of new divisor.
5. Find the largest possible digit to fill the blank which will also become the new digit in the quotient, such that when the new divisor is multiplied by this new digit in the quotient the product is less than or equal to dividend. In this case $103 \times 3 = 309$, so we choose the new digit as 3. Place 309 under 383 and get the remainder 74.
6. Repeat the process of steps 3, 4 and 5. Remainder is 0.

$\therefore \quad \sqrt{288369} = 537.$

Example 4. Find the square root of the following numbers.

(*i*) 12·0409 (*ii*) 0·00064516

Solution. **(*i*) Steps**

	3·47
3	12·04 09
	9
64	3 04
	2 56
687	48 09
	48 09
	0

1. Starting from decimal point, put arrows (or bars) on the pairs of integers from right to left (←) as usual and on the decimal part from left to right (→).
2. Take the first pair of digits. In this case, it is 12. Find the greatest number whose square is 12 or less than 12. Such a number is 3. Write 3 on the top in the quotient and also in the divisor.

   Subtract 3^2 *i.e.* 9 from 12. The remainder is 3.
3. Since the next pair of digits is after decimal point, therefore, write the decimal point in the quotient.
4. Bring down the pair of digits below the next bar (*i.e.* 04, in this case) to the right of the remainder. So the new dividend is 304.
5. Double the quotient (*i.e.* 3) to get 6 and enter it with a blank on its right at the place of new divisor.
6. Find the largest possible digit to fill the blank which will also become the new digit in the quotient, such that when the new divisor is multiplied by the new digit in the quotient the product is less than or equal to dividend. In this case, $64 \times 4 = 256$, so we choose the new digit as 4. Place 256 under 304. Subtract and get the remainder 48.

   Repeat the process of steps 4, 5 and 6. Remainder is 0.

$\therefore \quad \sqrt{12.0409} = 3.47$

***(ii)* Steps**

1. Integral part is zero, so start from decimal point and put bars (or arrows) on the pair of integers from left to right (→).
2. Since the first pairs of digits is after the decimal point, therefore, write the decimal point in the quotient.
3. As the first pair of digits after decimal point consists of both zero. Write 0 in the quotient to the right of the decimal point.
4. As the next pair of digits is 06. Find the greatest number whose square is 6 or less than 6. Such a number is 2. Write 2 in the quotient to the right of 0 and also write 2 in the divisor. Subtract 2^2 *i.e.* 4 from 6. The remainder is 2. Bring down the pair of digits below the next bar and proceed as in part (*i*).

	·0254
2	$0\cdot\overline{00}\ \overline{06}\ \overline{45}\ \overline{16}$
	4
45	245
	225
504	20 16
	20 16
	0

$\therefore \quad \sqrt{0\cdot00064516} = 0\cdot0254.$

Note. When the square root of a number is not exact, to find its approximate value correct to a certain place of decimal, we obtain the square root of the given number to one more place and then round off to the desired place.

Example 5. Find the square root of 9·81 correct to 2 decimal places.

	3·132
3	9·81 00 00
	9
61	81
	61
623	20 00
	18 69
6262	1 31 00
	1 25 24
	5 76

Solution. Since the addition of zero (zeros) at the end of a decimal number does not change its value, therefore, write 9·81 as 9·810000.

Now proceed as in example 4 part (*i*) to find the square root.

$\sqrt{9\cdot81} = 3\cdot132\ldots$

$= 3\cdot13$ correct to 2 decimal places.

Example 6. Find the square root of 5 correct to 2 decimal places. Hence find the value of $\sqrt{\dfrac{3+\sqrt{5}}{3-\sqrt{5}}}$ correct to three significant figures.

Solution. $\sqrt{5} = 2\cdot236\ldots$

$= 2\cdot24$ correct to 2 decimal places.

	2·236
2	5·00 00 00
	4
42	1 00
	84
443	16 00
	13 29
4466	271 00
	267 96
	304

$$\sqrt{\frac{3+\sqrt{5}}{3-\sqrt{5}}} = \sqrt{\frac{3+\sqrt{5}}{3-\sqrt{5}}\times\frac{3+\sqrt{5}}{3+\sqrt{5}}}$$

$$= \sqrt{\frac{(3+\sqrt{5})^2}{3^2-(\sqrt{5})^2}}$$

$$= \sqrt{\frac{(3+(\sqrt{5})^2}{9-5}} = \sqrt{\frac{(3+\sqrt{5})^2}{4}}$$

$$= \frac{3+\sqrt{5}}{2} = \frac{3+2\cdot236\ldots}{2} = \frac{5\cdot236}{2}$$

$= 2\cdot618\ldots = 2\cdot62$ correct to three significant figures.

Perfect square number

A number whose square root is exact is called a **perfect square number**.

For example :

(*i*) As $\sqrt{25} = 5$, 25 is a perfect square.

(*ii*) As $\sqrt{\frac{4}{9}} = \frac{2}{3}$, $\frac{4}{9}$ is a perfect square.

(*iii*) As $\sqrt{0{\cdot}16} = 0{\cdot}4$, 0·16 is a perfect square.

(*iv*) As $\sqrt{5} = 2{\cdot}236...$ is not exact, 5 is not a perfect square.

Example 7. Find the smallest natural number by which 980 should be multiplied to make it a perfect square.

Solution.

$$980 = 2 \times 2 \times 5 \times 7 \times 7$$
$$= 2^2 \times 7^2 \times 5$$

2	980
2	490
5	245
7	49
	7

Since the factor 5 does not have its pair, therefore, the smallest number by which the given number must be multiplied so that the product is a perfect square is 5.

Example 8. Find the smallest natural number by which 12168 should be divided to make it a perfect square.

Solution.

$$12168 = 2 \times 2 \times 2 \times 3 \times 3 \times 13 \times 13$$
$$= 2^2 \times 3^2 \times (13)^2 \times 2$$

2	12168
2	6084
2	3042
3	1521
3	507
13	169
	13

Since the factor 2 does not have its pair, therefore, the smallest number by which the given number must be divided so that the quotient is a perfect square is 2.

Example 9. Find the least number which should be subtracted from 984 to make it a perfect square.

Solution. We find the square root of 984.

	31
3	9 84
	9
61	84
	61
	23

We find that the remainder is 23. This means that if 23 is subtracted from 984, then the remainder will be zero and the new number will be a perfect square.

In fact, $984 - 23 = 961$ and $961 = (31)^2$.

Hence, the required least number = 23.

Example 10. What least number should be added to 6598 to make it a perfect square?

Solution. We find the square root of 6598.

	81
8	$\overline{65}$ $\overline{98}$
	64
161	1 98
	1 61
	37

So, we find that the given number 6598 is greater than $(81)^2$.

The next perfect square number is $(82)^2$.

Thus, $(81)^2 < 6598 < (82)^2$ *i.e.* $6561 < 6598 < 6724$.

Therefore, the least number that should be added to the given number to make it a perfect square

$$= (82)^2 - 6598 = 6724 - 6598$$
$$= 124.$$

Example 11. Find the greatest number of five digits which is a perfect square.

Solution. The greatest number of 5 digits is 99999

We find the square root of 99999

$\therefore$ The required number = 99999 – 143

= 99856.

	316
3	$\overline{9}\ \overline{99}\ \overline{99}$
	9
61	99
	61
626	38 99
	37 56
	1 43

CUBES AND CUBE ROOTS

Cube of a number

When a number is multiplied by itself three times, the product so obtained is called the ***cube*** *of that number.*

For example :

(*i*) $5 \times 5 \times 5 = 125$ *i.e.* $5^3 = 125$, so 125 is the cube of 5.

(*ii*) $\frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{8}{27}$ *i.e.* $\left(\frac{2}{3}\right)^3$, so $\frac{8}{27}$ is the cube of $\frac{2}{3}$.

(*iii*) $0{\cdot}7 \times 0{\cdot}7 \times 0{\cdot}7 = 0{\cdot}343$ *i.e.* $(0{\cdot}7)^3 = 0{\cdot}343$, so 0·343 is the cube of 0·7.

If a is any number, then a^3 is called a ***cube number*** *or a* ***perfect cube number****.*

Note. In this section, we shall consider only positive numbers.

Cube root of a number

The ***cube root*** *of a number is that number which when multiplied by itself three times gives the original number.*

For example :

(*i*) As $5 \times 5 \times 5 = 125$, cube root of 125 is 5.

(*ii*) As $\frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{8}{27}$, cube root of $\frac{8}{27}$ is $\frac{2}{3}$.

(*iii*) As $0{\cdot}7 \times 0{\cdot}7 \times 0{\cdot}7 = 0{\cdot}343$, cube root of 0·343 is 0·7.

The cube root of a is denoted by $\sqrt[3]{a}$.

Thus, $\sqrt[3]{125} = 5$, $\sqrt[3]{\frac{8}{27}} = \frac{2}{3}$, $\sqrt[3]{0{\cdot}343} = 0{\cdot}7$.

Cube root of a number by prime factorisation

Method to find the cube root of a number :

(*i*) *Express the given number as the product of primes.*

(*ii*) *Make groups in triplets of the same prime.*

(*iii*) *Take one factor from each triplets of primes. Multiply them together; the product so obtained is the required cube root of the given number.*

Remarks

- Instead of writing the prime factors in triplets, we can write them in the exponential notation. Then for finding the cube root of the number, take one-third of each index value.
- Cube root of a fraction = $\dfrac{\text{cube root of its numerator}}{\text{cube root of its denominator}}$.
- Here, we shall find the cube roots of only perfect cube numbers.

Example 12. Find the cube root of the following numbers :

(*i*) 1728 (*ii*) $4\frac{17}{27}$ (*iii*) 0·216 (*iv*) 9·261

2	1728
2	864
2	432
2	216
2	108
2	54
3	27
3	9
	3

Solution. (*i*)

$$1728 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 3$$
$$= (2 \times 2 \times 2) \times (2 \times 2 \times 2) \times (3 \times 3 \times 3)$$
$$\therefore \sqrt[3]{1728} = 2 \times 2 \times 3$$
$$= 12.$$

Alternative method

$$1728 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 3$$
$$= 2^6 \times 3^3$$

Take one-third of each index

$$\therefore \sqrt[3]{1728} = 2^2 \times 3^1$$
$$= 2 \times 2 \times 3 = 12.$$

(*ii*)
$$\sqrt[3]{4\frac{17}{27}} = \sqrt[3]{\frac{125}{27}} = \sqrt[3]{\frac{5\times5\times5}{3\times3\times3}}$$
$$= \frac{\sqrt[3]{5\times5\times5}}{\sqrt[3]{3\times3\times3}} = \frac{5}{3} = 1\frac{2}{3}.$$

(*iii*)
$$\sqrt[3]{0\cdot216} = \frac{\sqrt[3]{216}}{\sqrt[3]{1000}}$$

Converting decimal to fraction

$$= \sqrt[3]{\frac{27}{125}}$$

Expressing in lowest terms

$$= \frac{\sqrt[3]{27}}{\sqrt[3]{125}} = \frac{\sqrt[3]{3\times3\times3}}{\sqrt[3]{5\times5\times5}} = \frac{3}{5} = 0\cdot6.$$

(*iv*)
$$\sqrt[3]{9\cdot261} = \sqrt[3]{\frac{9261}{1000}}$$

Converting decimal to fraction

$$= \frac{\sqrt[3]{9261}}{\sqrt[3]{1000}} = \frac{\sqrt[3]{3\times3\times3\times7\times7\times7}}{\sqrt[3]{2\times2\times2\times5\times5\times5}}$$
$$= \frac{\sqrt[3]{(3\times3\times3)\times(7\times7\times7)}}{\sqrt[3]{(2\times2\times2)\times(5\times5\times5)}} = \frac{3\times7}{2\times5} = \frac{21}{10} = 2\cdot1.$$

Example 13. By what smallest number should 18252 be multiplied so that the product becomes a perfect cube? Also, find the cube root of the product.

2	18252
2	9126
3	4563
3	1521
3	507
13	169
	13

Solution. Splitting 18252 into prime factors, we have

$$18252 = 2 \times 2 \times 3 \times 3 \times 3 \times 13 \times 13$$
$$= (2 \times 2) \times (3 \times 3 \times 3) \times (13 \times 13)$$

We note that the prime factor 3 occurs thrice while prime factors 2 and 13 occur twice. Therefore, the smallest number by which the given number must be multiplied so that the product is a perfect cube is 2×13 *i.e.* 26.

Then product

$$= (2 \times 2 \times 2) \times (3 \times 3 \times 3) \times (13 \times 13 \times 13).$$

$\therefore$ The cube root of the product $= 2 \times 3 \times 13 = 78$.

Example 14. Divide 259875 by the smallest number so that the quotient is a perfect cube. Also, find the cube root of the quotient.

Solution. Splitting 259875 into prime factors, we have

$$259875 = 5 \times 5 \times 5 \times 3 \times 3 \times 3 \times 7 \times 11$$
$$= (5 \times 5 \times 5) \times (3 \times 3 \times 3) \times 7 \times 11$$

5	259875
5	51975
5	10395
3	2079
3	693
3	231
7	77
	11

We note that the prime factors 5 and 3 occur thrice while prime factors 7 and 11 occur once. Therefore, the smallest number by which the given number must be divided so that the quotient is a perfect cube is 7×11 *i.e.* 77.

Then quotient $= \dfrac{259875}{77} = 3375$.

Now $3375 = (5 \times 5 \times 5) \times (3 \times 3 \times 3)$

$\therefore$ $\sqrt[3]{3375} = 5 \times 3 = 15$.

Exercise 3.6

1. Find the square root of the following numbers by (prime) factors :
   (*i*) 676 (*ii*) 1024 (*iii*) 7396 (*iv*) $9\frac{67}{121}$
   (*v*) $17\frac{13}{36}$ (*vi*) 1·96 (*vii*) 0·0064 (*viii*) 0·4225
2. Find the value of the following :
   (*i*) $\sqrt{85-\sqrt{16}}$ (*ii*) $\sqrt{2\frac{14}{121}}-\sqrt{1\frac{21}{100}}$ (*iii*) $\sqrt{(1\cdot3)^2-(1\cdot2)^2}$
3. There are 3249 trees in a garden. The number of trees in each row is equal to the number of rows. Find the number of trees in each row.
4. Find the square root of the following numbers by division method :
   (*i*) 53824 (*ii*) 213444 (*iii*) 4214809
5. Find the square root of the following numbers by division method:
   (*i*) 18·4041 (*ii*) 5·774409 (*iii*) 935·1364
6. Find the square root of 2 correct to
   (*i*) two decimal places (*ii*) two significant figures.
7. Find $\sqrt{27}$ correct to two decimal places.
8. Find the square root of the following correct to two decimal places :
   (*i*) 645·8 (*ii*) 107·45 (*iii*) 5·462
9. Find the square root of 253·6 correct to three significant figures.
10. Find the square root of $\sqrt{3}$ correct to three significant figures. Hence find the value of $\sqrt{\dfrac{2+\sqrt{3}}{2-\sqrt{3}}}$ correct to two decimal places.

11. Find the smallest natural number by which the following numbers must be multiplied to make them a perfect square :
    (*i*) 588 (*ii*) 1682 (*iii*) 2178
12. Find the smallest natural number by which the following numbers must be divided to make them a perfect square :
    (*i*) 588 (*ii*) 2592 (*iii*) 16224
13. What least number must be subtracted from the following numbers to make them a perfect square?
    (*i*) 691 (*ii*) 8934 (*iii*) 11021
14. What least number must be added to the following numbers to make them perfect squares?
    (*i*) 3247 (*ii*) 5678 (*iii*) 8000
15. Find the greatest number of six digits which is a perfect square.
16. Find the cube root of each of the following numbers :
    (*i*) 13824 (*ii*) 250047 (*iii*) $\frac{64}{1331}$
    (*iv*) $5\frac{1182}{2197}$ (*v*) 0·003375 (*vi*) 19·683
17. Multiply 6561 by the smallest number so that the product is a perfect cube. Also find the cube root of the product.
18. Divide the number 8748 by the smallest number so that the quotient is a perfect cube. Also find the cube root of the quotient.

Summary

- All natural numbers 1, 2, 3,... together with 0 are called whole numbers. 0 is the smallest whole number and there is no largest whole number.
- *Division algorithm*. If a, $b \neq 0$ are whole numbers, then there exist unique whole numbers q and r such that

$$a = b \times q + r \text{ where } 0 \leq r < b.$$

- **Z** (set of integers) = {..., –3, –2, –1, 0, 1, 2, 3,...}.
- Every integer has a successor as well as a predecessor.
- Of the two given integers, the one which lies to the right on the number line is greater than the other.
- *Factors and multiples* — a (natural) number which divides a given (natural) number exactly is called a factor (or divisor) of that given number, and the given number is called a multiple of that number.
- **Types of whole numbers :**
  - *Even number* — a whole number which is divisible by 2.
  - *Odd number* — a whole number which is not divisible by 2.
  - *Prime number* — a natural number greater than 1 and having no factor other than 1 and itself.
  - *Composite number* — a natural number having atleast one more factor other than 1 and itself.
  - *Co-prime numbers* — two natural numbers having no common factor other than 1.

- Least Common Multiple (L.C.M.) of two natural numbers is the smallest natural number which is multiple of both the numbers.
- Highest Common Factor (H.C.F. or G.C.D.) of two natural numbers is the largest common factor (or divisor) of the given numbers.
- H.C.F. × L.C.M. of two natural numbers = product of the two numbers.
- H.C.F. of two natural numbers always divides their L.C.M.
- Fraction is a number which represents a part of whole; $\frac{2}{7}, \frac{13}{34}, \frac{31}{5}, -\frac{3}{8}$ are all fractions.
- **Types of fractions**
  - *Proper fraction* — its value lies between 0 and 1.
  - *Improper fraction* — its value is greater than or equal to 1.
  - *Mixed fraction* — it consists of two parts, a natural number and a proper fraction.
  - *Negative fractions* — $-\frac{5}{7}, -\frac{35}{6}, -7\frac{2}{9}$ etc. are negative fractions.
  - *Proper negative fraction* — its value lies between –1 and 0.
  - *Improper negative fraction* — its value is less than or equal to –1.
  - *Simple fraction* — whose numerator and denominator both are whole numbers.
  - *Complex fraction* — whose numerator or denominator or both are fractions.
  - *Decimal fraction* — whose denominator is of the form 10, 100, 1000,...
  - *Vulgar fraction* — whose denominator is not of the form 10, 100, 1000,...
  - *Like fractions* — have same denominator.
  - *Unlike fractions* — have different denominators.
  - *Equivalent fractions* — have equal value.
- A fraction is in simplest form if its numerator and denominator have no common factor (except 1).
- If $\frac{a}{b}$ and $\frac{c}{d}$ are any two different fractions, then the fraction $\frac{a+c}{b+d}$ lies between them *i.e.* if $\frac{a}{b} < \frac{c}{d}$ then $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.
- The place values of the digits that are on the left of decimal point in order are, 1, 10, 100, 1000,... and so on; and the place values of the digits on the right of decimal point are $\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}$,.... and so on.
- **Calculation rules**

  To simplify an expression involving fractions or decimals (or both), use rule of **BODMAS**. It helps us in remembering the order in which the several operations must be done.
- Recurring decimals are non-terminating decimal numbers in which a certain digit or a certain block of digits repeat over and over again.
- Every fraction can be written either as a terminating decimal or as a recurring decimal and conversely every terminating decimal or a recurring decimal can be written as a fraction.
- Any number that can be expressed as $\frac{m}{n}$, where m and n are integers and $n \neq 0$ is called a **rational number**. Every terminating decimal or a recurring decimal is a rational number. The set of rational numbers is denoted by **Q**.
- If a and b are any two different rational numbers, then $\frac{a+b}{2}$ is a rational number and it lies between them *i.e.* if $a < b$ then $a < \frac{a+b}{2} < b$.

- The numbers such as $\sqrt{2}$, $\sqrt{3}$, $5\sqrt{7}$, $-\sqrt{5}$, $\frac{2}{\sqrt{7}}$, $3 + \sqrt{11}$, $2 - \sqrt{7}$, $\sqrt{2} + \sqrt{13}$, $\sqrt[3]{2}$ etc. are **irrational numbers**. All non-terminating and non-recurring decimal numbers are irrational numbers.
- All rational numbers and all irrational numbers are **real numbers**. All decimal numbers (terminating, recurring, or non-terminating and non-recurring) are real numbers. The set of real numbers is denoted by **R**.
  Note that $\mathbf{N} \subset \mathbf{W} \subset \mathbf{Z} \subset \mathbf{Q} \subset \mathbf{R}$.
- Corresponding to every real number (rational or irrational) there exists one and only one point on a line (say l), and conversely, corresponding to every point on the line l there exists one and only one real number. The line l is called the **number line**.
- If the product of two irrational numbers is a rational number then each number is called the *rationalising factor* of the other number. The process of multiplying an irrational number by its rationalising factor is called **rationalisation**.
- When a number is multiplied by itself, the product so obtained is called the **square** of that number.
- The **square root** of a number is that number which when multiplied by itself gives the original number.
- A number whose square root is exact is called a **perfect square**.
- If a is any number, a^3 is a (perfect) cube number.
- The cube root of a number is that number which when multiplied by itself three times gives the original number.

Check Your Progress

1. Write all possible whole numbers that can be formed by using the digits 5, 0, 9; using each digit atmost once.
2. Find the smallest six digit number which is exactly divisible by 239.
3. Simplify the following :
   (*i*) (-5) of $(-7) - 8 \div (-4) + (41 - \overline{5-7})$
   (*ii*) $9 - [5 - 2 + \{(-7) \text{ of } (-2) - (6 \div 3 \times 2 - \overline{1-3})\}]$
4. Write the prime factorisation of the smallest six-digit number.
5. Find the least number which on dividing by each of 24, 36, 108 and 192 leaves the remainder 7 in each case.
6. Find the smallest number of five digits which is exactly divisible by each of 8, 24, 72, 126 and 168.
7. Find the greatest number which divides 2706, 7041 and 8263 leaving the remainder 6, 21 and 55 respectively.
8. Two natural numbers are co-prime numbers and their L.C.M. is 6300. If one of the numbers is 75, find the other number.
9. The L.C.M. of two numbers is 12 times their H.C.F. The sum of the H.C.F. and L.C.M. is 403. If one of the numbers is 93, find the other number.
   [Hint. Let H.C.F. of the two numbers be x, then their L.C.M. = $12x$. According to given information $x + 12x = 403 \Rightarrow 13x = 403 \Rightarrow x = 31$.**]**
10. Can you find two natural numbers with their H.C.F. 12 and L.C.M. 54? Justify your answer.
    [Hint. L.C.M. of two natural numbers must be divisible by their H.C.F.**]**

11. Arrange the given fractions in descending order :

(*i*) $\frac{3}{4}, \frac{7}{12}, \frac{11}{18}$ (*ii*) $\frac{5}{9}, \frac{11}{12}, \frac{7}{10}, \frac{3}{4}$

12. Insert three fractions between the fractions $\frac{3}{4}$ and $\frac{5}{12}$.

13. Simplify : $1\frac{1}{3} \div \left[\frac{3}{4} \div \left(2\frac{1}{4} - 1\frac{2}{5}\right)\right]$.

14. A wedding cake weighed 8 kg. If $\frac{2}{5}$ th of its weight was flour, $\frac{5}{16}$ th was sugar, $\frac{1}{4}$ th was cream and the rest were nuts, find the weight of nuts.

15. The highest score in a cricket test match in an innings was $\frac{5}{18}$ of the total, and the next highest was $\frac{4}{13}$ of the remaining. If the difference between two scores was 20 runs, what was the total score?

16. Write the following as decimal numbers :

(*i*) $2 \times 10^3 + 5 \times 10 + 9 \times 1 + \frac{3}{10} + \frac{7}{1000}$ (*ii*) $37 + \frac{7}{10^3} + \frac{9}{10^5}$

(*iii*) $3\frac{17}{40}$ (*iv*) $\frac{13}{625}$ (*v*) $\frac{1}{37}$

17. Simplify the following :

(*i*) $\frac{0{\cdot}035 \times 2{\cdot}6}{1{\cdot}4 \times 1{\cdot}3}$ (*ii*) $\frac{(-0{\cdot}3)^3 \times 4{\cdot}5}{2{\cdot}7 \times 1{\cdot}5}$

18. Insert two rational numbers between $\frac{1}{3}$ and $\frac{1}{4}$, and arrange in ascending order.

19. Rationalise the denominator of the following :

(*i*) $\frac{33}{2\sqrt{3}-1}$ (*ii*) $\frac{10}{2\sqrt{2}+\sqrt{3}}$

20. Insert three irrational numbers between 3 and 4.

21. Insert two irrational number between $\sqrt{20}$ and $\sqrt{24}$.

22. Find the square root of the following numbers by (prime) factors :

(*i*) 9604 (*ii*) $26\frac{22}{49}$ (*iii*) 0·1681

23. Find the square root of the following numbers by division method :

(*i*) 182329 (*ii*) 0·00002601 (*iii*) 0·018769

24. Find the square root of 5 correct to two decimal places. Hence find the value of $\frac{4}{7-\sqrt{5}}$ by rationalising its denominator.

25. Find the smallest natural number by which 3150 should be divided so that the quotient becomes a perfect square.

26. The soldiers of an army were to stand in a solid square. In doing so, 36 soldiers were left. If there are 8500 soldiers in the army, find the number of soldiers in the front row of the solid square.

27. Find the least natural number of four digits which is a perfect square.

[Hint. Least number of four digits is 1000, so find the least number which must be added to 1000 to make it a perfect square.**]**

28. Find the cube root of each of the following numbers :

(*i*) 592704 (*ii*) $1\frac{6364}{9261}$ (*iii*) 17·576

29. Divide 137592 by the smallest number so that the quotient is a perfect cube. Also find the cube root of the quotient.