Chapter 18

LOCI

POINTS TO REMEMBER

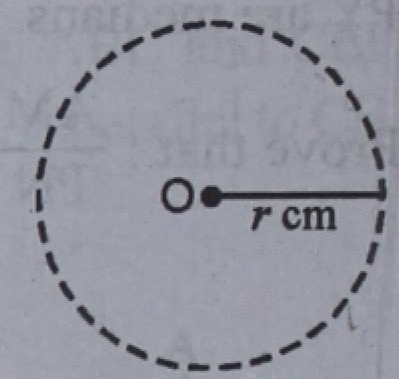
1. **LOCUS** *is the path traced out by a moving point which moves according to some given geometrical conditions.*

   Thus, (*i*) Every point which satisfies the given geometrical conditions will lie on the locus.

   And, (*ii*) Every point lying on the locus will satisfy the given geometrical conditions.

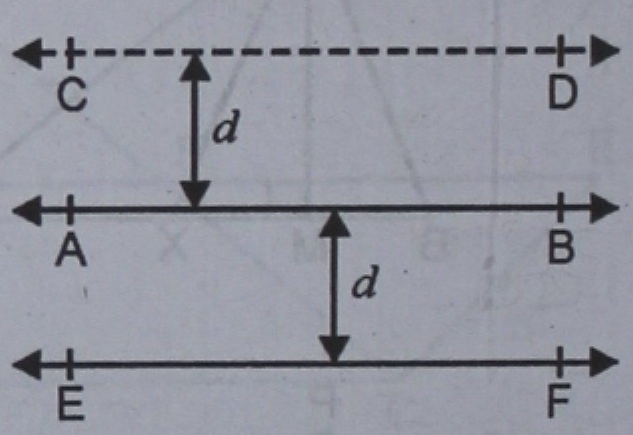
   Remark : *The plural of locus is loci, read as 'losai'.*

   Example 1. A circle with centre O and radius r cm is the locus of a point which moves in a plane in such a way that its distance from the fixed point O is always equal to r cm.



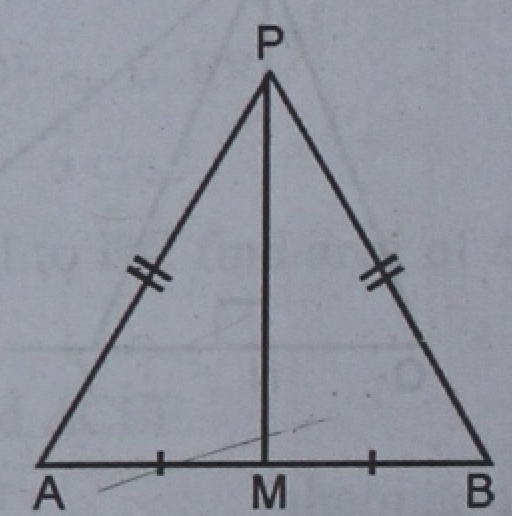
   Example 2. Let a point P move in such a way that its distance from a fixed line AB is always equal to d cm.

   Clearly, the locus of the moving point is a pair of straight lines CD and EF, each parallel to AB at a distance of d cm from it.



2. **Points Equidistant From Two Given Points**

   Theorem 1. *The locus of a point which is equidistant from two given fixed points, is the perpendicular bisector of the line segment joining the given fixed points.*



   Proof : We shall prove the theorem in two parts (*a*) and (*b*) given below.

   Part (*a*) : Every point which is equidistant from two fixed points A and B, lies on the perpendicular bisector of AB.

   Given : Two fixed points A and B and P is a point, such that PA = PB.

   To Prove : P lies on the perpendicular bisector of AB.

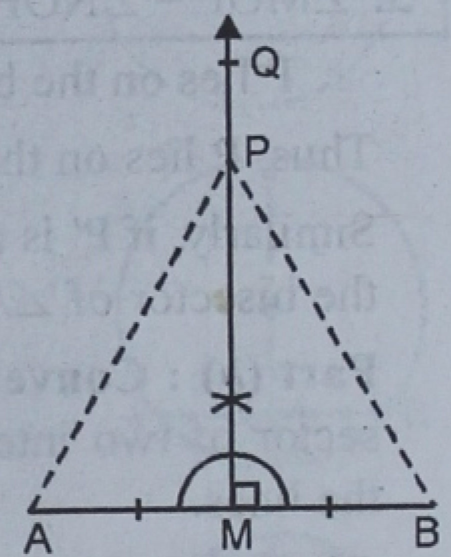
   Construction : Join AB. Find its middle point M and Join MP.

Proof.

Statement	Reason
1. In ΔPMA and ΔPMB,	
PA = PB	Given.
MA = MB	By construction, M is the mid-point of AB.
PM = PM	Common.
∴ ΔPMA ≅ ΔPMB	SSS–congruency axiom.
⇒ ∠AMP = ∠BMP ...I	c.p.c.t.
2. ∠AMP + ∠BMP = 180° ...II	AMB is a straight line.
3. ∠AMP = ∠BMP = 90°	From I and II.
4. PM is the right bisector of AB	

Hence, P lies on the perpendicular bisector of AB.

Part (*b*) : Conversely, every point on the perpendicular bisector of AB is equidistant from A and B.



Given : Two fixed points A and B, MQ is the perpendicular bisector of AB and P is any point on MQ.

To Prove : PA = PB.

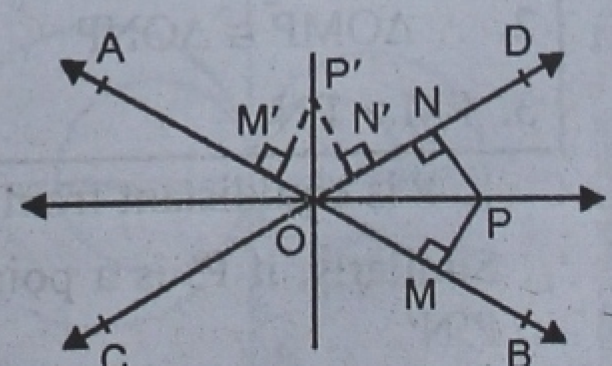
Construction : Join PA and PB.

Proof.

Statement	Reason
1. In ΔPMA and ΔPMB,	
MA = MB	M is the mid-point of AB (given).
PM = PM	Common.
∠PMA = ∠PMB	Each equal to 90° (given).
∴ ΔPMA ≅ ΔPMB	R.H.S. congruency axiom.
2. So, PA = PB	c.p.c.t.
Hence, PA = PB.	

Hence, the locus of a point which is equidistant from two fixed points A and B, is the perpendicular bisector of AB.

3. Point Equidistant From Two Intersecting Lines



Theorem 2. *The locus of a point which is equidistant from two intersecting lines is the pair of lines bisecting the angles formed by the given lines.*

Proof : We shall prove the theorem in two parts ; Part (*a*) and Part (*b*).

Part (*a*) : Every point which is equidistant from two intersecting lines, lies on the bisector of the angle between the given lines.

Given : Two straight lines, AB and CD, intersecting at a point O and P is a point in the interior of ∠BOD such that PM ⊥ AB, PN ⊥ OD and PM = PN.

To Prove : P lies on the bisector of ∠BOD.

Construction : Join OP.

Proof :

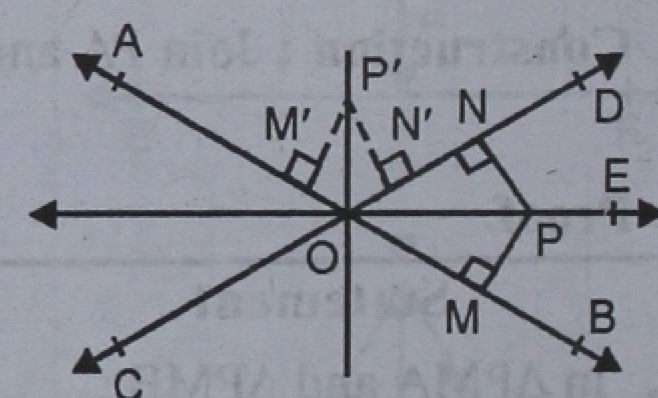
Statement	Reason
1. In ΔOMP and ΔONP,	
PM = PN	Given
∠OMP = ∠ONP	Each equal to 90° (Given)
OP = OP	Common.
2. ΔOMP ≅ ΔONP	RHS–axiom of congruency
3. ∠MOP = ∠NOP	c.p.c.t.

∴ P lies on the bisector of ∠BOD

Thus, P lies on the bisector of ∠BOD

Similarly, if P′ is a point such that P′M′ ⊥ OA and P′N′ ⊥ OD and P′M′ = P′N′, then P′ lies on the bisector of ∠AOD.

Part (*b*) : Conversely, every point on the angle bisector of two intersecting lines, is equidistant from the lines.



Given : Two lines AB and CD intersecting at a Point O ; OE is the bisector of ∠BOD and P is a point on OE ; PM ⊥ OB and PN ⊥ OD.

To Prove : PM = PN

Proof.

Statement	Reason
1. In ΔOMP and ΔONP, we have	
∠MOP = ∠NOP	Given, as OE is the bisector of ∠BOD.
∠OMP = ∠ONP	Each equal to 90°.
OP = OP	Common.
2. ∴ ΔOMP ≅ ΔONP	ASA–axiom of congruency.
3. PM = PN	c.p.c.t.

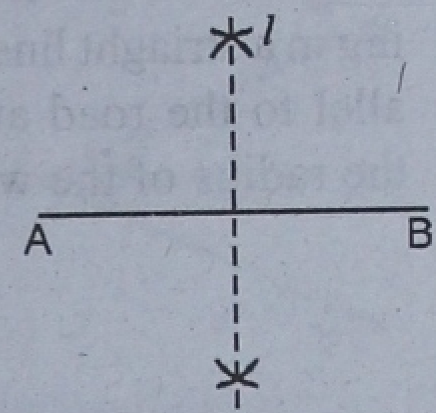
∴ P is equidistant from OB and OD, and therefore from AB and CD.

Similarly, if P′ is a point on the bisector of ∠AOD and P′M′ ⊥OA, P′N′ ⊥ OD, then P′M′ = P′N′.

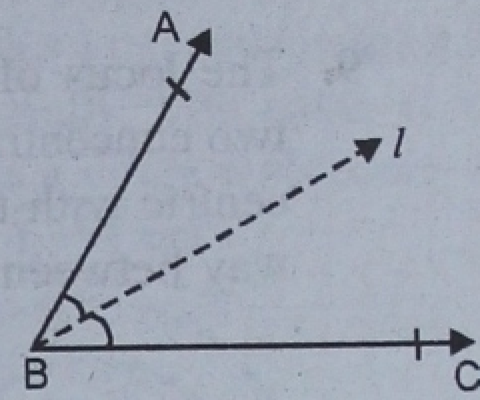
4. **Some Examples of Loci**

In each figure, we shall show the locus by dotted lines.

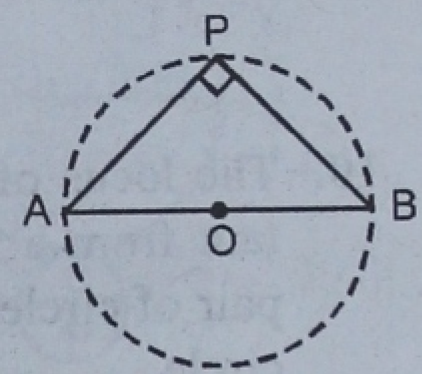
1. The locus of a point which is equidistant from two given points A and B is the perpendicular bisector of AB.



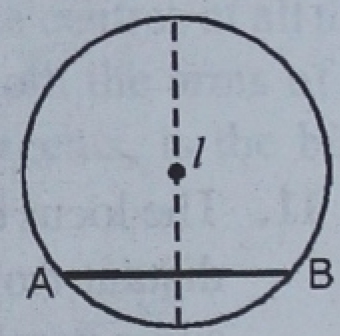
2. The locus of a point equidistant from two intersecting lines AB and BC is the bisector of ∠ABC.



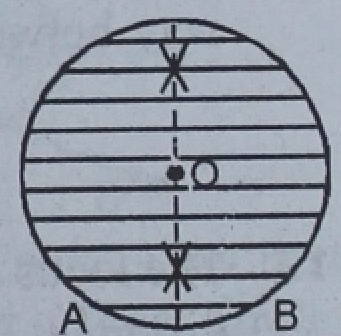
3. If A and B are two fixed points, then the locus of a point P such that ∠APB = 90° is the circle with AB as diameter.
(∵ Angle in a semi-circle is always a right angle)



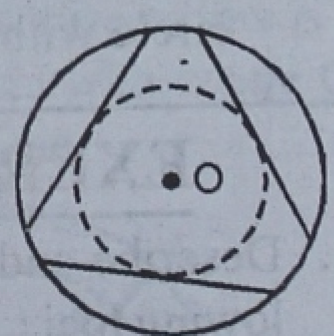
4. The locus of a point P inside a circle such that PA = PB, where A and B are two fixed points on the circle is a diameter of the circle bisecting the chord AB.



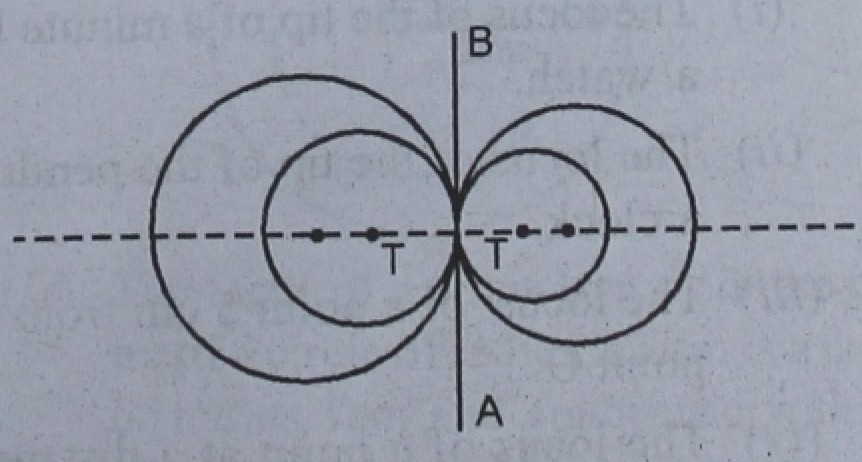
5. The locus of mid-points of all chords parallel to a chord AB of a circle is the diameter of the circle which is the right bisector of AB.



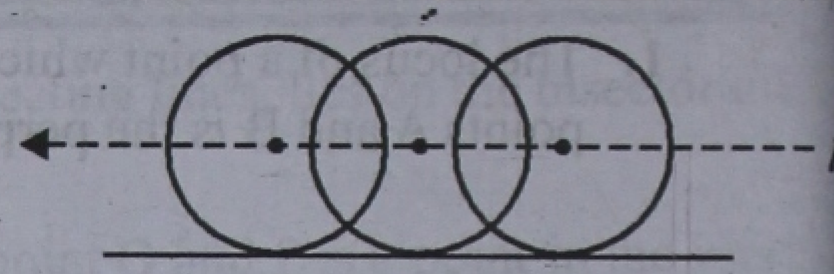
6. The locus of mid-points of all equal chords of a circle concentric with the given circle with radius equal to distance of each chord from the centre.



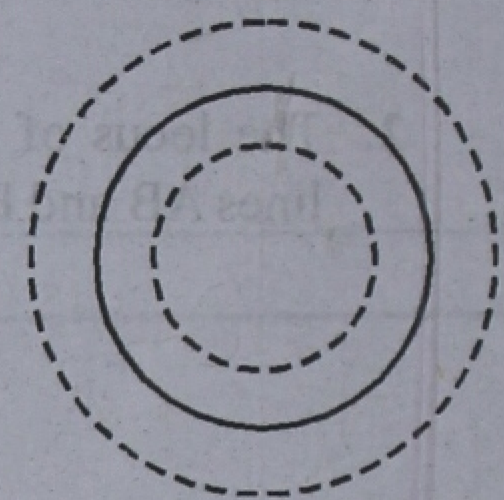
7. The locus of centres of circles touching a given line AB at a given point T on it is the straight line perpendicular to AB at T.



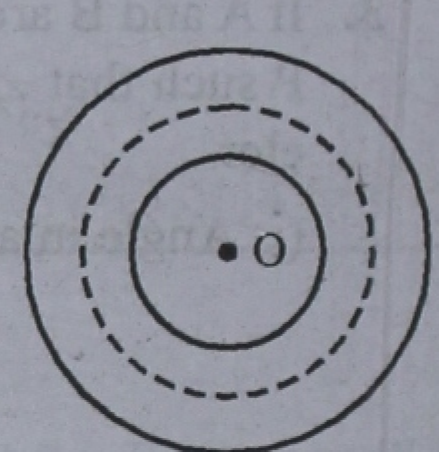
8. The locus of the centre of a wheel moving in a striaght line is a straight line parallel to the road at a distance equal to the radius of the wheel.



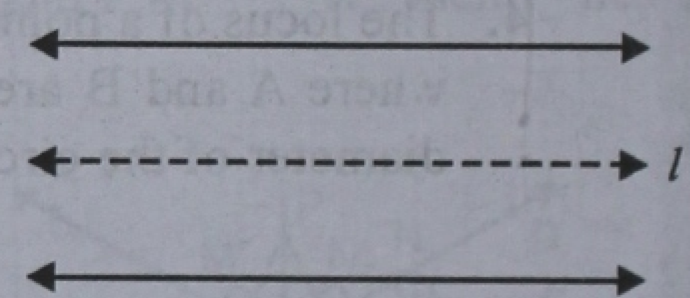
9. The locus of a point equidistant from two concentric circles is the circle concentric with the given circles and midway between them.



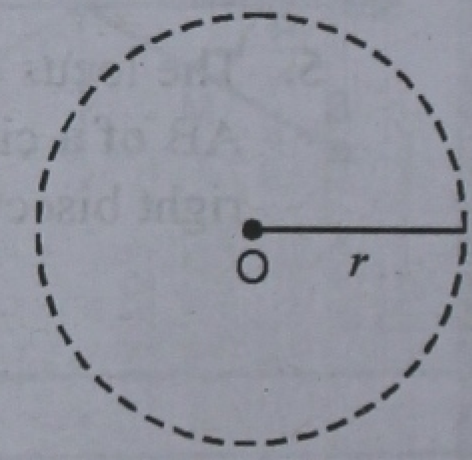
10. The locus of a point which is equidistant from a given circle consists of a pair of circle concentric with the given circle.



11. The locus of a point which remains equidistant from two given parallel lines, is a line parallel to the given lines and midway between them.



12. The locus of a point which is at a given distance r from a fixed point O, is a circle with centre O and radius r.



EXERCISE 18

Q.1. Describe and construct each of the following loci :

(*i*) The locus of the tip of a minute hand of a watch.

(*ii*) The locus of the tip of the pendulum of a clock.

(*iii*) The locus of a point 5 cm from a fixed point O.

(*iv*) The locus of a point at a distance of 3 cm from a fixed line AB.

(*v*) The locus of a point equidistant from the arms OA and OB of ∠AOB.

(*vi*) The locus of the centres of all circles, each of radius 1 cm and touching externally a fixed circle with centre O and radius 3 cm.

(*vii*) The locus of the centres of all circles to which both the arms of an angle ∠AOB are tangents.

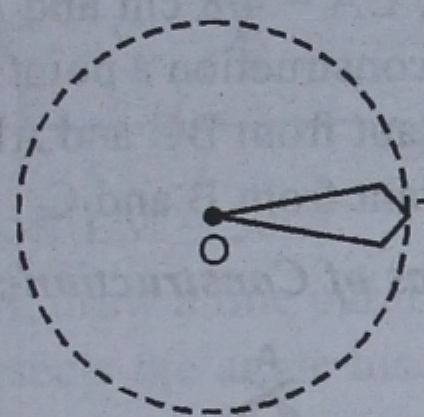
(*viii*) The locus of a point 1 cm from the circumference of a fixed circle towards the centre O, whose radius is 3 cm.

(*ix*) The locus of a point 1 cm from the centre of a circle of radius 2·5 cm.

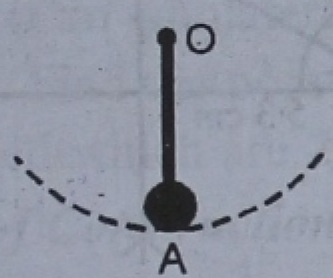
(*x*) The locus of a stone dropped from a tower.

(*xi*) AB is a fixed line. State the locus of a point P such that ∠APB = 90° ?

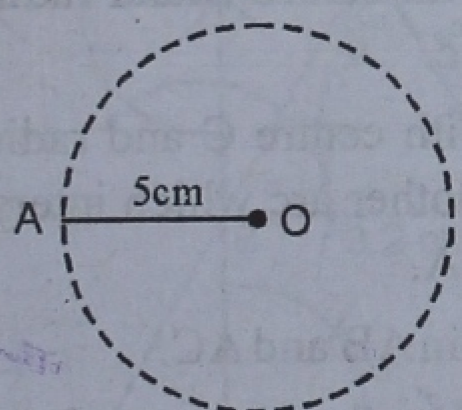
Sol. (*i*) The locus of the tip (T) of a minute hand of a watch is a circle whose radius is the length of the minute hand.



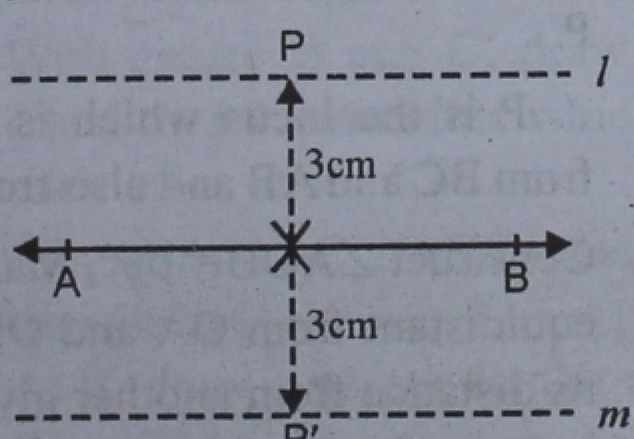
(*ii*) The locus of the tip (A) of the pendulum of a clock is an arc of a circle whose radius is OA.



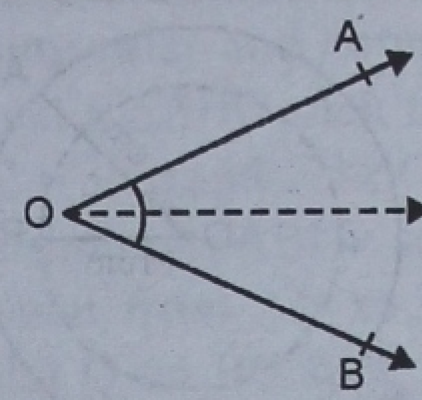
(*iii*) The locus of a point (A) is a circle with centre O and radius OA = 5cm,



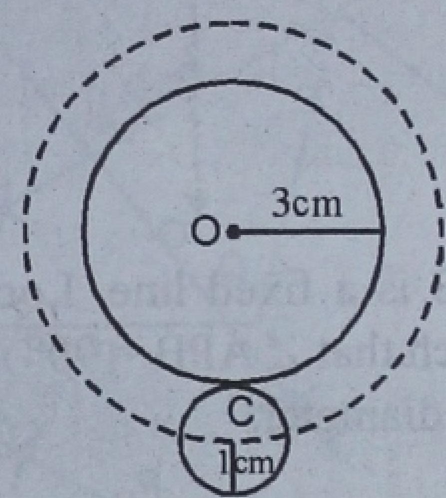
(*iv*) Locus of a point P is a pair of lines parallel to the given line AB at a distance of 3cm.



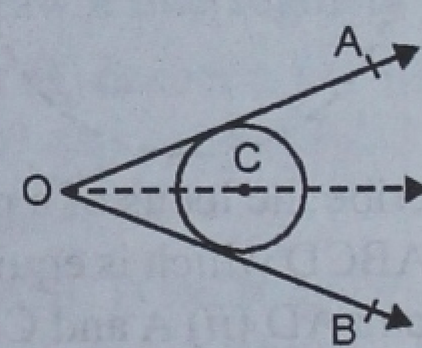
(*v*) Locus of a point P is the bisector of ∠AOB which is equidistant from the arms OA and OB of the angle ∠AOB.



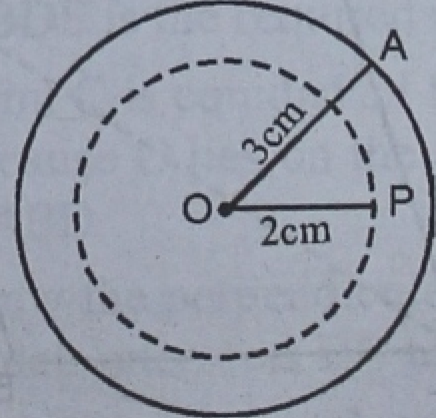
(*vi*) The locus of the centres (C) of all circles each of radius 1 cm and touching a fixed circle externally, with either O and radius 3cm is a circle with centre O and radius 3 + 1 = 4 cm.



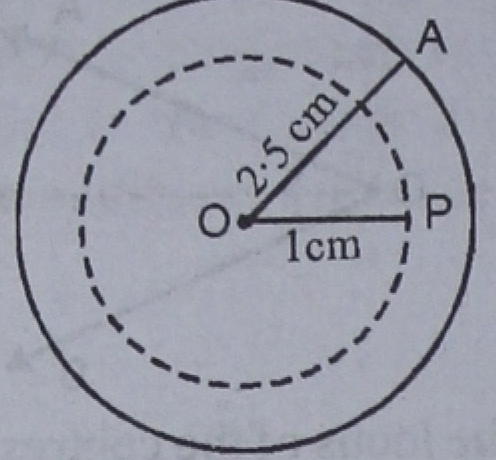
(*vii*) The locus of the centres of all the circles which touch both the arms of an angle ∠AOB are tangents, is the bisector of ∠AOB.



(*viii*) The locus of a point 1 cm from the circumference of a fixed circle towards the centre O, whose radius is 3cm is a concentric circle with centre O and radius 3 – 1 = 2 cm.

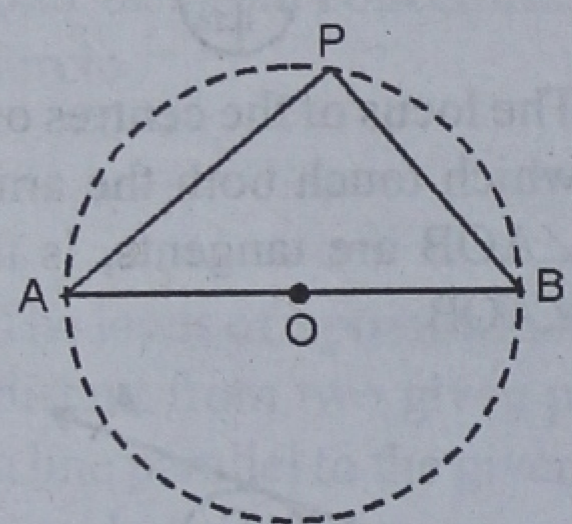


(*ix*) The locus of a point 1cm from the centre of a circle of radius 2·5 cm is a circle of radius 1cm and concentric with the given circle.



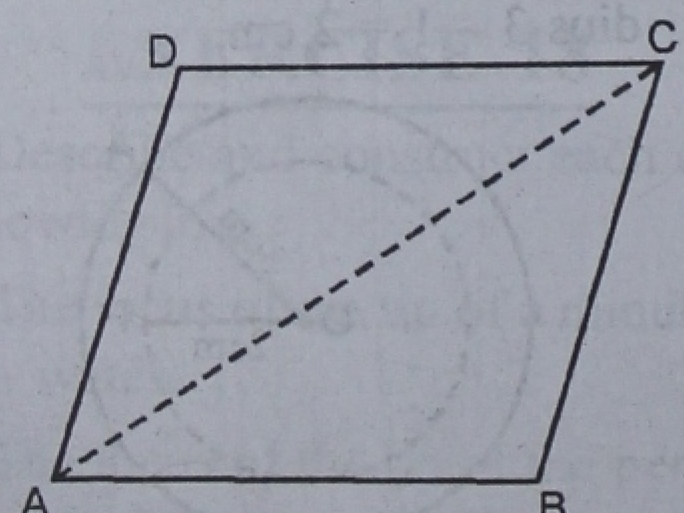
(*x*) The locus o a stone dropped from a tower is a vertical straight line.

(*xi*) AB is a fixed line. Locus of a point P such that $\angle APB = 90°$ is a circle on AB as diameter.

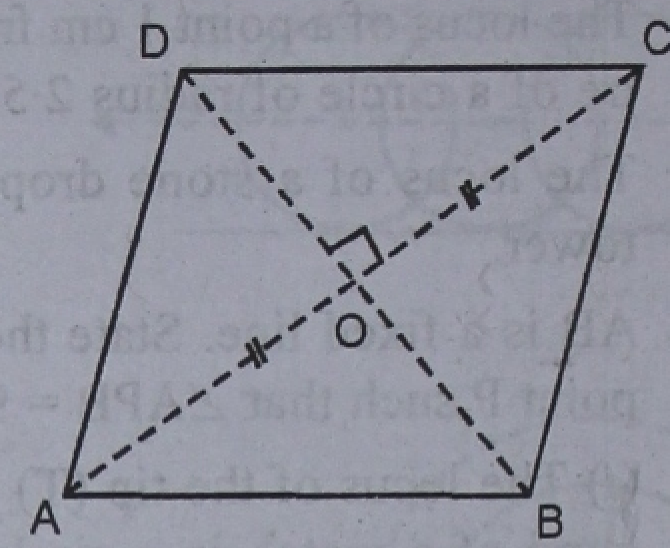


Q. 2. Describe the locus of a point in a rhombus ABCD which is equidistant from (*i*) AB and AD (*ii*) A and C.

Sol. (*i*) The locus of a point in a rhombus ABCD which is equidistant from AB and AD is the bisector of ∠DAB *i.e.* diagonal AC.

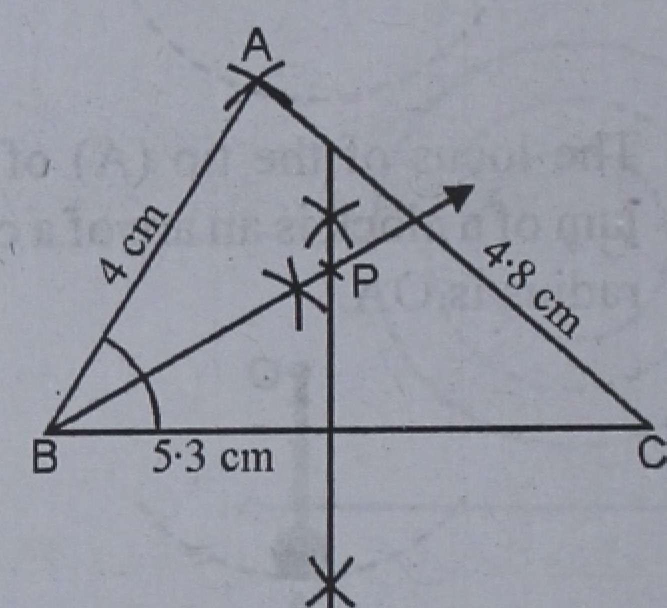


(*ii*) The locus of a point in rhombus ABCD, which is equidistant from A and C is the perpendicular bisector of AC *i.e.* diagonal BD.



Q. 3. Construct a ΔABC in which BC = 5·3 cm, CA = 4·8 cm and AB = 4cm. Find by construction a point P which is equidistant from BC and AB and also equidistant from B and C.

Sol. ***Steps of Construction :***



(*i*) Draw a line segment BC = 5·3cm.

(*ii*) With centre B and radius 4cm draw an arc.

(*iii*) With centre C and radius 4·8 cm draw another arc which intersect the first arc at A.

(*iv*) Join AB and AC.

(*v*) Draw the bisector of ∠ABC.

(*vi*) Draw the perpendicular bisector of side BC which intersect the angle bisector at P.

∴ P is the locus which is equidistant from BC and AB and also from B and C.

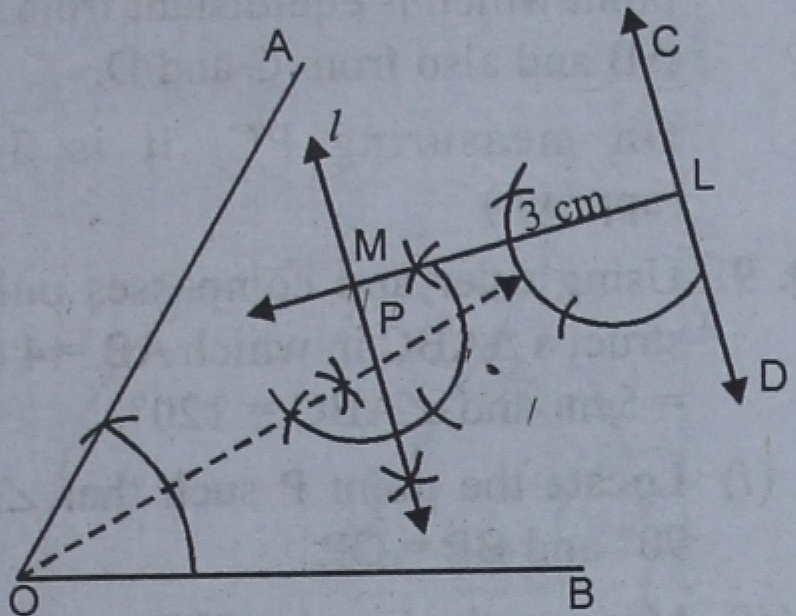
Q. 4. Construct $\angle AOB = 60°$. Mark a point P equidistant from OA and OB such that its distance from another given line CD is 3 cm.

Sol. ***Steps of Construction :***

(*i*) Draw an angle $AOB = 60°$ and a line CD.

(*ii*) Draw bisector of ∠AOB.

(*iii*) Take a point L on CD and draw a perpendicular from L.



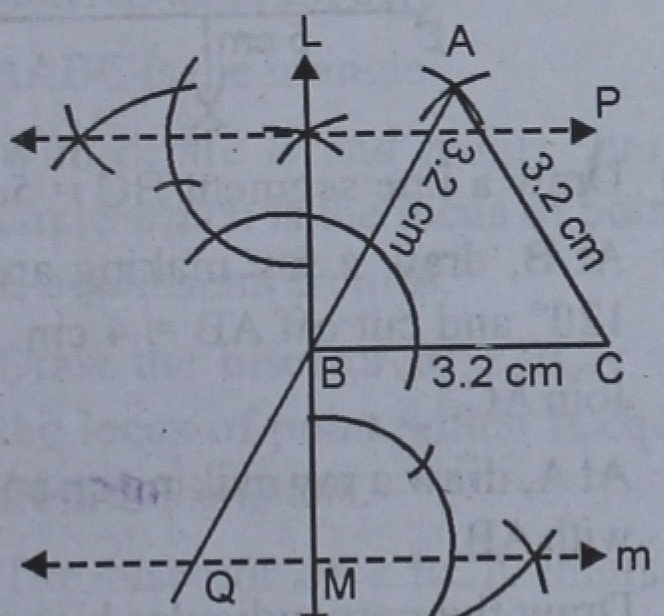
(*iv*) Cut off LM = 3cm.

(*v*) At M, draw a line parallel to CD which intersects the angle bisector at P.

P is the required point which is equidistant from OA and OB and is at a distance of 3cm from the given line CD.

Q.5. ΔABC is an equilateral triangle of side 3·2 cm. Find the points on AB and AB produced which are 2 cm from BC.

Sol. ***Steps of Construction :***



(*i*) Draw a line segment BC = 3·2 cm.

(*ii*) With centre B and C, draw arcs each equal to 3·2 cm radius which intersect each other at A.

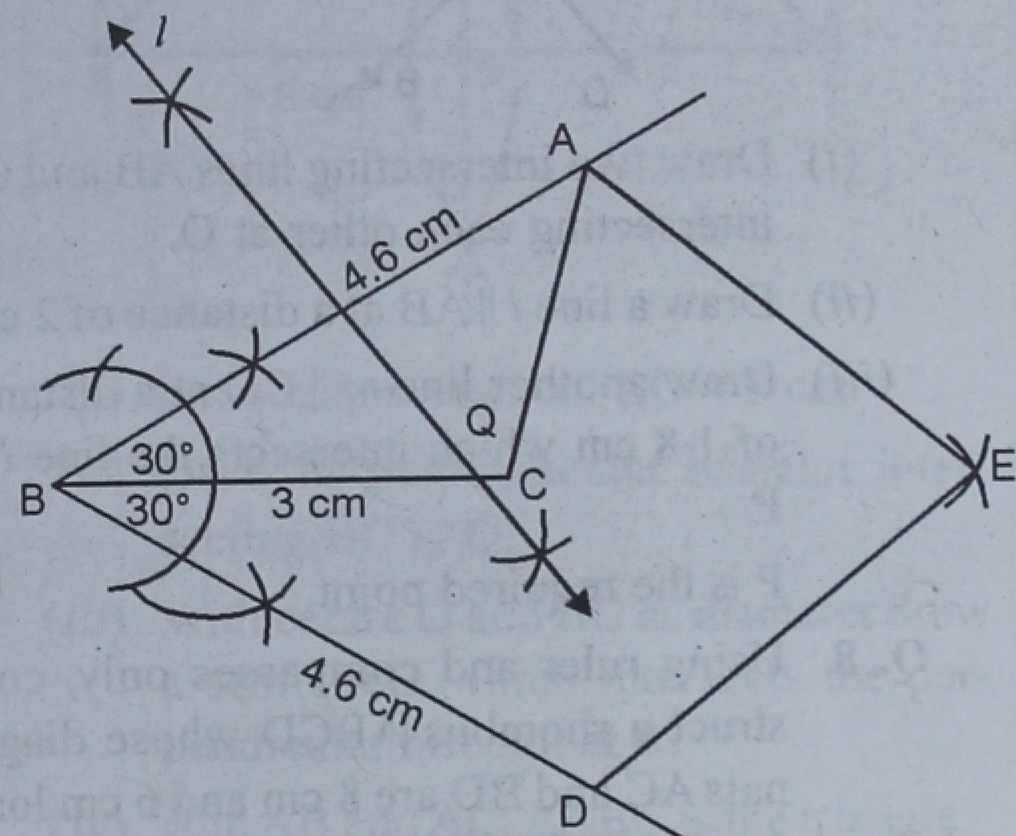
(*iii*) Join AB and AC. ΔABC is the equilateral triangle.

(*iv*) At B, draw a perpendicular and cut off BL = BM = 2cm.

(*v*) At L and M, draw lines parallel to BC which intersect AB at P and AB produced at Q. P and Q are the required points.

Q.6. Using ruler and compasses only, construct a ΔABC such that AB = 4·6 cm, BC = 3cm and ∠ABC = 30°. Complete the rhombus ABDE such that C is equidistant from AB and BD. Locate the point Q on BC such that Q is equidistant from A and B.

Sol. ***Steps of Construction :***



(*i*) Draw a line segment BC = 3cm

(*ii*) At B, drawn a ray making an angle of 30° and cut off BA = 4·6 cm

(*iii*) Join AC

ΔABC is the triangle.

(*iv*) At B, draw ∠CBD = 30° and cut off BD = 4·6 cm.

(*v*) With either D and A and radius 4·6 cm, draw arcs intersecting each other at E.

(*vi*) Join DE and EA

ABDE is the required rhombus.

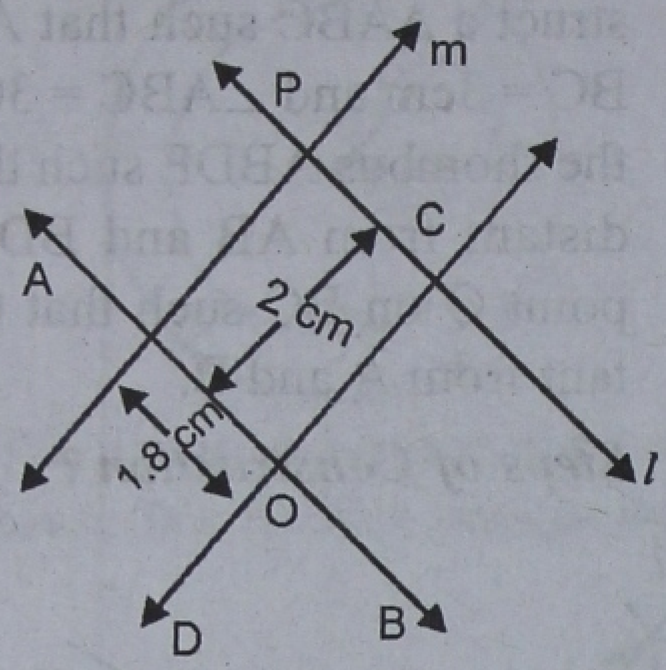
Point C is equidistant from AB and BD because C lies on the angle bisector of ∠ABD.

(*vii*) Draw the perpendicular bisector of AB which intersects BC at Q.

∴ Q is the required point which is equidistant from A and B.

Q.7. AB and CD are two intersecting lines. Find the position of the point distant 2cm from AB and 1·8 cm from CD.

Sol. ***Steps of Construction :***



(*i*) Draw two intersecting lines AB and CD intersecting each other at O.

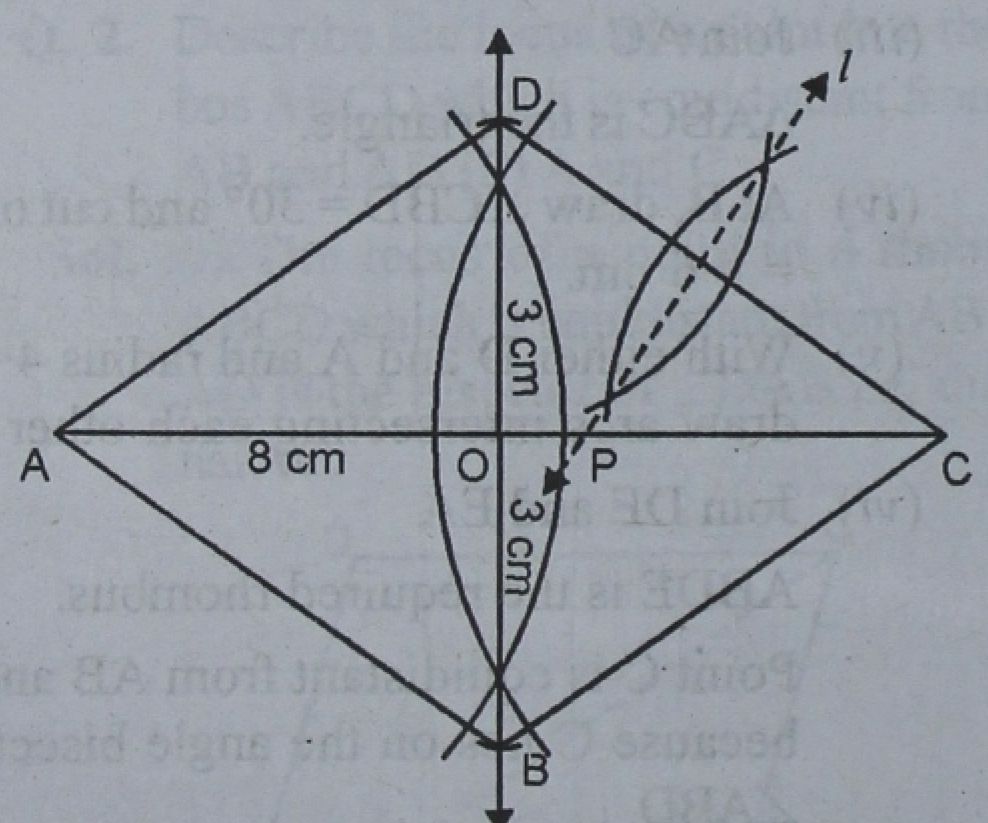
(*ii*) Draw a line *l* || AB at a distance of 2 cm.

(*iii*) Draw another line *m* || CD at a distance of 1·8 cm which intersects the line *l* at P.

P is the required point.

Q. 8. Using ruler and compasses only, construct a rhombus ABCD whose diagonals AC and BD are 8 cm and 6 cm long respectively. Find by construction, a point p equidistant from AB, AD and also equidistant from C and D. Measure PC.

Sol. ***Steps of Construction :***



(*i*) Draw a line segment AC = 8 cm.

(*ii*) Draw the perpendicular bisector of AC and cut off OB = OD = $\frac{6}{2}$ = 3 cm.

(*iii*) Join AB, BC, CD and DA.

ABCD is the rhombus

(*iv*) Draw the perpendicular bisector of CD intersecting AC at P. P is the required point which is equidistant from AD and AB and also from C and D.

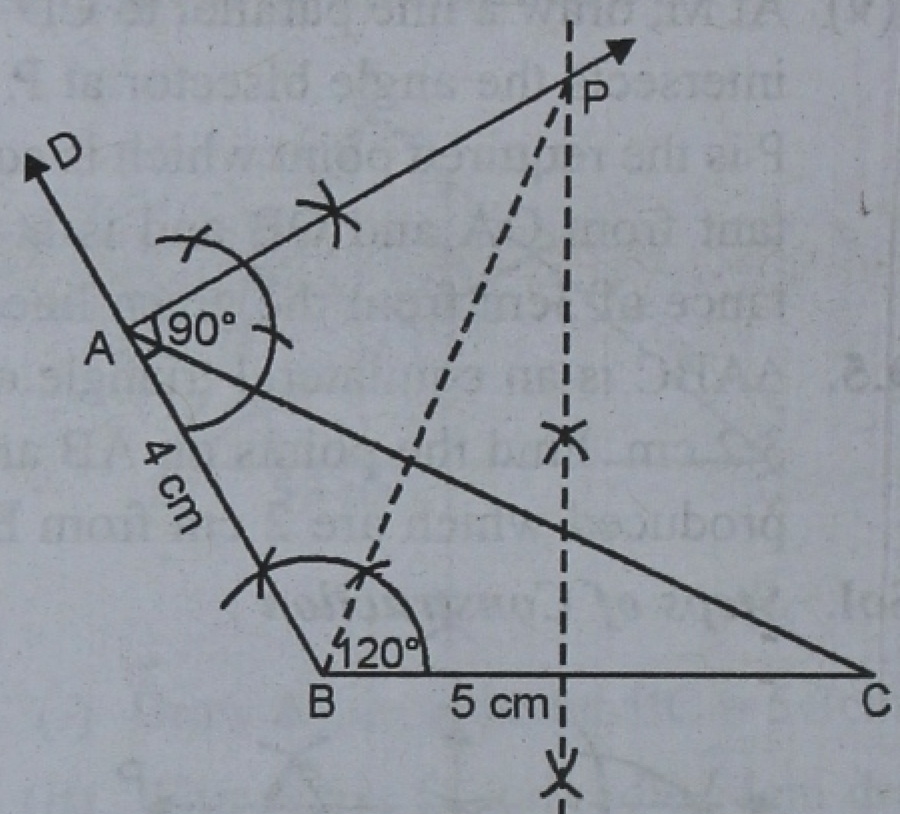
On measuring PC, it is 3·4 cm. (approx.)

Q. 9. Using ruler and compasses only, construct a ΔABC in which AB = 4 cm, BC = 5cm and ∠ABC = 120°.

(*i*) Locate the point P such that ∠BAP = 90° and BP = CP.

(*ii*) Measure the length of BP.

Sol. ***Steps of Construction :***



(*i*) Draw a line segment BC = 5cm.

(*ii*) At B, draw a ray making are angle of 120° and cut off AB = 4 cm.

(*iii*) Join AC.

(*iv*) At A, draw a ray making an angle of 90° with AB.

(*v*) Draw the perpendicular bisector of BC which intersects the 90° ray at P.

P is the required point which is at equidistant from B and C *i.e.* BP = CP and ∠BAP = 90°. On measuring BP, it is 7·3 cm. (approx)

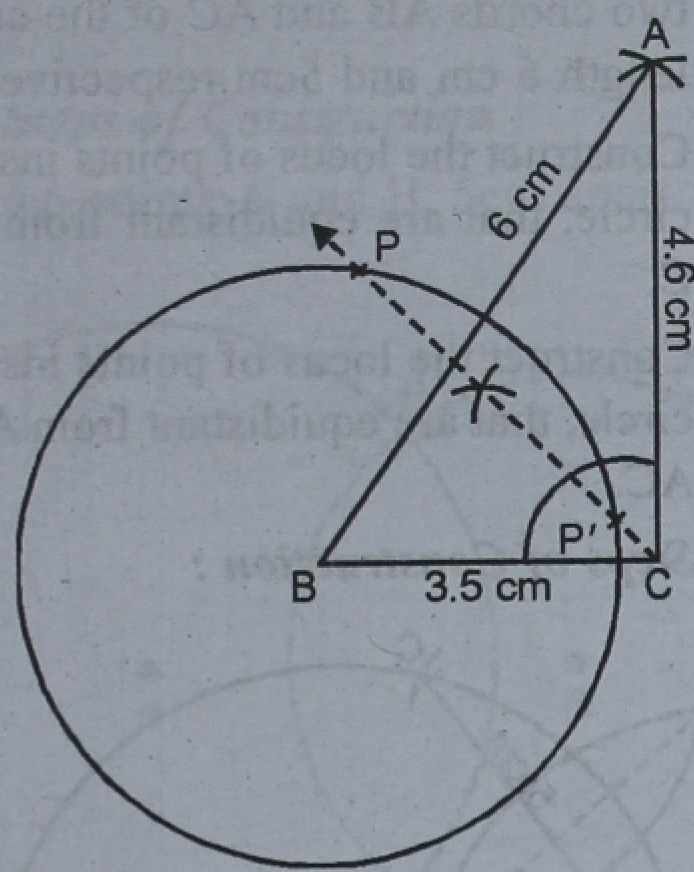
Q. 10. Using ruler and compasses only, construct a ΔABC in which AB = 6 cm, BC = 3·5 cm and CA = 4·6 cm.

(*i*) Draw the locus of a point P which moves so that it is always 3 cm from B.

(*ii*) Draw the locus of a point which moves so that it is equidistant from BC and CA.

(*iii*) Mark the point of intersection of the two loci obtained above. Measure PC.

Sol. ***Steps of Construction :***



(*i*) Draw a line segment BC = 3·5 cm.

(*ii*) With centre B and radius 6cm draw an arc.

(*iii*) With centre C and radius 4·6 cm, draw another arc which intersects the first arc at A.

(*iv*) Join AB and AC

ΔABC is the triangle.

(*a*) With centre B and radius 3cm draw a circle which is the locus of point which is equidistant from B.

(*b*) Draw the bisector of ∠BCA which is the locus of point which is equidistant from BC and CA.

(*c*) The bisector of ∠BCA intersects the circle at P and P′. Hence, P and P′ are two points which satisfy the above two conditions of locus.

On measuring PC, it is 4·2 cm (approx.).

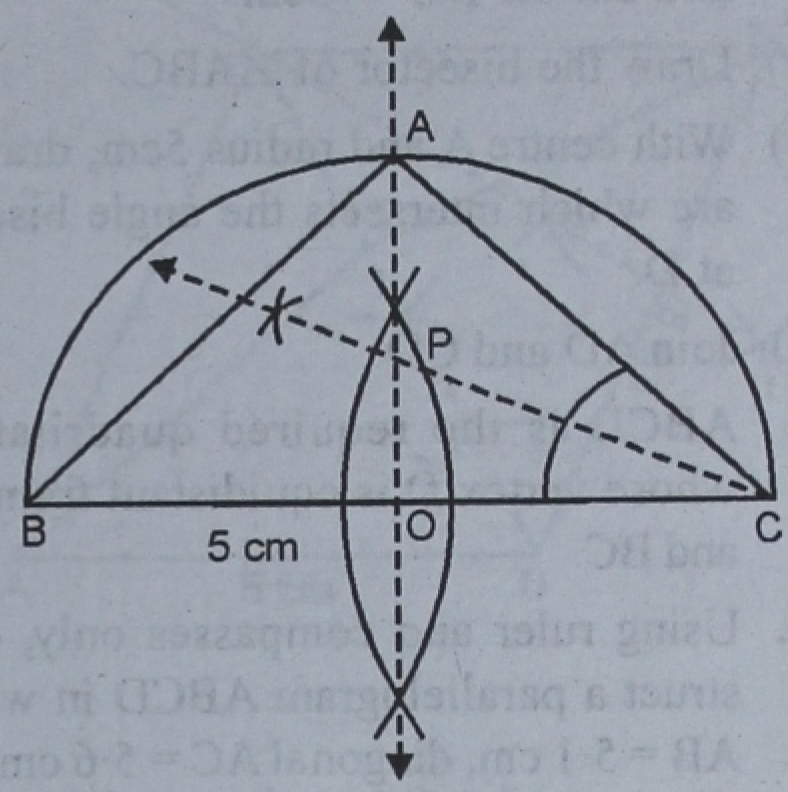
Q. 11. Using ruler and compasses only, construct an isosceles ΔABC in which BC = 5 cm, AB = AC and ∠BAC = 90°

Locate the point P such that :

(*i*) P is equidistant from BC and AC.

(*ii*) P is equidistant from B and C.

Sol. Steps of Construction :



(*i*) Draw a line segment BC = 5 cm.

(*ii*) Draw its perpendicular bisector intersecting BC at O.

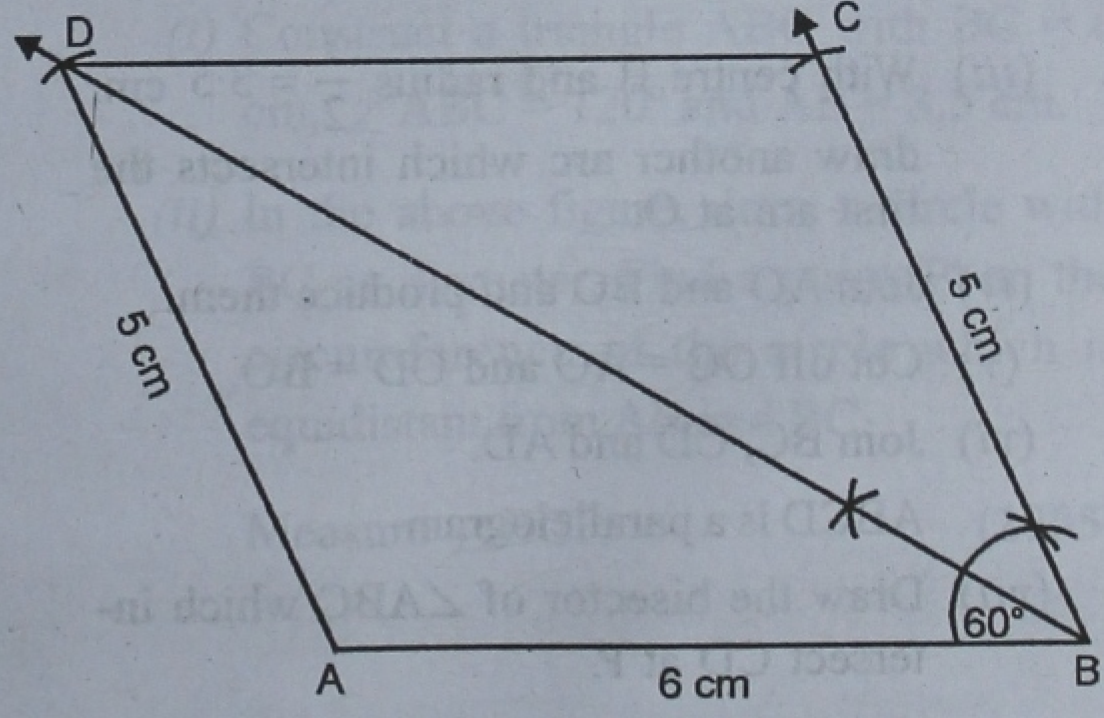
(*iii*) With centre O and BC as diameter draw a semicircle which intersects the perpendicular bisector at A.

(*iv*) Join AB and AC. ΔABC is the triangle.

(*v*) Draw the angle bisector of ∠ACB. Which intersects the perpendicular bisector at P. P is the required point, which is equidistant from BC and AC and also from B and C.

Q. 12. Using ruler and compasses only, construct a quadrilateral ABCD in which AB = 6cm, BC = 5cm, ∠B = 60° AD = 5cm and D is equidistant From AB and BC.

Sol. ***Steps of Construction :***



(*i*) Draw a line segment AB = 6cm.

(*ii*) At B draw a ray making an angle of 60° and cut off BC = 5cm.

(*iii*) Draw the bisector of ∠ABC.

(*iv*) With centre A and radius 5cm, draw an arc which intersects the angle bisector at D.

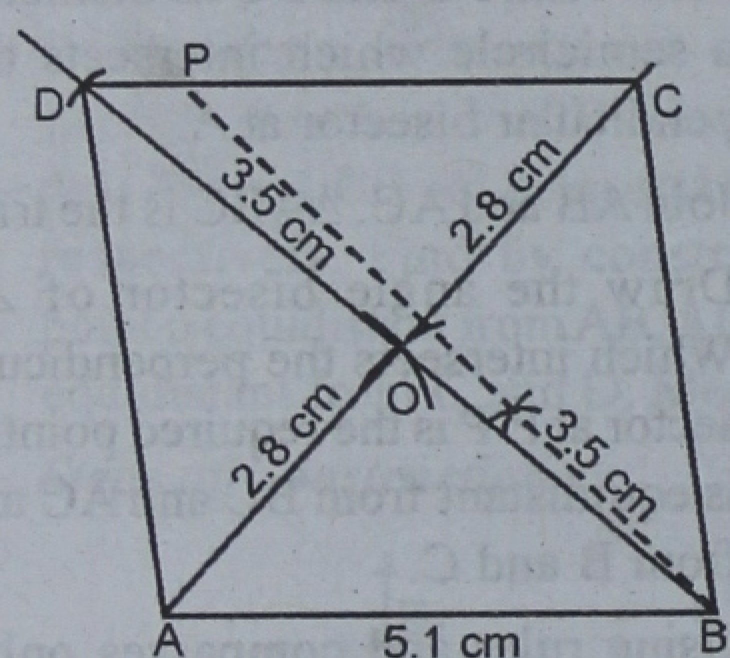
(*v*) Join AD and CD.

ABCD is the required quadrilateral whose vertex D is equidistant from AB and BC.

Q. 13. Using ruler and compasses only, construct a parallelogram ABCD in which AB = 5·1 cm, diagonal AC = 5·6 cm and diagonal BD = 7 cm.

Locate the point P on DC, which is equidistant from AB and BC.

Sol. ***Steps of Construction :***



(*i*) Draw a line segment AB = 5·1 cm.

(*ii*) With centre A and radius $\frac{5 \cdot 6}{2} = 2 \cdot 8$ cm, draw an arc.

(*iii*) With centre B and radius $\frac{7}{2} = 3 \cdot 5$ cm, draw another arc which intersects the first arc at O.

(*iv*) Join AO and BO and produce them.

(*v*) Cut off OC = AO and OD = BO.

(*vi*) Join BC, CD and AD.

ABCD is a parallelogram.

(*vii*) Draw the bisector of ∠ABC which intersect CD at P.

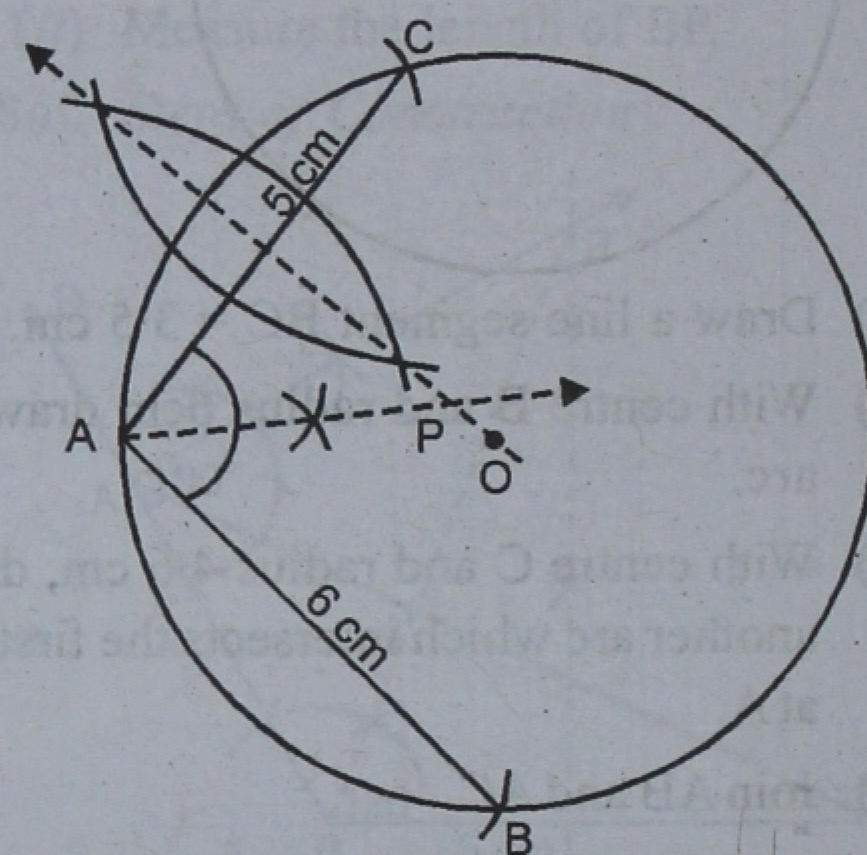
P is the required point which is equidistant from AB and BC.

Q. 14. Draw a circle of radius 4 cm and mark two chords AB and AC of the circle of length 6 cm and 5cm respectively.

(*i*) Construct the locus of points inside the circle, that are equidistant from A and C.

(*ii*) Construct the locus of points inside the circle, that are equidistant from AB and AC.

Sol. ***Steps of Construction :***



(*i*) Draw a circle of radius 4cm with centre O.

(*ii*) Take a point A on it.

(*iii*) With centre A and radius 6 cm, draw an arc which intersects the circle at and with radius 5cm, draw another arc cutting the circle at C.

(*iv*) Join AB and AC.

(*v*) Draw the angle bisector of ∠BAC.

(*vi*) Draw the perpendicular bisector of AC which intersects the angle bisector of P.

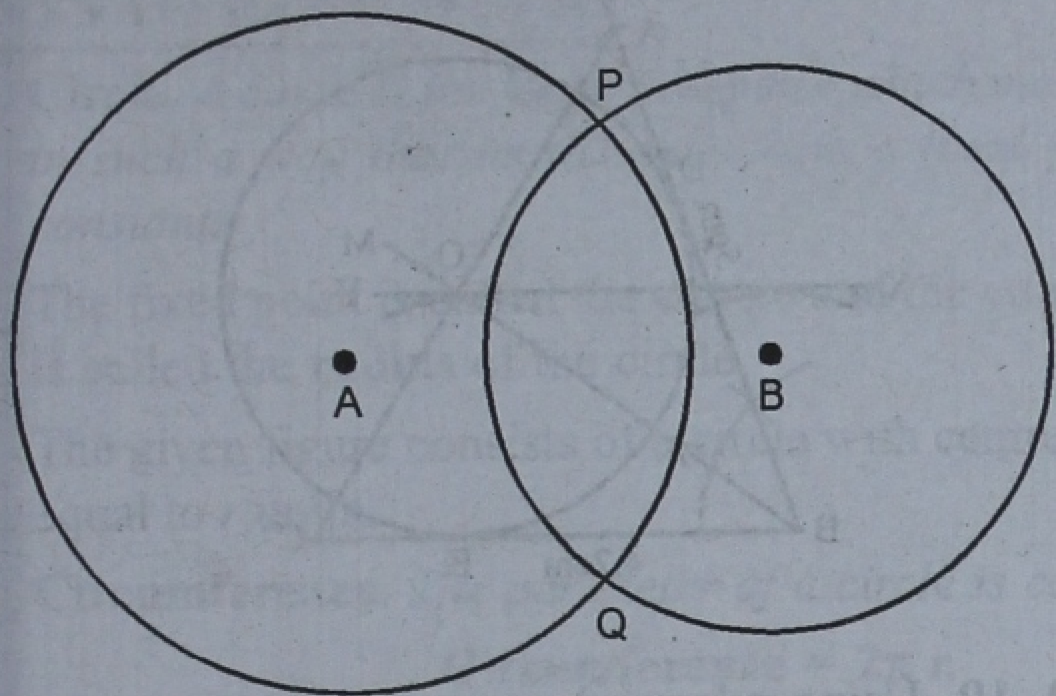
P is the required locus.

Q. 15. A and B are fixed points 5cm apart. The locus of the point P is the set of those points for which AP = 4cm and the locus of Q is the set of those points for which BQ = 3·5 cm.

Construct the loci of P and Q and the points of intersection of the two loci. How many such points are there ?

Sol. ***Steps of Construction :***

(*i*) Take two points A and B, 5cm apart.



(*ii*) With centre A and radius 4cm, draw a circle. This circle is the locus of points which are at a distance 4 cm from A.

(*iii*) With centre B, draw a circle of radius 3·5cm. This circle is the locus of points which are at a distance of 3·5cm from B.

There are two points P and Q where these two circles intersect each other which are the loci of P and Q.

Q. 16. Using only a ruler and compasses, construct ∠ABC = 120°; where AB = BC = 5cm.

(*a*) Mark two points D and E which satisfy the condition that they are equidistant from both BA and BC.

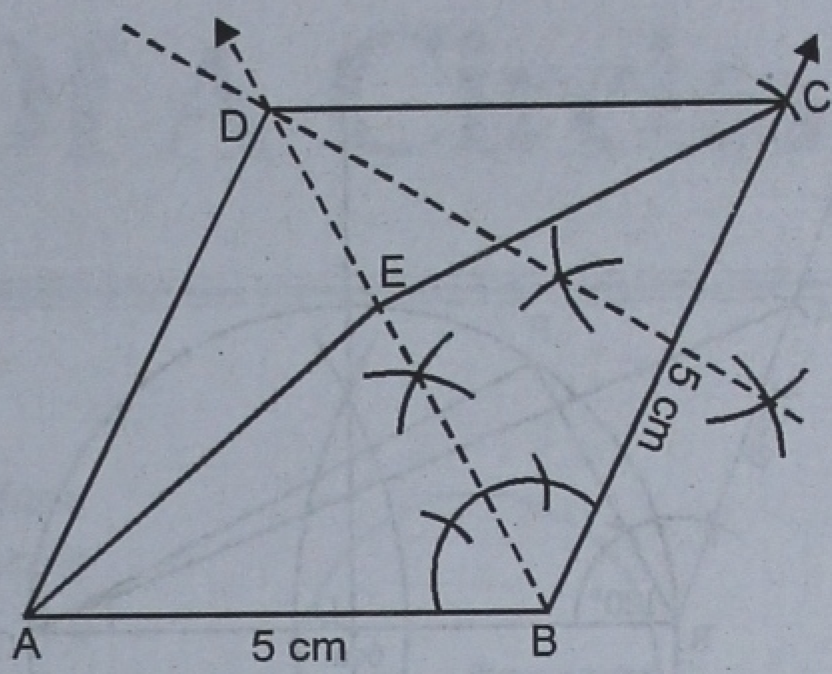
(*b*) In the above figure, join AE and EC. Describe the figures.

(*i*) ABCD

(*ii*) BD

(*iii*) ABE.

Sol. ***Steps of Construction :***



(*i*) Draw a line AB = 5cm.

(*ii*) At B, draw a ray making an angle of 120° and cut off BC = 5cm.

(*iii*) Draw the bisector of ∠ABC.

(*iv*) Draw the perpendicular bisector of BC meeting the angle bisector at D. They are equidistant from AB and BC both.

(*v*) Take any point E on the angle bisector of ∠ABC.

(*vi*) Join AD, DC, AE and EC.

(*b*) (*i*) ABCD is a rhombus.

(*ii*) BD is the angle bisector of ∠ABC.

(*iii*) ABE is a triangle.

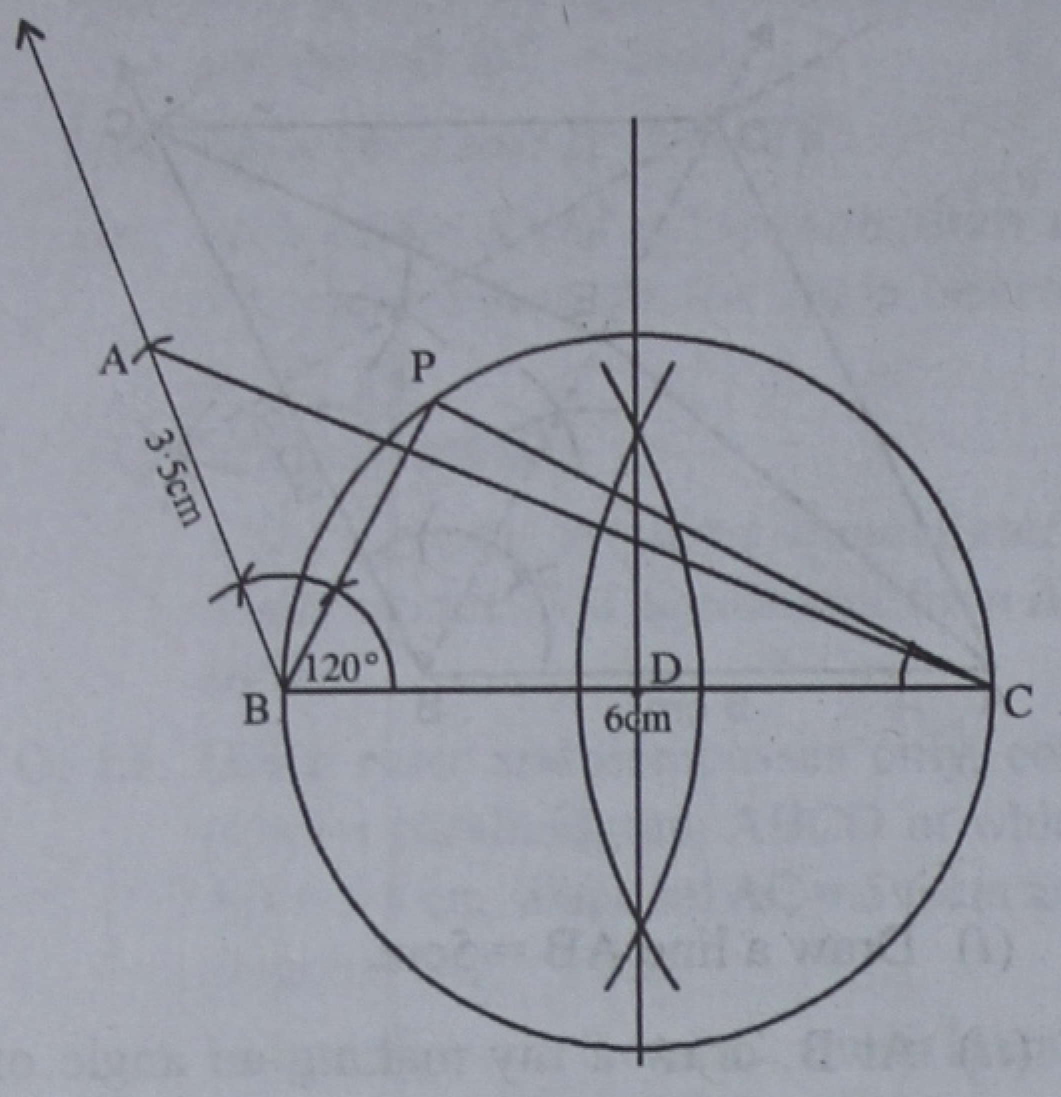
Q. 17. (a) Using a ruler and compass only :

(*i*) Construct a triangle ABC with BC = 6 cm, ∠ABC = 120° and AB = 3.5 cm.

(*ii*) In the above figure, draw a circle with BC as diameter. Find a point 'P' on the circumference of the circle which is equidistant from AB and BC.

Measure ∠BCP. **(2005)**

(a) **Steps of Constructions.**



(*i*) Draw a line segment BC = 6 cm.

(*ii*) Draw a ray at B making an angle of 120° and cut off AB = 3·5 cm.

(*iii*) Join AC.

(*iv*) Draw the perpendicular bisector of BC which intersects BC at D.

(*v*) With centre D and radius BD, draw a circle which passes through B and C.

(*vi*) Draw the bisector of ∠ABC which meets the circle at P.

∴ P is the required point which is equidistant from AB and BC.

(*vii*) Join PC.

On measuring the ∠BCB, it is 30°.

Q. 18. Use a ruler and a pair of compasses to construct Δ ABC in which BC = 4.2 cm, ∠ABC = 60° and AB = 5 cm. Construct a circle of radius 2 cm to touch both the arms of ∠ ABC of Δ ABC.

Sol. Steps of Construction :

(*i*) Draw a line segment BC = 4.2 cm.

(*ii*) At B, draw ∠CBL = 60°.

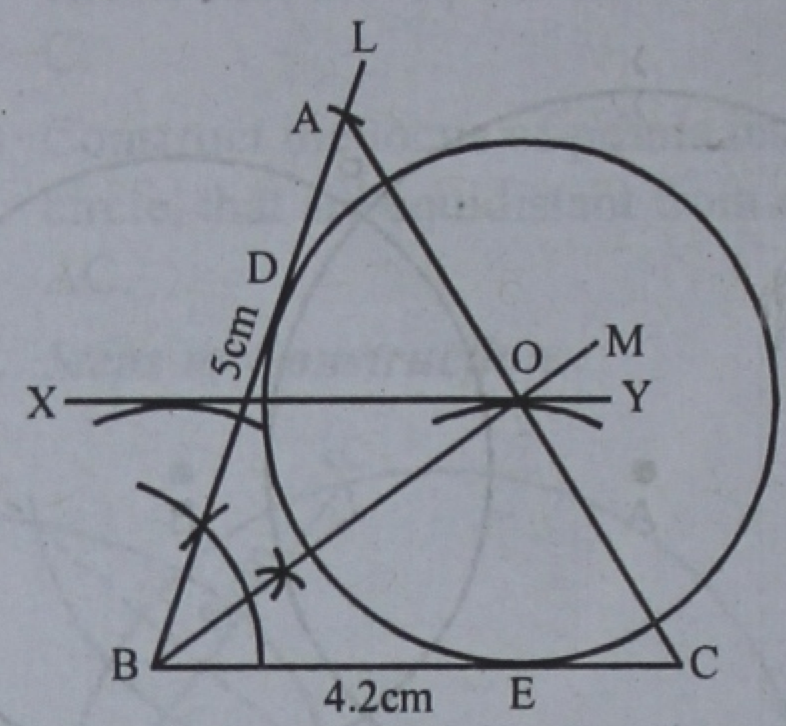
(*iii*) From BL, cut off BA = 5 cm.

(*iv*) Join AC, ABC is the required triangle.

(*v*) Draw BM, the bisector of ∠ ABC.

(*vi*) Draw a line XY ∥ BC at a distance of 2 cm from BC. XY meets BM at O.

(*vii*) With O as centre and radius equal to 2 cm, draw a circle, which touches BA at D and BC at E.



Q. 19. Using ruler and compasses construct :

(*i*) a triangle ABC in which AB = 5.5 cm, BC = 3.4 cm and CA = 4.9 cm.

(*ii*) the locus of points equidistant from A and C.

(*iii*) a circle touching AB at A and passing through C. **(2009)**

Sol. Steps : (*i*) Draw BC = 3.4 and mark the arcs of 5.5 and 4.9 cm from B and C. Join A, B and C.

ABC is the required triangle.

(*ii*) Draw ⊥ bisector of AC.

(*iii*) Draw an angle of 90° at AB at A which intersects ⊥ bisector at O. Draw circle taking O as centre and OA as radius.